Nonlinear hyperbolic equations
in surface theory:
integrable discretizations
and approximation results

A.I. Bobenko ∗  D. Matthes †  Yu.B. Suris ‡

Institut für Mathematik, Technische Universität Berlin,
Str. des 17. Juni 136, 10623 Berlin, Germany.

Abstract

A numerical scheme is developed for solution of the Goursat problem for a class of nonlinear hyperbolic systems with an arbitrary number of independent variables. Convergence results are proved for this difference scheme. These results are applied to hyperbolic systems of differential–geometric origin, like the sine–Gordon equation describing the surfaces of the constant negative Gaussian curvature ($K$–surfaces). In particular, we prove the convergence of discrete $K$–surfaces and their Bäcklund transformations to their continuous counterparts. This puts on a firm basis the generally accepted belief (which however remained unproved until this work) that the classical differential geometry of integrable classes of surfaces and the classical theory of transformations of such surfaces may be obtained from a unifying multi–dimensional discrete theory by a refinement of the coordinate mesh–size in some of the directions.

∗E–Mail: bobenko@math.tu-berlin.de
†E–Mail: matthes@math.tu-berlin.de
‡E–Mail: suris@sfb288.math.tu-berlin.de
1 Introduction

The development of the classical differential geometry led to introduction and studying various classes of surfaces which are of interest both for the internal differential–geometric reasons and for application in other sciences. We mention here minimal surfaces, constant curvature surfaces, isothermic surfaces, to name just a few general classes. A rich theory of such surface classes is, to a large extent, a classical heritage; more recently, also numerical methods appeared, stimulated by applications in sciences and in scientific computation, including visualization. Mostly, these numerical methods are of a variational nature, applicable to problems described by elliptic partial differential equations, like the Plateau problem in the theory of minimal surfaces (see, e.g., [PP, Hi, DH]). It seems that no general numerical methods have been developed for objects of the differential geometry described by hyperbolic differential equations, like the surfaces with constant negative Gaussian curvature.

Figure 1: Surfaces and their transformations as a limit of multidimensional lattices

The characteristic property of various special classes of surfaces studied by the classical differential geometry turns out to be their integrability, in the sense of the modern theory of solitons. One of the manifestations of integrability is the existence of a rich transformations theory, sometimes unified under the names of the Darboux–Bäcklund transformations. Classically, the theory of surfaces and the theory of their transformations were dealt with separately to a large extent. Recently, it became clear that both theories can be unified in the framework of the discrete differential geometry (cf. [S, BP2]). In this framework, multidimensional lattices with certain geometrical properties become the basic mathematical structures. As the lattice becomes more and more dense in some of the coordinates directions (the mesh size $\epsilon \to 0$), it approximates the smooth surface. The directions,
where the mesh size remains constant, correspond to the transformations of smooth surfaces (see Fig. 1).

The discrete differential geometry is nowadays a flourishing area which parallels to a large extent its classical (continuous) counterpart. Many important classes of surfaces have been discretized up to now, see a review in [BP2]. Their properties are well understood. Fig. 2 shows an example of a continuous Amsler surface (which is a surface with constant negative Gaussian curvature) and its discrete analog. The characteristic property of discrete surfaces $F : (\epsilon \mathbb{Z})^2 \mapsto \mathbb{R}^3$ with constant negative Gaussian curvature is that for each $(x, y) \in (\epsilon \mathbb{Z})^2$ the five points $F(x, y)$ and $F(x \pm \epsilon, y \pm \epsilon)$ lie in a plane. The subclass of Amsler surfaces is singled out by the condition that the surface should contain two straight lines.

Figure 2: A countinous and a discrete Amsler surfaces

Considering pictures like this, one is faced with a striking qualitative similarity of the continuous surfaces and their discrete counterparts. Moreover, from numerical experiments it became clear that the approximation is also quantitative: the points on the discrete surface converge to the corresponding

\footnote{We are thankful to Tim Hoffmann for producing these figures.}
points on the continuous surface as the mesh size $\epsilon$ goes to zero. The typical picture of the approximation error is shown on Fig. 3.

Figure 3: Maximum norm of the error $F^\epsilon - F$ vs. mesh size $\epsilon$. $F : [0, r] \times [0, r] \rightarrow \mathbb{R}^3$ is a smooth K-surface, $F^\epsilon : [0, r^\epsilon] \times [0, r^\epsilon] \rightarrow \mathbb{R}^3$ are discrete K-surfaces. (See section 2 for notations.) The lower line corresponds to $r = 1.0$, the upper one to $r = 4.0$.

For a “sufficiently typical” family of discrete surfaces, we have plotted their maximal point distances from the respective continuous counterpart. (See section 5 for details.) The slope of both curves is very close to one, which indicates linear convergence.

All this suggests that it might be possible to develop the classical differential geometry, including both the theory of surfaces and of their transformations, as a limit of the discrete constructions, just by refining the mesh size in some directions. This is a common belief now, having however the status of folklore only, since there are no rigorous mathematical statements supporting it. On the other hand, the good quantitative properties of approximations delivered by the discrete differential geometry suggest that they might be put at the basis of the practical numerical algorithms for computations in the differential geometry. Again, absence of mathematical results on the quality of approximation prevents one from doing this.

The present paper aims at closing this gap. We take a step in this direction, developing a numerical scheme for a class of nonlinear hyperbolic equations, and proving general results on its convergence. It should be said
that we develop the numerics, having in mind also the above mentioned theoretical applications to an alternative foundation of the differential geometry. That means, first, that the class of hyperbolic systems we consider here includes those coming from geometric applications, and, second, that our discretizations respect the geometric structures, i.e. belong to the field of discrete differential geometry. In particular, the notion of integrability, in the guise of the multi-dimensional compatibility of discrete equations [BS], will play a considerable role in our approach.

**Example.** We shall illustrate our constructions by the well-known Sine-Gordon equation:

\[
\partial_x \partial_y \phi = \sin \phi.
\] (1)

A naive discretization of the Sine-Gordon equation could be obtained from (1) by replacing partial derivatives by their difference analogs:

\[
\delta_x^\epsilon \delta_y^\epsilon \phi = \sin \phi,
\] (2)

where we introduced the following notation:

\[
\delta_x^\epsilon p(x, y) = \frac{1}{\epsilon} \left( p(x + \epsilon, y) - p(x, y) \right), \quad \delta_y^\epsilon p(x, y) = \frac{1}{\epsilon} \left( p(x, y + \epsilon) - p(x, y) \right).
\] (3)

In length, Eq. (2) reads:

\[
\phi(x + \epsilon, y + \epsilon) - \phi(x + \epsilon, y) - \phi(x, y + \epsilon) + \phi(x, y) = \epsilon^2 \sin \phi(x, y).
\]

We shall prove an approximation theorem which implies that, on finite domains, the solutions of a Goursat problem for (2) converge with \( \epsilon \to 0 \) to the solutions of a Goursat problem for (1), provided the initial data on the characteristic lines converge. However, this is not the whole story. Solutions of the Sine-Gordon equation correspond to surfaces with constant negative Gaussian curvature. The discretization (2) is non-geometric, and it not clear how to construct discrete surfaces from its solutions. This is closely related to the fact that this discretization does not inherit the integrability of the Sine-Gordon equation. There exists a different one, due to Hirota [H], which is itself a discrete integrable system. Its geometric meaning and relation to discrete surfaces with the constant negative Gaussian curvature was clarified by [BPT]. The Hirota’s discretization of the Sine-Gordon equation reads:

\[
\sin \frac{1}{4} \left( \phi(x + \epsilon, y + \epsilon) - \phi(x + \epsilon, y) - \phi(x, y + \epsilon) + \phi(x, y) \right)
= \epsilon^2 \sin \frac{1}{4} \left( \phi(x + \epsilon, y + \epsilon) + \phi(x + \epsilon, y) + \phi(x, y + \epsilon) + \phi(x, y) \right).
\] (4)
Our theory yields the convergence results for the solutions of this equation to the solutions of (1), which can be extended to the convergence of surfaces, their associated families, and their Bäcklund transformations. The existence of Bäcklund transformations is considered as a characteristic property of integrability [RS].

It should be noticed that there exists an extensive literature dealing with the numerical solution of the sine–Gordon and similar equations, see, e.g. [AHS1, AHS2, FV, FS, SV]. Most of these references have a physical background and motivation, and due to this the problems settled and solved there are different from those relevant to the differential geometry. In particular, neither reference contains convergence results for the Goursat problem. Notice, further, that our results are applicable to a large class of hyperbolic systems describing various further geometries.

The structure of the paper is the following. In Sect. 2 we formulate the continuous and discrete setup of the two–dimensional hyperbolic systems and the corresponding Goursat problems. The $C^1$–convergence result is proven which holds for all difference schemes with a local approximation property. The $C^r$–approximation under the appropriate conditions is established in Sect. 3. The theory is extended to the case of three independent variables in Sect. 4. At this point the notion of three–dimensional compatibility starts to play the key role; it turns out to be intimately related to the integrability. Therefore, the convergence result holds only for difference schemes with these properties. The theory is illustrated in Sect. 5, where we apply the convergence results to an integrable discretization of the sine–Gordon equation, and thus prove the convergence of discrete $K$–surfaces and their Bäcklund transformations to the continuous counterparts. Finally, in Sect. 6 the theory is extended to the case of an arbitrary number of independent variables. The Appendix (Sect. 7) contains the technical proofs of some statements of the main text.

Further pictures of discrete K-surfaces (like the ones that constitute the data for Fig.3), as well as a movie visualization of the convergence can be found on http://www-sfb288.math.tu-berlin.de/~bobenko.

2 Two-dimensional theory

In this section we prove an approximation theorem for a certain class of hyperbolic differential and difference equations in two dimensions. Later on we will consider also more general $d$–dimensional systems. The notations we will use for domains of independent variables are the following: let $r =$
\((r_1, \ldots, r_d)\) consist of positive numbers \(r_i > 0\), then
\[
\Omega(r) = [0, r_1] \times \ldots \times [0, r_d] \subset \mathbb{R}^d.
\] (5)

As domains for discrete equations, we use parts of rectangular lattices inside \(\Omega(r)\), with possibly different grid sizes along different coordinate axes \(\epsilon = (\epsilon_1, \ldots, \epsilon_d)\):
\[
\Omega^\epsilon(r) = [0, r_1]^\epsilon_1 \times \ldots \times [0, r_d]^\epsilon_d \subset \prod_{i=1}^d (\epsilon_i \mathbb{Z}).
\] (6)

where \([0, r]^\epsilon = [0, r] \cap (\epsilon \mathbb{Z})\). The dependent variables of the differential and difference equations under consideration are supposed to belong to a normed vector space \(X\), the norm in which will be denoted by \(| \cdot |\).

In the two–dimensional situation the notations will be, somewhat inconsistently, simplified: we denote continuous domains by \(\Omega(r) = [0, r] \times [0, r] \subset \mathbb{R}^2\), and discrete ones by \(\Omega^\epsilon(r) = [0, r]^\epsilon \times [0, r]^\epsilon \subset (\epsilon \mathbb{Z})^2\). Each \(\Omega^\epsilon(r)\) contains \(O(\epsilon^{-2})\) of grid points. If \(\epsilon_2\) is an integer multiple of \(\epsilon_1\), then \(\Omega^\epsilon_2(r) \subset \Omega^\epsilon_1(r)\).

It will be convenient to assume that \(\epsilon\) attain only values of the form \(2^{-k}\) with a positive integer \(k\); for such \(\epsilon\), the relation \(\epsilon_1 < \epsilon_2\) implies that \(\epsilon_2\) is an integer multiple of \(\epsilon_1\). We also define the limiting domains
\[
\Omega^0(r) = \bigcup_{\epsilon = 2^{-k}} \Omega^\epsilon(r),
\] (7)

which are everywhere dense in \(\Omega(r)\). Each point \(x \in \Omega^0(r)\) belongs to \(\Omega^\epsilon(r)\) with \(\epsilon = 2^{-k}\) for all \(k\) large enough, \(k \geq k_0(x)\). This allows us to speak about convergence by \(\epsilon = 2^{-k} \to 0\) of sequences of functions \(a^\epsilon\) defined on \(\Omega^\epsilon(r)\). In case of convergence, the limiting function may be thought of as defined on \(\Omega^0(r)\). If such a limiting function is continuous (Lipschitz) on \(\Omega^0(r)\), it can be extended to a continuous (resp. Lipschitz) function on \(\Omega(r)\).

Introduce the difference quotient operators \(\delta^x_x p\) and \(\delta^y_y p\), acting on functions \(p : \Omega^\epsilon(r) \to X\), by the formulas (3). The functions \(\delta^x_x p\) and \(\delta^y_y p\) are defined everywhere on \(\Omega^\epsilon(r)\) except for the points with maximal \(x\) or \(y\) coordinate.

**Definition 1** By a continuous 2D hyperbolic system we call a system of partial differential equations for functions \(a, b : \Omega(r) \to X\) of the form
\[
\partial_x a = f(a, b), \quad \partial_y b = g(a, b),
\] (8)

with smooth enough functions \(f, g : X \times X \to X\). A Goursat problem consists of prescribing the initial values
\[
a(x, 0) = a_0(x), \quad b(0, y) = b_0(y)
\] (9)

for \(x \in [0, r]\) and \(y \in [0, r]\), respectively. The functions \(a_0, b_0 : [0, r] \to X\) are also supposed to be smooth enough.
Definition 2 A discrete 2D hyperbolic system (or, better, a one-parameter family of such systems) consists of two partial difference equations for $a, b : \Omega^\epsilon(r) \to \mathcal{X}$ of the form

$$\delta_x^\epsilon a = f^\epsilon(a, b), \quad \delta_y^\epsilon b = g^\epsilon(a, b),$$

with smooth functions $f^\epsilon, g^\epsilon : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$. A Goursat problem for this system consists of prescribing the initial values

$$a(x, 0) = a_0^\epsilon(x), \quad b(0, y) = b_0^\epsilon(y)$$

for $x \in [0, r]^\epsilon$ and $y \in [0, r]^\epsilon$, respectively.

Example. Any equation of the type $\partial_x \partial_y u = F(u)$ can be brought into the form (8) by a variety of substitutions $a = A(u, \partial_x u)$ and $b = B(u, \partial_y u)$. For instance, a canonical way to do this for the Sine-Gordon equation (1) is to introduce two new dependent variables

$$a = \partial_x \phi, \quad b = \phi,$$

which have to satisfy a system of the form (8):

$$\partial_y a = \sin b, \quad \partial_x b = a.$$  
(13)

The second order difference equation (2) can be dealt with in a way which mimics the continuous situation: introduce two new dependent variables

$$a = \delta_x^\epsilon \phi, \quad b = \phi,$$

then they have to satisfy a discrete 2D hyperbolic system

$$\delta_y^\epsilon a = \sin b, \quad \delta_x^\epsilon b = a.$$  
(15)

A similar procedure can be performed with the discretization (4). Set

$$a = \delta_x^\epsilon \phi, \quad b = \phi + \frac{\epsilon}{2} \delta_y^\epsilon \phi.$$  
(16)

(The second equation may be written as $b(x, y) = (\phi(x, y + \epsilon) + \phi(x, y))/2$.) Then the following holds:

$$b(x + \epsilon, y) - b(x, y) = \frac{\epsilon}{2} (a(x, y + \epsilon) + a(x, y)),$$
$$e^{i\epsilon a(x, y + \epsilon)/2} - e^{i\epsilon a(x, y)/2} = \frac{\epsilon^2}{4} \left(e^{ib(x + \epsilon, y)} - e^{-ib(x, y)}\right).$$

(17)  
(18)
(the first equation guarantees the existence of \( \phi \) for a given pair of functions \( a, b \), satisfying (16), while the second one is equivalent to (4)). These equations can be solved for \( a(x, y + \epsilon) \) and \( b(x + \epsilon, y) \). The result reads:

\[
\delta_y a = \frac{2}{i \epsilon^2} \log \frac{1 - (\epsilon^2/4) \exp(-ib - iea/2)}{1 - (\epsilon^2/4) \exp(ib + iea/2)}, \quad \delta_x b = a + \frac{\epsilon}{2} \delta_y a. \tag{19}
\]

Both discrete 2D hyperbolic systems (13), (19) approximate the continuous one (13) in the sense of the next definition.

**Definition 3** A discrete 2D hyperbolic system (14) approximates the continuous one (8), if the functions \( f^\epsilon, g^\epsilon \) continuously depend on \( \epsilon \in (0, \epsilon_0) \), and

\[
f(a, b) = \lim_{\epsilon \to 0} f^\epsilon(a, b), \quad g(a, b) = \lim_{\epsilon \to 0} g^\epsilon(a, b), \tag{20}
\]

uniformly on any compact subset of \( X \times X \). If this convergence holds in the \( C^1 \)-topology, we speak about the \( C^1 \)-approximation. If

\[
f^\epsilon(a, b) = f(a, b) + O(\epsilon), \quad g^\epsilon(a, b) = g(a, b) + O(\epsilon) \tag{21}
\]

uniformly on any compact subset of \( X \times X \), we speak about the \( O(\epsilon) \)-approximation.

The following result is almost obvious:

**Proposition 1** The Goursat problem for a discrete 2D hyperbolic system (14) has a unique solution \((a^\epsilon, b^\epsilon)\) on \( \Omega^\epsilon(r) \).

**Proof.** At this point a useful remark should be done. Although our notations might suggest that the variables \((a, b)\) are attached to the points of the two-dimensional lattice \( \Omega^\epsilon(r) \), it is more natural to assume that they are attached to the edges of this lattice: \( a(x, y) \) – to the horizontal edge connecting the vertices \((x, y)\) and \((x + \epsilon, y)\), and \( b(x, y) \) – to the vertical edge connecting the vertices \((x, y)\) and \((x, y + \epsilon)\). So, the equations (10) give the fields sitting on the right and on the top edges of an elementary square, provided the fields sitting on the left and on the bottom ones are known. See Fig. 4. By induction, the whole solution can be calculated, starting with the fields sitting on all edges on the coordinate axes. ☐

Now the main result of this section can be formulated.

**Theorem 1** Let a family of discrete 2D hyperbolic systems (14) \( O(\epsilon) \)-approximate the continuous 2D hyperbolic system (8) in the \( C^1 \) sense. Let also the discrete initial data (11) approximate the continuous ones (9):

\[
a_0'(x) = a_0(x) + O(\epsilon), \quad b_0'(y) = b_0(y) + O(\epsilon) \tag{22}
\]


uniformly for $x \in [0, r]^*$ and $y \in [0, r]^*$, respectively. Then the sequence of solutions $(a^\epsilon, b^\epsilon)$ converges uniformly in the following sense: there exist $\bar{r} \in (0, r]$ and a pair of Lipschitz-continuous functions $(a, b)$ on $\Omega(\bar{r})$ such that

$$a^\epsilon(x, y) = a(x, y) + O(\epsilon), \quad b^\epsilon(x, y) = b(x, y) + O(\epsilon)$$

for all $(x, y) \in \Omega^\epsilon(\bar{r})$. The functions $(a, b)$ solve the continuous Goursat problem for (8) on $\Omega(\bar{r})$. If the functions $f, g$ admit global Lipschitz constants, then one can choose $\bar{r} = r$.

**Proof.** In general one cannot expect $\bar{r} = r$ in the Theorem because the solutions of the limiting equations may have blow-ups that are absent in the discretization. Consequently, one essential step in proving the theorem is to attain $\epsilon$-independent \textit{a priori} bounds on $a^\epsilon$ and $b^\epsilon$.

**Lemma 1** Let the norms of initial data $a_0^\epsilon$, $b_0^\epsilon$ be bounded by $\epsilon$-independent constants. Then there exists $\bar{r} \in (0, r]$ such that the norms of the solutions $(a^\epsilon, b^\epsilon)$ are bounded on the respective $\Omega^\epsilon(\bar{r})$ independently of $\epsilon$. Furthermore, if $f$ and $g$ have a global Lipschitz constant $L$ on the whole of $X \times X$, then one can choose $\bar{r} = r$.

**Proof of Lemma 1.** Let $|a_0^\epsilon|, |b_0^\epsilon| \leq M_0$ with $M_0 > 0$. We show that, fixing an arbitrary $P_0 > M_0$, it is possible to find $\bar{r} \leq r$ (independent of $\epsilon$) such that one has $|a^\epsilon(x, y)|, |b^\epsilon(x, y)| \leq P_0$ for $(x, y) \in \Omega^\epsilon(\bar{r})$. Actually, it is enough to take

$$\bar{r} = \frac{P_0 - M_0}{\mathcal{F}(P_0)},$$

where

$$\mathcal{F}(M) = \sup_{\epsilon} \sup_{|a|, |b| < M} \{ |f^\epsilon(a, b)|, |g^\epsilon(a, b)| \}. \quad (25)$$
Indeed, from the difference equations (10), written in length as

\[ a^\epsilon(x, y) = a^\epsilon(x, y - \epsilon) + \epsilon f^\epsilon(a^\epsilon(x, y - \epsilon), b^\epsilon(x, y - \epsilon)), \]

\[ b^\epsilon(x, y) = b^\epsilon(x - \epsilon, y) + \epsilon g^\epsilon(a^\epsilon(x - \epsilon, y), b^\epsilon(x - \epsilon, y)), \]

we can conclude by induction that

\[ |a^\epsilon(x, y)| \leq M_0 + yF(P_0), \quad |b^\epsilon(x, y)| \leq M_0 + xF(P_0), \]  

(28)

at least as long as the left-hand sides remain \( \leq P_0 \). Since \( M_0 + \bar{r}F(P_0) = P_0 \), they remain \( \leq P_0 \) for all \( x, y \leq \bar{r} \).

The argument for the case when \( f, g \), and therefore \( f^\epsilon, g^\epsilon \) have a global Lipschitz constant \( L \), is a little bit different. Set

\[ \Delta(x, y) = \max\{|a^\epsilon(x, y)|, |b^\epsilon(x, y)|, M_0\}. \]  

(29)

From (26), (27) it follows readily that

\[ \Delta(x, y) \leq \left(1 + \epsilon J(\Delta(x', y'))\right) \cdot \Delta(x', y'), \]  

(30)

where \( (x', y') = (x, y - \epsilon) \) or \( (x', y') = (x - \epsilon, y) \), depending on which of \( |a^\epsilon(x, y)| \) or \( |b^\epsilon(x, y)| \) is greater, and where

\[ J(M) = \frac{1}{M} \sup_{\epsilon} \sup_{|a|, |b| \leq M} \{|f^\epsilon(a, b)|, |g^\epsilon(a, b)|\}. \]  

(31)

Now, this function admits for \( M \geq M_0 \) an estimate by an absolute constant:

\[ J(M) \leq \frac{1}{M_0}(|f^\epsilon(0, 0)| + |g^\epsilon(0, 0)|) + 2L := K. \]

Hence,

\[ \Delta(x, y) \leq (1 + \epsilon K) \cdot \Delta(x', y'). \]

Using Lemma 2 below (or a simple induction), we find:

\[ \Delta(x, y) \leq M_0 \exp(2K(x + y)) \leq M_0 \exp(4Kr). \]  

(32)

This finishes the proof. ■

On the last step we used the simple particular case \( d = 2, \kappa = 0 \) of the following lemma, which will be used repeatedly later on. Its proof is given in the Appendix.
Lemma 2 Let the function \( \Delta : \Omega^\varepsilon(r) \to \mathbb{R}_+ \) satisfy the following condition: for any \( x \in \Omega^\varepsilon(r) \) different from the origin, there exists an index \( 1 \leq i \leq d \) such that
\[
\Delta(x) \leq (1 + \varepsilon_i \kappa)\Delta(x - \varepsilon_i e_i) + \varepsilon_i \kappa,
\]
where \( \kappa, \kappa \) are some non-negative numbers, and \( e_i \) is the unit vector of the \( i \)-th coordinate axis. Then
\[
\Delta(x) \leq \max \left( \Delta(0), \frac{\kappa}{\kappa} \right) \exp \left( 2\kappa \sum_{j=1}^d x_j \right).
\]

We demonstrate further that the solutions \((a^\varepsilon, b^\varepsilon)\) are not only uniformly bounded, but are actually Lipschitz continuous with a Lipschitz constant independent of \( \varepsilon \). In virtue of the equations \((10)\) and Lemma 1 it is clear that the difference quotients \( \delta_y^\varepsilon a^\varepsilon \) and \( \delta_y^\varepsilon b^\varepsilon \) are uniformly bounded. This turns out to be true also for \( \delta_x^\varepsilon a^\varepsilon \) and \( \delta_x^\varepsilon b^\varepsilon \).

Lemma 3 Let the initial data of the continuous Goursat problem \( a_0, b_0 : [0, r] \to \mathcal{X} \) be \( C^1 \) functions, with the \( C^1 \)–norm less then \( M > 0 \), and let the initial data of the discrete Goursat problem \( a_0^\varepsilon, b_0^\varepsilon : [0, r] \to \mathcal{X} \) satisfy
\[
|a_0^\varepsilon(x) - a_0(x)| \leq M\varepsilon, \quad |b_0^\varepsilon(y) - b_0(y)| \leq M\varepsilon.
\]

With \( \bar{r} \in (0, r] \) chosen according to Lemma 1, the expressions \( \delta_x^\varepsilon a^\varepsilon \) and \( \delta_y^\varepsilon b^\varepsilon \) are bounded on the \( \Omega^\varepsilon(\bar{r}) \) by \( \varepsilon \)-independent constants.

Proof. Let \( M_1 \) be a common bound on the values of the solutions of the discrete Goursat problems. Set
\[
M_2 = \sup_{\varepsilon} \sup_{|a|, |b| \leq M_1} \left\{ |f^\varepsilon(a, b)|, |g^\varepsilon(a, b)|, |\partial_a f^\varepsilon(a, b)|, \ldots, |\partial_b g^\varepsilon(a, b)| \right\}
\]
(recall that \( f^\varepsilon \to f \) and \( g^\varepsilon \to g \) locally uniformly in \( C^1 \)). One can assume that \( M \) from the condition of the Lemma is greater than \( M_1 \) and \( M_2 \). By the mean value theorem, we find an estimate for \( \delta_x^\varepsilon a^\varepsilon(x, y) \) with \( y = 0 \):
\[
|\delta_x^\varepsilon a_0^\varepsilon(x)| \leq |\delta_x^\varepsilon a_0(x)| + \varepsilon^{-1}|a_0^\varepsilon(x + \varepsilon) - a_0(x + \varepsilon)| + \varepsilon^{-1}|a_0^\varepsilon(x) - a_0^\varepsilon(x)| \leq 3M.
\]

Proceeding from \( y \) to \( y + \epsilon \), we find:
\[
|\delta_x^\varepsilon a^\varepsilon(x, y + \varepsilon)| \leq |\delta_x^\varepsilon a^\varepsilon(x, y)| + \varepsilon|\delta_x^\varepsilon f^\varepsilon(a^\varepsilon(x, y), b^\varepsilon(x, y))| \leq |\delta_x^\varepsilon a^\varepsilon(x, y)| + \varepsilon M(\delta_x^\varepsilon a^\varepsilon(x, y) + |\delta_x^\varepsilon b^\varepsilon(x, y)|) \leq (1 + \varepsilon M)|\delta_x^\varepsilon a^\varepsilon(x, y)| + \varepsilon M^2.
\]
Now Lemma 3 yields the desired estimate:

\[ |\delta^e a^e(x, y)| \leq 4M \exp(M\bar{r}). \]

The same reasoning applies to \( \delta^e b^e \).

**Proof of Theorem 1 continued.** We have to show that the \( \epsilon \)-dependent solutions of difference equations have limits \((a^0, b^0)\) on \( \Omega^0(\bar{r}) \) with the convergence rates

\[
\sup_{(x, y) \in \Omega^0(\bar{r})} |a^e(x, y) - a^0(x, y)|, |b^e(x, y) - b^0(x, y)| = \mathcal{O}(\epsilon),
\]

and that these limiting functions \((a^0, b^0)\) can be extended to continuous functions \((a, b)\) on \( \Omega(\bar{r}) \) satisfying the differential equations.

Take \( M > M_2 \) (cf. (33)) such that it bounds also \( a^e \) and \( b^e \) along with their respective difference quotients (see Lemma 3). We will prove that \( a^e(x, y) \) and \( b^e(x, y) \) are Cauchy sequences at any point \((x, y) \in \Omega^0(\bar{r})\). To this end, fix \( \epsilon, \epsilon' \) which are of the usual form \( 2^{-k} \); we assume \( \epsilon > \epsilon' \), so that \( \epsilon/\epsilon' \) is a natural number. Below we use the following abbreviations: \( \Delta^{\epsilon\epsilon'} a(x, y) \) for \( |a^e(x, y) - a^{\epsilon\epsilon'}(x, y)| \), and \( f^e[a^e, b^e](x, y) \) for \( f^\epsilon(a^e(x, y), b^e(x, y)) \), etc. We have:

\[
\Delta^{\epsilon\epsilon'} a(x, y) \leq \\
\Delta^{\epsilon\epsilon'} a(x, y - \epsilon) + \left| \epsilon f^e[a^e, b^e](x, y - \epsilon) - \epsilon' \sum_{k=1}^{\epsilon/\epsilon'} f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \kappa\epsilon') \right| = \\
\Delta^{\epsilon\epsilon'} a(x, y - \epsilon) + \epsilon' \sum_{k=1}^{\epsilon/\epsilon'} \left| f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \epsilon) - f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \kappa\epsilon') \right|. \quad (39)
\]

For the terms in the last sum with a fixed \( 0 < \kappa \leq \epsilon/\epsilon' \) we have:

\[
|f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \epsilon) - f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \kappa\epsilon')| \leq \\
|f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \epsilon) - f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \epsilon)| \quad (40) \\
+ |f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \epsilon) - f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \epsilon)| \quad (41) \\
+ |f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \epsilon) - f^{\epsilon'}[a^{\epsilon'}, b^{\epsilon'}](x, y - \kappa\epsilon')|. \quad (42)
\]

Here (41) = \( \mathcal{O}(\epsilon) \) because \( f^\epsilon = f + \mathcal{O}(\epsilon) \), (12) = \( \mathcal{O}(\epsilon) \) due to Lemma 3, and

\[
(11) \leq M \left( \Delta^{\epsilon\epsilon'} a(x, y - \epsilon) + \Delta^{\epsilon\epsilon'} b(x, y - \epsilon) \right).
\]

Putting all this on the right–hand side of (33), and taking into account that the sum there contains \( \mathcal{O}(\epsilon/\epsilon') \) terms, we find:

\[
\Delta^{\epsilon\epsilon'} a(x, y) \leq (1 + \epsilon M)\Delta^{\epsilon\epsilon'} a(x, y - \epsilon) + \epsilon M \Delta^{\epsilon\epsilon'} b(x, y - \epsilon) + \mathcal{O}(\epsilon^2).
\]
The analogous estimate holds for $b$:

$$\Delta^\epsilon b(x, y) \leq (1 + \epsilon M)\Delta^\epsilon b(x - \epsilon, y) + \epsilon M \Delta^\epsilon a(x - \epsilon, y) + O(\epsilon^2).$$

Introducing the quantities

$$m_0 := \sup_{[0, \bar{r}]} \left\{ |a^\epsilon_0(\cdot) - a^\epsilon_0(\cdot)|, |b^\epsilon_0(\cdot) - b^\epsilon_0(\cdot)| \right\},$$

$$\Delta(x, y) := \max\{\Delta^\epsilon a(x, y), \Delta^\epsilon b(x, y), m_0\},$$

we get the estimate which holds also for the points of the boundary of $\Omega^\epsilon(\bar{r})$ different from the origin:

$$\Delta(x, y) \leq (1 + \epsilon M)\Delta(x', y') + O(\epsilon^2),$$

where either $(x', y') = (x, y - \epsilon)$, or $(x', y') = (x - \epsilon, y)$. Now Lemma 3 yields the final estimate: $\Delta(x, y) = O(\epsilon)$. Thus $a^\epsilon(x, y)$ and $b^\epsilon(x, y)$ are proved to form Cauchy sequences at every point $(x, y) \in \Omega^\epsilon(\bar{r})$. Their limits $a^0(x, y)$ and $b^0(x, y)$ satisfy (38). These limiting functions obviously have the same Lipschitz constants as all the $a^\epsilon$ and $b^\epsilon$. Therefore there is a unique (Lipschitz) continuous extension of these functions from $\Omega^0(\bar{r})$ to $\Omega(\bar{r})$; we call these extensions $a$ and $b$.

It remains to be shown that $(a, b)$ solve the differential equations (8). To do this, we prove that certain integral equations hold. Let $(x, y) \in \Omega^0(\bar{r})$, then:

$$a^\epsilon(x, y) = a^\epsilon_0(x) + \epsilon \sum_{k=0}^{y/\epsilon-1} f^\epsilon[a^\epsilon, b^\epsilon](x, k\epsilon) =$$

$$a^\epsilon_0(x) + \epsilon \sum_{k=0}^{y/\epsilon-1} f[a, b](x, k\epsilon) + \epsilon \sum_{k=0}^{y/\epsilon-1} \left( f^\epsilon[a^\epsilon, b^\epsilon](x, k\epsilon) - f[a, b](x, k\epsilon) \right).$$

All the terms inside the last sum are uniformly $O(\epsilon)$, therefore the whole sum (with the pre-factor $\epsilon$) is also $O(\epsilon)$. The first sum on the right-hand side is, up to $O(\epsilon)$, equal to $\int_0^y f(a(x, \eta), b(x, \eta))d\eta$, which exists since $f$ is smooth and $a, b$ are Lipschitz. Therefore,

$$a(x, y) = a_0(x) + \int_0^y f[a, b](x, \eta)d\eta$$

on $\Omega^0(\bar{r})$. By continuity, this holds on all of $\Omega(\bar{r})$. It follows that $a$ is everywhere differentiable with respect to $x$, and $\partial_x a = f(a, b)$. The function $b$ is treated in the same manner. \( \blacksquare \)
Corollary 1 The two dimensional hyperbolic Goursat problem (8), (9) possesses a unique classical solution.

Proof. The existence part of this statement is an immediate consequence of Theorem [4]; uniqueness is easy to show as follows. Let \((a, b)\) and \((\hat{a}, \hat{b})\) be two solutions of the system of integral equations consisting of (45) and

\[
b(x, y) = b_0(y) + \int_0^x g[a, b](\xi, y)d\xi.
\] (46)

Subtracting the corresponding equations, we find after some simple manipulations:

\[
\Delta(x, y) \leq L \left( \int_0^x \Delta(\xi, y)d\xi + \int_0^y \Delta(x, \eta)d\eta \right),
\]

where we introduced the deviation function

\[
\Delta(x, y) = |a(x, y) - \hat{a}(x, y)| + |b(x, y) - \hat{b}(x, y)|,
\]

and \(L\) is a common Lipschitz constant of \(f, g\) over the range of values of \(a, b\). Now the 2D version of the classical Gronwall inequality (\(d = 2\) case of Lemma [4] below) implies \(\Delta(x, y) \equiv 0\). ■

Lemma 4 Let the continuous function \(\Delta : \Omega(\mathbb{R}) \mapsto \mathbb{R}\) satisfy

\[
\Delta(x_1, \ldots, x_d) \leq L \sum_{j=1}^d \int_0^{x_j} \Delta(x_1, \ldots, x_{j-1}, \xi_j, x_{j+1}, \ldots, x_d)d\xi_j + Q,
\] (47)

with some constants \(L, Q \geq 0\). Then

\[
\Delta(x_1, \ldots, x_d) \leq 2Q \exp \left( 2dL \sum_{i=1}^d x_i \right).
\] (48)

Proof of this lemma is put in the Appendix.

3 Additional Smoothness

In this section we show that the discrete techniques can be used to prove regularity of the solutions to the hyperbolic equations.
**Theorem 2** Let the assumptions of Theorem 1 hold; additionally, let the partial difference quotients up to the order $k + 1$ of the discrete initial data be uniformly bounded independently of $\epsilon$:

$$|((\delta_x^\epsilon)^m a_0^\epsilon(x))|, |((\delta_y^\epsilon)^n b_0^\epsilon(y))| \leq M, \quad m, n \leq k + 1,$$

(49)

Suppose that the convergence $f^\epsilon \to f$, $g^\epsilon \to g$ is locally uniform in $C^{k+1}$. Then the limit functions $a = \lim_{\epsilon \to 0} a^\epsilon$, $b = \lim_{\epsilon \to 0} b^\epsilon$ belong to $C^k$, and moreover these limits are uniform in $C^k$:

$$\sup_{\Omega^\epsilon(r-K\epsilon)} |((\delta_x^\epsilon)^m (\delta_y^\epsilon)^n a^\epsilon - \partial_x^m \partial_y^n a)|, |((\delta_x^\epsilon)^m (\delta_y^\epsilon)^n b^\epsilon - \partial_x^m \partial_y^n b)| \to 0, \quad m + n \leq k.$$

(50)

**Remark.** Assume that the convergence $f^\epsilon \to f$, $g^\epsilon \to g$ is locally uniform in $C^\infty$ and that the initial data $a_0, b_0$ are $C^\infty$–smooth. Then the canonical choice of the discrete initial data $a_0^\epsilon(x) = a_0(x)$ and $b_0^\epsilon(y) = b_0(y)$ guarantees the $C^k$–convergence for any $k$. One may then loosely speak of $C^\infty$-approximation.

The ideas of the proof and even the essential estimates are basically the same as one would most likely use in the continuous setting. However, the discrete setting has the advantage that in contrast to higher-order partial derivatives all the difference quotients automatically exist, commute with each other etc.

First, one obtains à priori estimates not only for the values of $a^\epsilon$, $b^\epsilon$, but for their higher order difference quotients. We will need discrete analogues of the $C^k$-norms. Let $\mathcal{Y}$ be a normed linear space; for a function $u : \Omega^\epsilon(r) \to \mathcal{Y}$, define

$$\|u\|_0 = \sup_{\Omega^\epsilon(r)} |u|,$$

(51)

and

$$\|u\|_K = \max_{k + \ell \leq K} \sup_{\Omega^\epsilon(r-K\epsilon)} |((\delta_x^\epsilon)^k (\delta_y^\epsilon)^\ell u)|.$$

(52)

The following statement comes to replace the chain rule, which is no more available in the discrete context.

**Lemma 5** Let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a smooth function, and consider two functions $a, b : \Omega^\epsilon(r) \to \mathcal{X}$. Then the $K$-th order difference quotients of $f(a, b)$ can be estimated as follows ($m + n = K$):

$$|((\delta_x^\epsilon)^m (\delta_y^\epsilon)^n f[a, b](x, y))| \leq A \cdot \left( |((\delta_x^\epsilon)^m (\delta_y^\epsilon)^n a(x, y))| + |((\delta_x^\epsilon)^m (\delta_y^\epsilon)^n b(x, y))| \right) + P(\|a\|_{K-1}, \|b\|_{K-1}) + B\epsilon.$$

(53)
Here $P$ is a polynomial on two variables of total degree $\leq K$ with positive coefficients; the constants $A$, $B$, and the coefficients of $P$ depend only on $K$, on the Lipschitz constant of the functions $a$ and $b$, and on the expressions $\|(D^k_a D^\ell_b f)[a,b]\|_0$ for $k + \ell \leq K + 1$.

The technical proof of the lemma is put in the Appendix.

**Remark.** In applications of Lemma 5, we have to handle a whole $\epsilon$-dependent family of functions $a^\epsilon, b^\epsilon, f^\epsilon, g^\epsilon$ at once. Thus the constants $A$, $B$ and the polynomial $P$ become, in principle, $\epsilon$-dependent, too. However, under the conditions like those of Theorem 2, one can choose these constants independently of $\epsilon$. In particular, if $\sup_\epsilon \|a^\epsilon\|_K$ and $\sup_\epsilon \|b^\epsilon\|_K$ are finite, then $(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n f^\epsilon(a^\epsilon, b^\epsilon)$ is bounded for $m + n = K$, independently of $\epsilon$.

**Lemma 6** Under the conditions of Theorem 2, 

$$\sup_\epsilon \|a^\epsilon\|_{K+1} < \infty, \quad \sup_\epsilon \|b^\epsilon\|_{K+1} < \infty. \tag{54}$$

**Proof** goes by induction with respect to the total degree $m + n = K \leq k + 1$ of the difference quotient. So assume that $\sup_\epsilon \|a^\epsilon\|_{K-1}, \sup_\epsilon \|b^\epsilon\|_{K-1} < \infty$ is already proved.

First, look at the difference quotients of $a^\epsilon$ with $n > 0$. Then

$$(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n a^\epsilon(x, y) = (\delta_x^\epsilon)^m(\delta_y^\epsilon)^{n-1} f^\epsilon(a^\epsilon, b^\epsilon)(x, y), \tag{55}$$

and the statement follows from the induction hypothesis by using Lemma 5, as pointed out in the preceeding remark. Analogously, the difference quotients of $b^\epsilon$ with $m > 0$ are readily estimated. If $n = 0$, then we have:

$$(\delta_x^\epsilon)^m a^\epsilon(x, y) = (\delta_x^\epsilon)^m a^\epsilon(x, y - \epsilon) + \epsilon(\delta_x^\epsilon)^m f^\epsilon[a^\epsilon, b^\epsilon](x, y - \epsilon). \tag{56}$$

Apply Lemma 5 to find:

$$|(\delta_x^\epsilon)^m a^\epsilon(x, y)| \leq \epsilon A \|(\delta_x^\epsilon)^m a^\epsilon(x, y - \epsilon)\| + \epsilon A \|(\delta_x^\epsilon)^m b^\epsilon(x, y)\| + \epsilon \|a^\epsilon\|_{K-1} \|b^\epsilon\|_{K-1} \leq \epsilon^2 B.$$ 

By the induction hypotheses, the case considered above and the remark following Lemma 5, the quantities on the right-hand side are controlled, so that Lemma 5 can be applied, which yields a uniform, $\epsilon$-independent estimate for $(\delta_x^\epsilon)^m a^\epsilon$. Completely the same reasoning applies to $(\delta_y^\epsilon)^n b^\epsilon$. $lacksquare$

Theorem 2 is now a direct consequence of the next lemma.
Lemma 7 Let a sequence of functions \( \{u^\epsilon : \Omega^\epsilon(r) \to X\} \) be bounded in the discrete \( C^{k+1} \)-norm, independently of \( \epsilon \):
\[
\sup_\epsilon \|u^\epsilon\|_{k+1} = M < \infty. \tag{57}
\]
Then there exists a function \( u \in C^k(\Omega(r), X) \), and a subsequence \( \{u^\epsilon\} \subseteq E_\infty \) for which
\[
\sup_{\Omega^\epsilon(r)} |(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n u^\epsilon(x, y) - \partial_x^m \partial_y^n u(x, y)| \to 0 \quad \text{as} \quad \epsilon \to 0, \epsilon \in E_\infty. \tag{58}
\]

Proof. The set \( \Omega^0(r) \) consists of countably many points; choose any enumeration
\[
\Omega^0(r) = \{(x_n, y_n)|n = 1, 2, 3, \ldots\}. \tag{59}
\]
Since all difference quotients up to the order \( k \) are bounded at \((x_1, y_1)\), there exists a subsequence \( \{u^\epsilon\} \subseteq E_1 \) such that
\[
(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n u^\epsilon(x_1, y_1) \to u^{(m,n)}(x_1, y_1) \quad \text{as} \quad \epsilon \to 0, \epsilon \in E_1. \tag{60}
\]
for all \( m + n \leq k \). Repeat this procedure at \((x_2, y_2)\), where a convergent subsubsequence with \( E_2 \subseteq E_1 \) is selected, and so on. We get a series of infinite sets \( E_{i+1} \subseteq E_i \). Eventually, for all \( m + n \leq k \) the limits \( u^{(m,n)}(x, y) \) are defined everywhere on \( \Omega^0(r) \). Since \( \|u^\epsilon\|_{k+1} \leq M \) holds \( \epsilon \)-uniformly, the functions \( u^{(m,n)} \) thus defined have the Lipschitz property on \( \Omega^0(r) \), so they possess unique continuous extensions to \( \Omega(r) \) with \( M \) as a Lipschitz constant.

We are ready to describe the set \( E_\infty \) from assertion of the lemma. We construct it as an infinite sequence of numbers \( \epsilon_j \) converging to zero, in such a way that
\[
\sup_{\Omega^{\epsilon_j}} |(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n u^\epsilon(x, y) - u^{(m,n)}(x, y)| \leq 2^{-j}(2M + 1). \tag{61}
\]
Let \( \epsilon_0 = 1 \), and take \( \epsilon_j < \epsilon_{j-1} \) of the usual form (integer power of 1/2), satisfying
\[
|(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n u^\epsilon(x', y') - u^{(m,n)}(x', y')| \leq 2^{-j} \tag{62}
\]
for all \( m + n \leq k \) at all points \((x', y') \in \Omega^{2^{-j}}(r)\) (we have with necessity \( \epsilon_j \leq 2^{-j} \), therefore \( \Omega^{2^{-j}}(r) \subseteq \Omega^{\epsilon_j}(r) \)). Such \( \epsilon_j \) exists, because for finitely many lattice sites of \( \Omega^{2^{-j}} \) there holds a pointwise convergence argument from the beginning of this proof. In order to establish the estimate \( \text{[61]} \) on the larger set \( \Omega^{\epsilon_j}(r) \), we make use of the Lipschitz properties. Let \((x, y) \in \Omega^{\epsilon_j}(r) \setminus \Omega^{\epsilon_0}(r) \), we make use of the Lipschitz properties.
\( \Omega^\epsilon(r) \) be arbitrary; then there always exists a point \((x', y') \in \Omega^{2^{-j}}(r) \) with 
\[ |x' - x| + |y' - y| \leq 2^{-j}. \]
Thus we conclude
\[ |(\delta^\epsilon_x)^m(\delta^\epsilon_y)^n u^{\epsilon_j}(x, y) - u^{(m,n)}(x, y)| \]
\[ \leq |(\delta^\epsilon_x)^m(\delta^\epsilon_y)^n u^{\epsilon_j}(x, y) - (\delta^\epsilon_x)^m(\delta^\epsilon_y)^n u^{\epsilon_j}(x', y')| \]
\[ + |(\delta^\epsilon_x)^m(\delta^\epsilon_y)^n u^{\epsilon_j}(x', y') - u^{(m,n)}(x', y')| \]
\[ + |u^{(m,n)}(x', y') - u^{(m,n)}(x, y)| \]
\[ \leq M 2^{-j} + 2^{-j} + M 2^{-j}. \]

It remains to show that \( u^{(m,n)} \) are indeed the respective partial derivatives of \( u \). This can be done by proving the corresponding integral equations: from
\[ (\delta^\epsilon_x)^m(\delta^\epsilon_y)^n u^{\epsilon}(x, y) = (\delta^\epsilon_x)^m(\delta^\epsilon_y)^n u^{\epsilon}(0, y) + \epsilon \sum_{k=0}^{x/\epsilon-1} (\delta^\epsilon_x)^{m+1}(\delta^\epsilon_y)^n u^{\epsilon}(k\epsilon, y) \quad (63) \]
there follows in the limit \( \epsilon \to 0, \epsilon \in E_\infty \)
\[ u^{(m,n)}(x, y) = u^{(m,n)}(0, y) + \int_0^x u^{(m+1,n)}(\xi, y)d\xi, \quad (64) \]
so that \( \partial x u^{(m,n)}(x, y) = u^{(m+1,n)}(x, y) \).

**Proof of Theorem 4.** With Lemma 7 at hand, the convergence of the difference quotients of \((a^\epsilon, b^\epsilon)\) is proved as follows. Any subsequence of \((a^\epsilon, b^\epsilon)\) has a subsubsequence which converges uniformly with all its difference quotients to a \(C^k\) limit function. But by Theorem 1, this limit function is the unique solution \((a, b)\) of the correspondent continuous Goursat problem. If any subsequence of a given sequence has a subsequence converging to one and the same limit, then the whole sequence must converge to that same limit.

4 Three–dimensional theory: approximating Bäcklund transformations

The Sine-Gordon equation (1) is a very special one, in that it possesses Bäcklund transformations. This means that from a given solution \( \phi \) we can construct new solutions by solving ordinary differential equations only. The famous formula for a family of elementary Bäcklund transformations \( \phi \mapsto \tilde{\phi} \) for (1) reads:
\[ \partial_x \tilde{\phi} + \partial_x \phi = 2\alpha \sin \frac{\tilde{\phi} - \phi}{2}, \quad \partial_y \tilde{\phi} - \partial_y \phi = \frac{2}{\alpha} \sin \frac{\tilde{\phi} + \phi}{2}. \quad (65) \]
A direct calculation shows that this system is compatible (i.e. \( \partial_y(\partial_x \tilde{\phi}) = \partial_x(\partial_y \phi) \)), provided \( \phi \) is a solution of the Sine-Gordon equation, and then \( \tilde{\phi} \) is also a solution. An equivalent way to express this state of affairs is to introduce, along with the variables \( a, b \) from Equation (12), also the auxiliary function \( \theta = (\tilde{\phi} - \phi)/2 \), which satisfies the following system of ordinary differential equations:

\[
\begin{align*}
\partial_x \theta &= -a + \alpha \sin \theta, \\
\partial_y \theta &= \frac{1}{\alpha} \sin(b + \theta).
\end{align*}
\]

This system is compatible in the sense that \( \partial_y(\partial_x \theta) = \partial_x(\partial_y \theta) \), provided \((a, b)\) solve the system (13) equivalent to the Sine–Gordon equation, and the initial value

\[
\theta(0, 0) = \theta_0
\]

defined uniquely a solution of (66). Then the formulas

\[
\begin{align*}
\tilde{a} &= a + 2\partial_x \theta = -a + 2\alpha \sin \theta, \\
\tilde{b} &= b + 2\theta
\end{align*}
\]

deliver a new solution \((\tilde{a}, \tilde{b})\) of the 2D hyperbolic system (13) equivalent to the Sine–Gordon equation. Clearly, Bäcklund transformations can be iterated in a straightforward manner.

**Definition 4** A continuous 2D hyperbolic system with a Bäcklund transformation is a compatible system of partial differential and difference equations

\[
\begin{align*}
\partial_y a &= f(a, b), \\
\partial_x b &= g(a, b), \\
\partial_x \theta &= u(a, \theta), \\
\partial_y \theta &= v(b, \theta), \\
\delta_z a &= \xi(a, \theta), \\
\delta_z b &= \eta(b, \theta)
\end{align*}
\]

for functions \( a, b, \theta : \Omega(r, R) \mapsto \mathcal{X} \), where

\[
\Omega(r, R) = \{(x, y, z) \mid (x, y) \in \Omega(r), \ z = 0, 1, \ldots, R\}.
\]

Here \( f, g, u, v, \xi, \eta : \mathcal{X} \times \mathcal{X} \mapsto \mathcal{X} \) are assumed to be smooth functions. A Goursat problem is posed by the requirement

\[
a(x, 0, 0) = a_0(x), \ b(0, y, 0) = b_0(y), \ \theta(0, 0, z) = \theta_0(z)
\]

for \( x \in [0, r], \ y \in [0, r], \) and \( z \in \{0, 1, \ldots, R\} \), respectively, with given smooth functions \( a_0(x), b_0(y) \) and a sequence \( \theta_0(0), \ldots, \theta_0(R - 1) \).
The compatibility conditions mentioned in this definition, are to assure
the existence of solutions of the above Goursat problem. They follow from
\[ \partial_y(\partial_x \theta) = \partial_x(\partial_y \theta), \quad \partial_y(\delta_z a) = \delta_z(\partial_y a), \quad \partial_x(\delta_z b) = \delta_z(\partial_x b). \]
In length, these conditions for (69) to be Bäcklund transformations of the
2D hyperbolic system read:
\[
\begin{align*}
D_a u(a, \theta) \cdot f(a, b) + D_b u(a, \theta) \cdot v(b, \theta) &= D_b v(b, \theta) \cdot g(a, b) + D_a v(b, \theta) \cdot u(a, \theta), \\
D_a \xi(a, \theta) \cdot f(a, b) + D_b \xi(a, \theta) \cdot v(b, \theta) &= f(a + \xi(a, \theta), b + \eta(b, \theta)) - f(a, b), \\
D_b \eta(b, \theta) \cdot g(a, b) + D_\theta \eta(b, \theta) \cdot u(a, \theta) &= g(a + \xi(a, \theta), b + \eta(b, \theta)) - g(a, b).
\end{align*}
\]
(72)
The existence of Bäcklund transformations may be regarded as one of the
possible definitions of the integrability of a given 2D continuous hyperbolic
system. For a given 2D continuous hyperbolic system with Bäcklund trans-
formations, not every discretization possesses the analogous property. For
instance, the naive discretization (4) of the Sine-Gordon equation does not
admit Bäcklund transformations, while the integrable discretization (4) does.
The difference analogs of the formulas (65) read:
\[
\begin{align*}
\sin \frac{1}{4}(\tilde{\phi}(x + \epsilon, y) - \tilde{\phi}(x, y) + \phi(x + \epsilon, y) - \phi(x, y)) &= \\
&= \epsilon \alpha \sin \frac{1}{4}(\tilde{\phi}(x + \epsilon, y) + \phi(x, y) - \phi(x + \epsilon, y) - \phi(x, y)). \\
\sin \frac{1}{4}(\tilde{\phi}(x, y + \epsilon) - \tilde{\phi}(x, y) + \phi(x, y + \epsilon) - \phi(x, y)) &= \\
&= \epsilon \sin \frac{1}{4}(\tilde{\phi}(x, y + \epsilon) + \phi(x, y) + \phi(x, y + \epsilon) + \phi(x, y)).
\end{align*}
\]
(73) (74)
Obviously, in the limit \( \epsilon \to 0 \) these equations approximate (65). A very
remarkable feature is that these equations closely resemble the original dif-
fERENCE equation (4), if one considers the tilde as the shift in the third z-
direction. Upon introducing the quantity \( \theta = (\bar{\phi} - \phi)/2 \), one rewrites (73),
(74) in the form of the system of first order equations approximating (66),
(68):
\[
\begin{align*}
\delta_x \epsilon \theta &= -a + \frac{1}{i \epsilon} \log \frac{1 - (\epsilon \alpha/2) \exp(-i \theta + i \alpha \theta/2)}{1 - (\epsilon \alpha/2) \exp(i \theta - i \epsilon \alpha/2)}, \\
\delta_y \epsilon \theta &= \frac{1}{i \epsilon} \log \frac{1 - (\epsilon/2 \alpha) \exp(-i \theta - i \epsilon \theta)}{1 - (\epsilon/2 \alpha) \exp(i \theta + i \theta)},
\end{align*}
\]
(75) (76)
\[ \tilde{a} = a + 2\delta_x \theta, \quad \tilde{b} = b + 2\theta + \epsilon\delta_y \theta. \]  

(77)

This suggests the following definition.

**Definition 5** A discrete 3D hyperbolic system is a collection of compatible partial difference equations of the form

\[
\begin{align*}
\delta_y a &= f^\epsilon(a, b), \\
\delta_x b &= g^\epsilon(a, b), \\
\delta_x \theta &= u^\epsilon(a, \theta), \\
\delta_y \theta &= v^\epsilon(b, \theta), \\
\delta_z a &= \xi^\epsilon(a, \theta), \\
\delta_z b &= \eta^\epsilon(b, \theta),
\end{align*}
\]

for functions \( a, b, \theta : \Omega^\epsilon(r, R) \rightarrow X \), where

\[
\Omega^\epsilon(r, R) = \{(x, y, z) \mid (x, y) \in \Omega^\epsilon(r), z = 0, 1, \ldots, R\}. \hspace{2cm} (78)
\]

Here the functions \( f^\epsilon, g^\epsilon, u^\epsilon, v^\epsilon, \xi^\epsilon, \eta^\epsilon : X \times X \rightarrow X \) are smooth enough. A Goursat problem consists of prescribing the initial data

\[
\begin{align*}
a(x, 0, 0) &= a_0^\epsilon(x), \\
b(0, y, 0) &= b_0^\epsilon(y), \\
\theta(0, 0, z) &= \theta_0^\epsilon(z)
\end{align*}
\]

(80)

for \( x \in [0, r]^\epsilon \), \( y \in [0, r]^\epsilon \), and \( z \in \{0, 1, \ldots, R\} \), respectively.

Compatibility conditions are necessary for solutions of (78) to exist. These conditions express the following identities that have to be fulfilled for the solutions:

\[
\begin{align*}
\delta_y(\delta_x \theta) &= \delta_x(\delta_y \theta), \\
\delta_y(\delta_z a) &= \delta_z(\delta_y a), \\
\delta_x(\delta_z b) &= \delta_z(\delta_x b).
\end{align*}
\]

In length, these formulas read:

\[
\begin{align*}
u^\epsilon\left( a + \epsilon f^\epsilon(a, b), \theta + \epsilon v^\epsilon(b, \theta) \right) - u^\epsilon(a, \theta) &= \\
= \epsilon v^\epsilon\left( b + \epsilon g^\epsilon(a, b), \theta + \epsilon u^\epsilon(a, \theta) \right) - v^\epsilon(b, \theta), \\
\xi^\epsilon\left( a + \epsilon f^\epsilon(a, b), \theta + \epsilon v^\epsilon(b, \theta) \right) - \xi^\epsilon(a, \theta) &= \\
= \epsilon f^\epsilon\left( a + \xi^\epsilon(a, \theta), b + \epsilon \eta^\epsilon(b, \theta) \right) - \epsilon f^\epsilon(a, b), \\
\eta^\epsilon\left( b + \epsilon g^\epsilon(a, b), \theta + \epsilon u^\epsilon(a, \theta) \right) - \eta^\epsilon(b, \theta) &= \\
= \epsilon g^\epsilon\left( a + \xi^\epsilon(a, \theta), b + \epsilon \eta^\epsilon(b, \theta) \right) - \epsilon g^\epsilon(a, b).
\end{align*}
\]

(81)

This has to be satisfied identically in \( a, b, \theta \in X \).

As demonstrated in [BS], the compatibility of a discrete 3D hyperbolic system is closely related to its integrability in the sense of the soliton theory. Moreover, such a key attribute of integrability as a discrete zero curvature representation with a spectral parameter can be derived from the fact of compatibility.
Proposition 2 The Goursat problem for a discrete 3D hyperbolic system \((78)\) satisfying the compatibility conditions \((81)\) has a unique solution \((a', b', \theta')\) on \(\Omega'(r, R)\).

Proof. Like in the proof of Proposition 1, it is enough to demonstrate that the solution can be propagated along an elementary “cube” of the three–dimensional lattice, then the induction will end the proof. Again, it is convenient to assume that the variables \(a(x, y, z), b(x, y, z), \theta(x, y, z)\) are attached not to the points \((x, y, z) \in \Omega(r, R)\), but rather to the edges \([(x, y, z), (x + \epsilon, y, z)], [(x, y, z), (x, y + \epsilon, z)], [(x, y, z), (x, y, z + 1)]\), respectively. Denote (in this proof only) shifts of the edge variables in the directions of \(x, y, z\) axes by the subscripts 1, 2, 3, respectively. (See Fig. 5.) Then the values \((a_2, b_1)\) are determined by the first equation in \((78)\), the values \((\theta_1, \theta_2)\) – by the second one, and the values \((a_3, b_3)\) – by the third one. For each one of the values \((a_{23}, b_{13}, \theta_{12})\) sitting on the edges adjacent to \((x + \epsilon, y + \epsilon, z + 1)\) we get two possible values (from two equations defined on two facets sharing the corresponding edge). The compatibility conditions \((81)\) guarantee that these two values for each of the edge variables actually coincide.

Figure 5: Three-dimensional consistency

Theorem 3 Let the family of discrete 3D hyperbolic systems \((78)\) satisfying the compatibility conditions \((81)\) approximate the continuous 2D hyperbolic system with a Bäcklund transformation \((69)\). Let the approximation be of the order \(O(\epsilon)\) in the \(C^1\)–sense, so that the relations

\[ f'(a, b) = f(a, b) + O(\epsilon), \quad \text{etc.} \quad (82) \]
and similar relations for the first partial derivatives hold uniformly on any bounded set of \((a, b, \theta) \in \mathcal{X}^3\). Let also the initial data be approximated according to
\[
a_0^\epsilon(x) = a_0(x) + \mathcal{O}(\epsilon), \quad b_0^\epsilon(y) = b_0(y) + \mathcal{O}(\epsilon), \quad \theta_0^\epsilon(z) = \theta_0(z) + \mathcal{O}(\epsilon)
\]
uniformly for \(x \in [0, r]^\epsilon, y \in [0, r]^\epsilon, z \in \{0, 1, \ldots, R\}\), respectively.

Then, for some \(\bar{r} \in (0, r]\), the sequence of solutions \((a^\epsilon, b^\epsilon, \theta^\epsilon)\) has a uniform limit of Lipschitz–continuous functions \((a, b, \theta)\) on \(\Omega(r, R)\) in the sense that the relations
\[
a^\epsilon(x, y, z) = a(x, y, z) + \mathcal{O}(\epsilon),
\]
\[
b^\epsilon(x, y, z) = b(x, y, z) + \mathcal{O}(\epsilon),
\]
\[
\theta^\epsilon(x, y, z) = \theta(x, y, z) + \mathcal{O}(\epsilon)
\]
hold uniformly in \((x, y, z)\) from the respective \(\Omega^\epsilon(\bar{r}, R)\). Furthermore, \((a, b, \theta)\) solve the Goursat problem for the continuous 2D system with a sequence of Bäcklund transformations.

Proof parallels the proof of Theorem 1, and starts with an \(\acute{a} \text{ priori}\) estimate for \(a^\epsilon, b^\epsilon, \theta^\epsilon\).

Lemma 8 Let the norms of initial data \(a_0^\epsilon, b_0^\epsilon, \theta_0^\epsilon\) be bounded by \(\epsilon\)-independent constants. Then there exists \(\bar{r} \in (0, r]\) such that the norms of the solutions \((a^\epsilon, b^\epsilon, \theta^\epsilon)\) are bounded on the respective \(\Omega^\epsilon(\bar{r}, R)\) independently of \(\epsilon\). If the right–hand sides of Eqs. (78) possess a global Lipschitz constant \(L\) on the whole of \(\mathcal{X} \times \mathcal{X} \times \mathcal{X}\), then one can choose \(\bar{r} = r\).

Proof of Lemma 8. Let \(|a_0^\epsilon|, |b_0^\epsilon|, |\theta_0^\epsilon| \leq M_0\) with \(M_0 > 0\). Set
\[
\mathcal{F}(M) = \sup_{\epsilon} \sup_{|a|, |b|, |\theta| < M} \{|f^\epsilon(a, b)|, \ldots, |\eta^\epsilon(b, \theta)|\}.
\]
Choose an arbitrary \(P_0 > M_0\), and define inductively \(P_{j+1} = P_j + \mathcal{F}(P_j)\) for \(j = 0, 1, \ldots, R - 1\). Finally, set
\[
\bar{r} = \min_{j=0,1,\ldots,R} \frac{P_j - M_0}{2\mathcal{F}(P_j)},
\]
so that
\[
M_0 + 2\bar{r}\mathcal{F}(P_j) \leq P_j \quad \text{for all} \quad j = 0, 1, \ldots, R.
\]
We show now by induction in \(z\) that for all \((x, y, z) \in \Omega^\epsilon(r, R)\) there holds:
\[
|a^\epsilon(x, y, z)|, |b^\epsilon(x, y, z)|, |\theta^\epsilon(x, y, z)| \leq P_2.
\]
Indeed, like in Lemma 4 (see (28)), from the first pair of equations in (78) we conclude that

\[ |a'(x, y, 0)| \leq M_0 + yF(P_0), \quad |b'(x, y, 0)| \leq M_0 + xF(P_0), \tag{87} \]

and similarly from the second pair of equations in (78) we conclude that

\[ |\theta'(x, y, 0)| \leq M_0 + (x + y)F(P_0), \tag{88} \]

as long as the left–hand sides of these inequalities remain \( \leq P_0 \). Due to (85), they remain \( \leq P_0 \) for all \((x, y) \in \Omega^\epsilon(\tilde{r})\). Now assuming (86) for a given \( z \), the third pair of equations in (78) immediately implies:

\[ |a'(x, y, z + 1)|, \ |b'(x, y, z + 1)| \leq P_z + F(P_z) \leq P_{z+1}, \tag{89} \]

while from the second pair we derive:

\[ |\theta'(x, y, z + 1)| \leq M_0 + (x + y)F(P_{z+1}) \leq P_{z+1}. \tag{90} \]

This proves (86), and therefore the first statement of lemma. The second one is proved similarly to the analogous statement of Lemma 4. ■

Further, like in the two–dimensional case, we have to demonstrate that not only the solutions \((a^\epsilon, b^\epsilon, \theta^\epsilon)\) are bounded on \( \Omega^\epsilon(\tilde{r}, R) \), but also their difference quotients do.

**Lemma 9** Let the initial data of the continuous Goursat problem \( a_0, b_0 : [0, r] \to \mathcal{X} \) be \( C^1 \) functions, with the \( C^1 \)–norm less then \( M > 0 \), and let the initial data of the discrete Goursat problem \( a_0^\epsilon, b_0^\epsilon : [0, r]^\epsilon \to \mathcal{X} \) satisfy

\[ |a_0^\epsilon(x) - a_0(x)| \leq M\epsilon, \quad |b_0^\epsilon(y) - b_0(y)| \leq M\epsilon. \tag{91} \]

Let \( \tilde{r} \in (0, r] \) be chosen according to Lemma 8. Then all the difference quotients \( \delta_x^\epsilon a^\epsilon, \delta_y^\epsilon a^\epsilon, \delta_x^\epsilon b^\epsilon, \delta_y^\epsilon b^\epsilon, \delta_x^\epsilon \theta^\epsilon, \text{ and } \delta_y^\epsilon \theta^\epsilon \) are bounded on all of the \( \Omega^\epsilon(\tilde{r}, R) \) by \( \epsilon \)-independent constants.

**Proof of Lemma 9.** Let the estimates (84) hold for all \((x, y, z) \in \Omega^\epsilon(\tilde{r}, R)\). Then from the equations (78) it is immediately seen that

\[ |\delta_x^\epsilon a^\epsilon|, \ |\delta_y^\epsilon b^\epsilon|, \ |\delta_x^\epsilon \theta^\epsilon|, \ |\delta_y^\epsilon \theta^\epsilon| \leq F(P_z). \]

Therefore, like in Lemma 8, we only have to estimate the quotients \( \delta_x^\epsilon a^\epsilon \) and \( \delta_y^\epsilon b^\epsilon \). Let \( M \) be chosen to be greater than the common Lipschitz constant of
the functions \( f^\varepsilon, g^\varepsilon, \xi^\varepsilon \) and \( \eta^\varepsilon \) for all \((a, b, \theta)\) with \(|a|, |b|, |\theta| \leq P_R\). Then, proceeding as in the proof of Lemma 3, we have:

\[
|\delta^*_x a^\varepsilon(x, y, 0)| \leq |\delta^*_x a^\varepsilon(x, y - \varepsilon, 0)| + \varepsilon|\delta^*_x f^\varepsilon(a^\varepsilon, b^\varepsilon)(x, y - \varepsilon, 0)|
\]

\[
\leq |\delta^*_x a^\varepsilon(x, y - \varepsilon, 0)| + \varepsilon M(|\delta^*_x a^\varepsilon(x, y - \varepsilon, 0)| + |\delta^*_x b^\varepsilon(x, y - \varepsilon, 0)|)
\]

\[
\leq (1 + \varepsilon M)|\delta^*_x a^\varepsilon(x, y - \varepsilon, 0)| + \varepsilon M \mathcal{F}(P_0).
\]

By (37) and Lemma 2, we obtain:

\[
|\delta^*_x a^\varepsilon(x, y, 0)| \leq (3M + \mathcal{F}(P_0))e^{2My}.
\] (92)

Proceeding from \( z - 1 \) to \( z \), we find:

\[
|\delta^*_x a^\varepsilon(x, y, z)| \leq |\delta^*_x a^\varepsilon(x, y, z - 1)| + |\delta^*_y \xi^\varepsilon[a^\varepsilon, \theta^\varepsilon](x, y, z - 1)|
\]

\[
\leq |\delta^*_x a^\varepsilon(x, y, z - 1)| + M(|\delta^*_x a^\varepsilon(x, y, z - 1)| + |\delta^*_y \theta^\varepsilon(x, y, z - 1)|)
\]

\[
\leq (1 + M)|\delta^*_x a^\varepsilon(x, y, z - 1)| + M \mathcal{F}(P_{z-1}).
\]

Applying again Lemma 4, we find:

\[
|\delta^*_x a^\varepsilon(x, y, z)| \leq (|\delta^*_x a^\varepsilon(x, y, 0)| + \mathcal{F}(P_{R-1}))e^{2Mz} \leq (3M + 2P_R)e^{2M(y+z)}. \] (93)

An estimate for \(|\delta^*_y b^\varepsilon(x, y, z)|\) is obtained in a completely similar way. ■

**Proof of Theorem 3, continued.** Proceeding parallel to the proof of the two–dimensional Theorem 1, we introduce the quantities \( \Delta^{\alpha \varepsilon}a = |a^\varepsilon - a^\varepsilon| \) etc., and

\[
\Delta(x, y, z) = \max\{\Delta^{\alpha \varepsilon}a(x, y, z), \Delta^{\alpha \varepsilon}b(x, y, z), \Delta^{\alpha \varepsilon}\theta(x, y, z), m_0\},
\]

where \( m_0 \) is the supremum of the analogous quantities \( \Delta^{\alpha \varepsilon}a_0 = |a_0^\varepsilon - a_0^\varepsilon| \) for the initial data (cf. (43)). The following inequality comes to replace (44):

\[
\Delta(x, y, z) \leq (1 + \delta M)\Delta(x', y', z') + \mathcal{O}(\varepsilon \delta),
\] (94)

where either \((x', y', z') = (x - \varepsilon, y, z)\) or \((x', y', z') = (x, y - \varepsilon, z)\), in which cases \( \delta = \varepsilon \), or else \((x', y', z') = (x, y, z - 1)\), and then \( \delta = 1 \). In any case Lemma 2 is applicable and yields the final estimate: \( \Delta(x, y, z) = \mathcal{O}(\varepsilon) \). This proves that \( a^\varepsilon(x, y, z), b^\varepsilon(x, y, z), \theta^\varepsilon(x, y, z) \) form Cauchy sequences at every point \((x, y, z) \in \Omega^0(\bar{r}, R)\). The end of the proof is completely analogous to that of Theorem 1. ■
5 Approximation theorems for K-Surfaces

In the present section we apply the theory developed so far to prove that the
known construction of discrete surfaces of constant negative Gauss curvature
\( K = -1 \) (\( K\)-surfaces, for short) may be actually used not only to modelling
the geometric properties of their continuous counterparts, but also to a quanti-
tative approximation. First we briefly recall the correspondent geometric
notions.

Let \( F \) be a K-surface parametrized by its asymptotic lines:
\[
F : \Omega (r) \mapsto \mathbb{R}^3.
\] (95)
This means that the vectors \( \partial_x F, \partial_y F, \partial^2_x F, \partial^2_y F \) are orthogonal to the normal
vector \( N : \Omega (r) \mapsto S^2 \). Reparametrizing the asymptotic lines, if necessary ,
we assume that \( |\partial_x F| = 1 \) and \( |\partial_y F| = 1 \). The angle \( \phi = \phi (x, y) \)
between the vectors \( \partial_x F, \partial_y F \) satisfies the sine–Gordon equation (1). Moreover, 
a K-surface is determined by a solution to (1) essentially uniquely . The
 correspondent construction is as follows. Consider the matrices \( U, V \) defined
by the formulas
\[
U(a; \lambda) = \frac{i}{2} \begin{pmatrix} a & -\lambda \\ -\lambda & -a \end{pmatrix},
\] (96)
\[
V(b; \lambda) = \frac{i}{2} \begin{pmatrix} 0 & \lambda^{-1} \exp(ib) \\ \lambda^{-1} \exp(-ib) & 0 \end{pmatrix},
\] (97)
taking values in the twisted loop algebra
\[
g[\lambda] = \{ \xi : \mathbb{R}_* \mapsto \text{su}(2) : \xi(-\lambda) = \sigma_3 \xi(\lambda) \sigma_3 \}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Suppose now that \( a \) and \( b \) are some real–valued functions on \( \Omega (r) \). Then the
zero curvature condition
\[
\partial_y U - \partial_x V + [U, V] = 0
\] (98)
is satisfied identically in \( \lambda \), if and only if \( (a, b) \) satisfy the system (13), or, 
in other words, if \( a = \partial_x \phi \) and \( b = \phi \), where \( \phi \) is a solution of (1). Given a
solution \( \phi \), that is, a pair of matrices (96), (97) satisfying (98), the following
system of linear differential equations is uniquely solvable:
\[
\partial_x \Phi = U \Phi, \quad \partial_y \Phi = V \Phi, \quad \Phi(0, 0, \lambda) = 1.
\] (99)
Here \( \Phi : \Omega (r) \mapsto G[\lambda] \) takes values in the twisted loop group
\[
G[\lambda] = \{ \Xi : \mathbb{R}_* \mapsto \text{SU}(2) : \Xi(-\lambda) = \sigma_3 \Xi(\lambda) \sigma_3 \}.
\]
The solution $\Phi(x, y; \lambda)$ yields the immersion $F(x, y)$ by the **Sym formula**:

$$F(x, y) = (2\lambda \Phi(x, y; \lambda)^{-1} \partial_\lambda \Phi(x, y; \lambda)) \bigg|_{\lambda=1}.$$  \hspace{1cm} (100)

(Here the canonical identification of $\text{su}(2)$ with $\mathbb{R}^3$ is used.) Moreover, the right–hand side of (100), at the values of $\lambda$ different from $\lambda = 1$ delivers a whole family of immersions $F_\lambda : \Omega(r) \mapsto \mathbb{R}^3$, all of which turn out to be asymptotic lines parametrized K–surfaces. These surfaces $F_\lambda$ constitute the so–called **associated family** of $F$.

Now we turn to **discrete K-surfaces**. Let $F^\varepsilon$ be a discrete surface parametrized by asymptotic lines, i.e. an immersion

$$F^\varepsilon : \Omega^\varepsilon(r) \rightarrow \mathbb{R}^3$$  \hspace{1cm} (101)

such that for each $(x, y) \in \Omega^\varepsilon(r)$ the five points $F^\varepsilon(x, y)$, $F^\varepsilon(x \pm \varepsilon, y)$, and $F^\varepsilon(x, y \pm \varepsilon)$ lie in a single plane $\mathcal{P}(x, y)$. It is required that all edges of the discrete surface $F^\varepsilon$ have the same length $\varepsilon \ell$, that is $|\delta^\varepsilon_{x} F^\varepsilon| = |\delta^\varepsilon_{y} F^\varepsilon| = \ell$, and it turns out to be convenient to assume that $\ell = (1 + \varepsilon^2/4)^{-1}$. The same relation we presented between K-surfaces and solutions to the (classical) sine–Gordon equation (1) can be found between discrete K-surfaces and solutions to the sine–Gordon equation in Hirota’s discretization (4):

Consider the matrices $U^\varepsilon$, $V^\varepsilon$ defined by the formulas

$$U^\varepsilon(a; \lambda) = \left(1 + \varepsilon^2 \lambda^2/4\right)^{-1/2} \begin{pmatrix} \exp(i\varepsilon a/2) & -i\varepsilon \lambda/2 \\ -i\varepsilon \lambda/2 & \exp(-i\varepsilon a/2) \end{pmatrix},$$  \hspace{1cm} (102)

$$V^\varepsilon(b; \lambda) = \left(1 + \varepsilon^2 \lambda^{-2}/4\right)^{-1/2} \begin{pmatrix} 1 & (i\varepsilon \lambda^{-1}/2) \exp(ib) \\ (i\varepsilon \lambda^{-1}/2) \exp(-ib) & 1 \end{pmatrix}. $$  \hspace{1cm} (103)

Let $a$, $b$ be real–valued functions on $\Omega^\varepsilon(r)$, and consider the discrete zero curvature condition

$$U^\varepsilon(x, y + \varepsilon; \lambda) \cdot V^\varepsilon(x, y; \lambda) = V^\varepsilon(x + \varepsilon, y; \lambda) \cdot U^\varepsilon(x, y; \lambda)$$  \hspace{1cm} (104)

(where $U^\varepsilon$ and $V^\varepsilon$ depend on $(x, y) \in \Omega^\varepsilon(r)$ through the dependence of $a$ and $b$ on $(x, y)$, respectively). A direct calculation shows that (104) is equivalent to the system (19), or, in other words, to the equation (4) for the function $\phi$ defined by (16). The formula (104) is the compatibility condition of the following system of linear difference equations:

$$\Psi^\varepsilon(x + \varepsilon, y; \lambda) = U^\varepsilon(x, y; \lambda) \Psi^\varepsilon(x, y; \lambda),$$

$$\Psi^\varepsilon(x, y + \varepsilon; \lambda) = V^\varepsilon(x, y; \lambda) \Psi^\varepsilon(x, y; \lambda),$$

$$\Psi^\varepsilon(0, 0; \lambda) = 1.$$  \hspace{1cm} (105)
So, any solution of (4) uniquely defines a matrix $\Psi : \Omega \rightarrow G[\lambda]$ satisfying (103). This can be used to finally construct the immersion by an analog of the Sym formula:

$$F^\epsilon(x, y) = \left(2\lambda \Psi^\epsilon(x, y; \lambda)^{-1} \partial_\lambda \Psi^\epsilon(x, y; \lambda)\right) \bigg|_{\lambda=1}. \quad (106)$$

The geometric meaning of the function $\phi$ is the following: the angle between the edges $F^\epsilon(x + \epsilon, y) - F^\epsilon(x, y)$ and $F^\epsilon(x, y + \epsilon) - F^\epsilon(x, y)$ is equal to $(\phi(x + \epsilon, y) + \phi(x, y + \epsilon))/2$; the angle between the edges $F^\epsilon(x, y + \epsilon) - F^\epsilon(x, y)$ and $F^\epsilon(x - \epsilon, y) - F^\epsilon(x, y)$ is equal to $\pi - (\phi(x, y + \epsilon) + \phi(x - \epsilon, y))/2$; the angle between the edges $F^\epsilon(x - \epsilon, y) - F^\epsilon(x, y)$ and $F^\epsilon(x, y - \epsilon) - F^\epsilon(x, y)$ is equal to $(\phi(x - \epsilon, y) + \phi(x, y - \epsilon))/2$; and the angle between the edges $F^\epsilon(x, y - \epsilon) - F^\epsilon(x, y)$ and $F^\epsilon(x + \epsilon, y) - F^\epsilon(x, y)$ is equal to $\pi - (\phi(x, y - \epsilon) + \phi(x + \epsilon, y))/2$. In particular, these angles sum up to $2\pi$, so that the four neighboring vertices of $F^\epsilon(x, y)$ lie in one plane, as they should.

Again, the right-hand side of (106) at the values of $\lambda$ different from $\lambda = 1$ delivers an associated family $F^\lambda$ of discrete asymptotic lines parametrized K–surfaces.

Finally, we discuss the Bäcklund transformations for continuous and discrete K–surfaces. Introduce the matrix

$$W(\theta; \lambda) = \begin{pmatrix} \alpha \exp(i\theta) & -i\lambda \\ -i\lambda & \alpha \exp(-i\theta) \end{pmatrix}. \quad (107)$$

It is easy to see that the matrix differential equations

$$\partial_x W = \tilde{U} W - W U, \quad \partial_y W = \tilde{V} W - W V \quad (108)$$

are equivalent to the formulas (66), (68). On the other hand, these matrix differential equations constitute a sufficient condition for the solvability of the system consisting of (99) and

$$\tilde{\Phi} = W \Phi. \quad (109)$$

So, frames $\Phi$ can be in a consistent way extended into the third direction $z$ (shift in which is encoded by the tilde), which results also in the transformation of the K–surfaces $F \mapsto \tilde{F}$, and moreover of the whole associated family, via (100).

Similarly, the matrix equations

$$W(x + \epsilon, y; \lambda) \tilde{U}^\epsilon(x, y; \lambda) = \tilde{U}^\epsilon(x, y; \lambda) W(x, y; \lambda), \quad (110)$$

$$W(x, y + \epsilon; \lambda) \tilde{V}^\epsilon(x, y; \lambda) = \tilde{V}^\epsilon(x, y; \lambda) W(x, y; \lambda) \quad (111)$$

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are equivalent to the formulas (75), (76), (77), and, on the other hand, assure the solvability of the system consisting of (105) and

$$\tilde{\Psi}^\epsilon = \mathcal{W}\Psi^\epsilon.$$  \hfill (112)

Therefore, also the frames $\Psi^\epsilon$ of the discrete surfaces can be extended in the third direction $z$. This leads to the transformation of discrete K-surfaces and their associated families, according to (106).

Geometrical meaning of Bäcklund transformations is the following (see, e.g., [BP1]). The asymptotic lines parametrizations of $F$ and $\tilde{F}$ correspond, and the vector $\Delta F = \tilde{F} - F$ lies in the intersection of tangential planes of $F$ and of $\tilde{F}$ in the corresponding points. This vector $\Delta F$ has a constant length $\alpha$, and $\theta$ is the angle between this vector and one of the asymptotic directions. Classically, a Bäcklund transformation is completely defined by $\alpha$ and the value of $\theta$ in one point. The definitions in the discrete case are completely analogous.

Now we are prepared to state the approximation theorem for K-surfaces.

**Theorem 4** Let $a_0 : [0, r] \mapsto \mathbb{R}$ and $b_0 : [0, r] \mapsto S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be two smooth functions. Then:

- There exists a unique asymptotic line parametrized K-surface $F : \Omega(r) \mapsto \mathbb{R}^3$ such that its characteristic angle $\phi : \Omega(r) \mapsto S^1$ on the coordinate axes satisfies:

$$\partial_x \phi(x, 0) = a_0(x), \quad \phi(0, y) = b_0(y), \quad x, y \in [0, r].$$  \hfill (113)

- For any $\epsilon > 0$ there exists a unique asymptotic line parametrized discrete K-surface $F^\epsilon : \Omega^\epsilon(r) \mapsto \mathbb{R}^3$ such that its characteristic angle $\phi^\epsilon : \Omega^\epsilon(r) \mapsto S^1$ on the coordinate axes satisfies:

$$\phi^\epsilon(x+\epsilon, 0) - \phi^\epsilon(x, 0) = \epsilon a_0(x), \quad \phi^\epsilon(0, y+\epsilon) + \phi^\epsilon(0, y) = 2b_0(y),$$  \hfill (114)

for $x, y \in [0, r - \epsilon]^\epsilon$.

- There holds:

$$\sup_{\Omega^\epsilon(r)} |F^\epsilon - F| \leq C\epsilon,$$  \hfill (115)

where $C$ does not depend on $\epsilon$. Moreover, for pair $(m, n)$ of nonnegative integers, there holds

$$\sup_{\Omega^\epsilon(r-\epsilon^\epsilon)} |(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n F^\epsilon - \partial_x^m \partial_y^n F| \to 0 \quad \text{as} \quad \epsilon \to 0.$$  \hfill (116)
• The estimates (115), (116) still hold, uniformly for \( \lambda \in [\Lambda^{-1}, \Lambda] \) with any \( \Lambda > 1 \), if one replaces in these estimates the immersions \( F, F^\epsilon \) by their associated families \( F_\lambda, F^\epsilon_\lambda \), respectively.

• Finally, if \( \theta_1, \ldots, \theta_R \in S^1 \) are parameters of a sequence of Bäcklund transformations, then for the transformed surfaces there also hold estimates analogous to (115), (116).

Remark: Local uniform convergence of the discrete surfaces to the continuous still holds if we replace \( a_0 \) and \( b_0 \) in (114) by some \( \epsilon \)-close initial data. However, we lose the \( C^\infty \)-convergence of the \( F^\epsilon \); it is weakened to local uniform convergence of the frames in the matrix norm. Replacing the discrete initial data by sequences \( a_0^\epsilon, b_0^\epsilon \) that converge in the discrete \( C^k \)-sense to \( a_0, b_0 \) leads to local \( C^k \)-convergence of \( F^\epsilon \) to the limit; this is a direct consequence of the following proof in combination with Theorem 2.

Eventually, we can be explicit about what is shown in Fig. 3. The family of K-surfaces under consideration is in correspondence to solutions of the discrete Sine-Gordon-Equation for the same initial data \( a_0, b_0 \) on different domains \( \Omega^\epsilon(r) \). The lower curve corresponds to \( r = 1 \), and the upper curve to \( r = 4 \). The mesh size was chosen \( \epsilon = 2^{-k} \) for \( k = 5, 6, \ldots, 11 \). As initial data we prescribed two smooth functions with no (apparent) special properties. The most precise approximation \( F^\epsilon_* \), with \( \epsilon_* \approx 0.002 \) is used as the reference point (the limiting smooth surface). The error in Fig. 3 is the \( C^0 \) distance of the map \( F^\epsilon \) to \( F^\epsilon_* \).

For special classes of K-surfaces possessing additional symmetries the convergence may be much faster. In particular, the rate of convergence for the Amsler surface in Fig. 4 is apparently quadratic in \( \epsilon \). This “superconvergence” might be explained by the fact that the discretization preserves the defining geometric (surface contains two straight asymptotic lines) and analytic (for a relation to the Painlevé III equation, see [Ho]) properties.

Proof. Theorems 1 and 2 yield the existence and the uniqueness of solutions \( (a^\epsilon, b^\epsilon) \) to the difference equations on the whole of \( \Omega^\epsilon(r) \), the existence and uniqueness of the solutions \( (a, b) \) to the differential equations, and the \( C^\infty \) approximation of the latter by the former.

It remains to prove that similar approximation holds also for the immersions \( F^\epsilon, F \). To do this, we prove the approximation property for the frames \( \Psi^\epsilon, \Phi \), uniformly in \( \lambda \in [\Lambda^{-1}, \Lambda] \), and then use the Sym formula. Recall that these frames are defined as solutions of the Cauchy problems for the system of linear difference equations (105), respectively for the system of linear
differential equations (99). Since the zero curvature conditions (104), (98) are satisfied, the existence of $\Psi^\epsilon$, $\Phi$ is guaranteed by standard ODE theory. Furthermore, at any point $(x, y)$, $\Psi^\epsilon(\lambda)$ and $\Phi(\lambda)$ are analytic functions of $\lambda \in D$, where $D$ is a closed disc in the complex plane of $\lambda$ that contains all points having distance $(2\Lambda)^{-1}$ from the interval $[\Lambda^{-1}, \Lambda]$. Since $D$ has positive distance to $i\mathbb{R}$ and $\infty$, all the matrices $U^\epsilon, V^\epsilon, U, V$ are bounded uniformly with respect to $\lambda \in D$ and $(x, y) \in \Omega(r)$. It is easy to see that

\[ U^\epsilon(a; \lambda) = 1 + \epsilon U(a; \lambda) + \mathcal{O}(\epsilon^2), \quad V^\epsilon(b; \lambda) = 1 + \epsilon V(b; \lambda) + \mathcal{O}(\epsilon^2). \quad (117) \]

Even more is true: If $a^\epsilon = a + \mathcal{O}(\epsilon)$ and $b^\epsilon = b + \mathcal{O}(\epsilon)$, then

\[ U^\epsilon(a^\epsilon; \lambda) = 1 + \epsilon U(a; \lambda) + \mathcal{O}(\epsilon^2), \quad V^\epsilon(b^\epsilon; \lambda) = 1 + \epsilon V(b; \lambda) + \mathcal{O}(\epsilon^2). \quad (118) \]

To estimate $\Psi^\epsilon - \Phi$, observe first that

\[ \Phi(x + \epsilon, y) = \Phi(x, y) + \int_x^{x+\epsilon} U(\xi, y)\Phi(\xi, y)d\xi = (1 + \epsilon U(x, y))\Phi(x, y) + \mathcal{O}(\epsilon^2). \]

On the other hand, due to (118),

\[ \Psi^\epsilon(x + \epsilon, y) = U^\epsilon(x, y)\Psi^\epsilon(x) = (1 + \epsilon U(x, y))\Psi^\epsilon(x, y) + \mathcal{O}(\epsilon^2). \]

Therefore,

\[ \Psi^\epsilon(x + \epsilon, y) - \Phi(x + \epsilon, y) = (1 + \epsilon U(x, y))(\Psi^\epsilon(x, y) - \Phi(x, y)) + \mathcal{O}(\epsilon^2). \quad (119) \]

Similarly,

\[ \Psi^\epsilon(x, y + \epsilon) - \Phi(x, y + \epsilon) = (1 + \epsilon V(x, y))(\Psi^\epsilon(x, y) - \Phi(x, y)) + \mathcal{O}(\epsilon^2). \quad (120) \]

Due to the zero curvature condition (98), one can find $\Phi(x, y)$ by first integrating the first equation in (99) from $(0, 0)$ to $(x, 0)$ along the $x$–axis, and then integrating the second equation in (99) from $(x, 0)$ to $(x, y)$ parallel to the $y$–axis. The similar holds for $\Psi^\epsilon$. Therefore the classical Gronwall inequality can be used to conclude that

\[ \Psi^\epsilon(x, y) - \Phi(x, y) = \mathcal{O}(\epsilon). \quad (121) \]

For real $\lambda$, the frames $\Psi^\epsilon$ and $\Phi$ belong to SU(2), so it is immediate that also

\[ (\Psi^\epsilon)^{-1}(x, y) - \Phi^{-1}(x, y) = \mathcal{O}(\epsilon). \quad (122) \]
Furthermore, since \( \Phi(\lambda) \) und \( \Psi(\lambda) \) are analytic functions of \( \lambda \in D \), we have for any \((x, y) \in \Omega^\epsilon(r)\) by Cauchy’s theorem:

\[
\sup_{\lambda \in [\Lambda^{-1}, \Lambda]} \| \partial_\lambda \Psi^\epsilon(x, y; \lambda) - \partial_\lambda \Phi(x, y; \lambda) \| \leq 2\Lambda \sup_{\lambda \in D} \| \Psi^\epsilon(x, y; \lambda) - \Phi(x, y; \lambda) \| = O(\epsilon). \tag{123}
\]

The last two estimates imply that, for all \( \lambda \in [\Lambda^{-1}, \Lambda] \) and uniformly on the respective \( \Omega^\epsilon(r) \),

\[
F^\epsilon - F = 2\lambda (\Psi^\epsilon(\lambda))^{-1} \partial_\lambda \Psi^\epsilon(\lambda) - 2\lambda (\Phi(\lambda))^{-1} \partial_\lambda \Phi(\lambda) = O(\epsilon). \tag{124}
\]

It remains to prove the approximation of the higher order partial derivatives of \( F \) by the correspondent difference quotients of \( F^\epsilon \). Introducing the notations

\[
U^\epsilon = (U^\epsilon - 1)/\epsilon = U + O(\epsilon), \quad V^\epsilon = (V^\epsilon - 1)/\epsilon = V + O(\epsilon),
\]

it is easy to see that for corresponding solutions \((a^\epsilon, b^\epsilon)\) and \((a, b)\) one has discrete \( C^k \)-approximation for all \( k > 0 \) and all \( \lambda \in D \):

\[
\|U^\epsilon - U\|_k \to 0 \text{ and } \|V^\epsilon - V\|_k \to 0
\]
as \( \epsilon \to 0 \). We find for \( m + n = k + 1 \) with \( m > 0 \):

\[
|\delta_x^m(\delta_y^n(\Psi^\epsilon - \Phi))| = |\delta_x^{m-1}(\delta_y^n(U^\epsilon \Psi^\epsilon - U \Phi))| + O(\epsilon) \\
\leq C\|U\|_k \cdot \|\Psi^\epsilon - \Phi\|_k + C\|U^\epsilon - U\|_k \cdot \|\Psi^\epsilon\|_k + O(\epsilon).
\]

Here we used that for discrete \( C^k \)-norms of matrix products

\[
\|A \cdot B\|_k \leq C_k\|A\|_k \cdot \|B\|_k \tag{125}
\]

holds (cf. the remark after the proof of Lemma 3 in the Appendix). If \( m = 0 \), we can do the same calculations with the roles of \( x \) and \( y \) interchanged and \( V, V^\epsilon \) in place of \( U, U^\epsilon \). From this estimate, we conclude by induction in \( k \) that

\[
\|\Psi^\epsilon - \Phi\|_k \to 0,
\]

and therefore

\[
\limsup_{\epsilon \to 0} \Omega^\epsilon(r) |(\delta_x^m(\delta_y^n(\Psi^\epsilon - \partial_x^m \partial_y^n \Phi))| = 0.
\]

Again by the Cauchy estimate, we get also the similar result for the respective \( \lambda \)-derivatives for all values \( \lambda \in [\Lambda^{-1}, \Lambda] \). From the Sym formulas, we get (116). Finally, the statement about the approximation of Bäcklund
transformed surfaces follows in a completely similar way with the reference to Theorem 3.

As a corollary, we automatically get the non-trivial classical theorem on the permutability of Bäcklund transformations, due to Bianchi: given a K–surface $F$ and its two Bäcklund transformations $F_1$, $F_2$, there exists a unique K–surface $F_{12}$ which is a Bäcklund transformation of $F_1$ and of $F_2$. This follows now by a continuous limit in two directions of a four–dimensional compatible lattice system.

6 General hyperbolic systems

The theory developed so far can be generalized to higher dimensions without difficulties. We want here formulate the most general setting in which our techniques can be used to prove the approximation results for discretizations of Goursat problems for nonlinear hyperbolic systems. As an illustrative example, the reader can think of the equation

$$\partial_x \partial_y \partial_z u = F(u, \partial_x u, \partial_y u, \partial_z u, \partial_x \partial_y u, \partial_x \partial_z u, \partial_y \partial_z u).$$

The Goursat problem for it consists of prescribing the values of $u(x, y, 0)$, $u(x, 0, z)$, $u(0, y, z)$ for $0 \leq x, y, z \leq r$.

The above equation can be rewritten as a hyperbolic system:

$$\begin{align*}
\partial_x u &= a, \\
\partial_y u &= b, \\
\partial_z u &= c, \\
\partial_x a &= h, \\
\partial_x b &= f, \\
\partial_x c &= g, \\
\partial_y a &= g, \\
\partial_y b &= h, \\
\partial_y c &= f, \\
\partial_z f &= \partial_y g = \partial_z h = F(u, a, b, c, f, g, h).
\end{align*}$$

A natural (naive) way to discretize it consists of replacing all partial derivatives $\partial_x$ etc. by the correspondent difference quotients $\delta_x$ etc. In the so-obtained hyperbolic difference system it is natural to assume that the variables $a, b, c$ live on the edges of the cubic lattice starting from the point $(x, y, z)$ in the direction of the axes $x, y, z$, respectively, and that the variables $f, g, h$ are defined on the two–cells (elementary squares) adjacent to the point $(x, y, z)$ and orthogonal to the axes $x, y, z$, respectively. The generalization of this construction is as follows.

We use the notations (5) for the domain of the differential system, and (6) for the domain of its discretization. We denote by $e_i$ the vector whose only nonvanishing component is 1 in the $i$th position. The dependent variables are denoted by $\vec{a} = (a_1, \ldots, a_N) \in \mathcal{X}_1 \times \ldots \mathcal{X}_N$. 

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Definition 6 For each $1 \leq k \leq N$, let there be chosen a nonempty subset $\mathcal{E}_k \subset \{1, \ldots, d\}$ of independent variables; let $\mathcal{D}_k$ be its complement, so that there is a disjoint union

$$\{1, \ldots, d\} = \mathcal{E}_k \cup \mathcal{D}_k.$$  \hfill (126)

A **discrete $d$–dimensional hyperbolic system** is a collection of compatible difference equations

$$\delta_{x_i} a_k = f_{(k,i)}(\vec{a}), \quad i \in \mathcal{E}_k,$$  \hfill (127)

for the functions $a_k : \Omega^e(r) \mapsto X_k$, $1 \leq k \leq N$. A **Goursat problem** consists of prescribing the values $a_k(x) = a_{k0}(x)$ on the subsets

$$G^e_k = \left\{ \sum_{i \in \mathcal{D}_k} \mu_i \epsilon_i e_i : \mu_i \in \mathbb{Z}, \ 0 \leq \mu_i \leq r_i/\epsilon_i \right\} \subset \Omega^e(r).$$  \hfill (128)

The dependent variables $a_k(x) = a_k(x_1, \ldots, x_d)$ are thought of as attached to the cubes of dimension $\#(\mathcal{D}_k)$ adjacent to the point $x$:

$$c(x; \mathcal{D}_k) := \left\{ x + \sum_{i \in \mathcal{D}_k} \mu_i \epsilon_i e_i : \ 0 \leq \mu_i \leq 1 \right\}.$$  \hfill (129)

We use an abbreviation $\vec{f}_i(\vec{a})$ for the $N$–vector with the components

$$(\vec{f}_i(\vec{a}))_k = \begin{cases} f_{(k,i)}(\vec{a}) & \text{if } i \in \mathcal{E}_k, \\ \text{not defined} & \text{otherwise}. \end{cases}$$  \hfill (130)

The **compatibility conditions** mentioned in the above definition express the following requirement:

$$\delta_{x_j}^\epsilon \delta_{x_i}^\epsilon a_k = \delta_{x_i}^\epsilon \delta_{x_j}^\epsilon a_k$$  \hfill (131)

for any choice of $i \neq j$ from the respective $\mathcal{E}_k$. They read:

1. The functions $f_{(k,i)}$ depend only on those $a_\ell$ for which

$$\mathcal{E}_k \setminus \{i\} \subset \mathcal{E}_\ell.$$

2. For any $k = 1, \ldots, N$ and any pair $i, j \in \mathcal{E}_k$, the identity

$$\epsilon_i f_{(k,i)}(\vec{a}) + \epsilon_j f_{(k,j)}(\vec{a} + \epsilon_i \vec{f}_i(\vec{a})) = \epsilon_j f_{(k,j)}(\vec{a}) + \epsilon_i f_{(k,i)}(\vec{a} + \epsilon_j \vec{f}_j(\vec{a}))$$  \hfill (133)

holds identically in $\vec{a} \in X_1 \times \ldots \times X_N$. 

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Indeed, in order for Eq. (131) to make sense, it is necessary that its both sides are well defined. In order for the left–hand side, say, to be defined, the function \( f(k,i) \) is allowed to depend only on those \( a_\ell \) for which \( j \in E_\ell \). This has to hold for all \( j \in E_k, j \neq i \), so we come to (132). Then (133) is nothing but the in–length translation of (131).

Proposition 3 A Goursat problem for a compatible discrete hyperbolic system admits a unique solution on \( \Omega^\varepsilon(r) \).

When a discrete hyperbolic system is considered as a discretization of a continuous system supplied by Bäcklund transformations, it is naturally supposed that the independent variables are divided into \((x_1, \ldots, x_n)\) discretized with the step \( \varepsilon_1 = \ldots = \varepsilon_n = \varepsilon \), while the rest ones \((x_{n+1}, \ldots, x_d)\) are intrinsically discrete with \( \varepsilon_{n+1} = \ldots = \varepsilon_d = 1 \). We will write in this case \( \Omega^\varepsilon(r) \) for \( \Omega^\varepsilon(r) \), so that in the continuous limit \( \varepsilon \to 0 \) the domain of independent variables becomes

\[
\Omega^0(\bar{r}) = [0, r_1] \times \ldots [0, r_n] \times \{0, 1, \ldots, r_{n+1}\} \times \ldots \times \{0, 1, \ldots, r_d\}.
\]

The functions \( f(k,i) = f_0^\varepsilon(k,i) \) are supposed to depend continuously on \( \varepsilon \in [0, \varepsilon_0] \).

In the limit \( \varepsilon \to 0 \) the first \( \leq n \) of equations (127) will turn into differential ones:

\[
\partial_x a_k = f_0^\varepsilon(k,i)(\bar{a}), \quad i \in E_k, \quad 1 \leq i \leq n,
\]

while the rest ones will remain difference:

\[
\delta_x a_k = f_0^\varepsilon(k,i)(\bar{a}), \quad i \in E_k, \quad n + 1 \leq i \leq d.
\]

The sets (128) on which the Goursat data are prescribed will turn into

\[
G_k^0 = \left\{ \sum_{i \in D_k, i \leq n} \mu_i e_i + \sum_{i \in D_k, i > n} \nu_i e_i : \mu_i \in [0, r_i], \nu_i \in \{0, 1, \ldots, r_i\} \right\}.
\]

Theorem 5 Let there be given an \( \varepsilon \)-family of Goursat problems for compatible discrete hyperbolic systems (127) on \( \Omega^\varepsilon(r) \); denote their solutions by \( \bar{a}^\varepsilon \).

Suppose that

\[
f_0^\varepsilon(k,i)(\bar{a}) = f_0^0(k,i)(\bar{a}) + O(\varepsilon)
\]

uniformly on any compact subset of \( X_1 \times \ldots \times X_N \), and that

\[
a_0^\varepsilon(x) = a_0^0(x) + O(\varepsilon)
\]

uniformly on \( G_k^0 \). Then there exist \( \bar{r}_i \in (0, r_i] \) for \( 1 \leq i \leq n \) and Lipschitz–continuous functions \( \bar{a}^0 \) on \( \Omega^0(\bar{r}_1, \ldots, \bar{r}_n, r_{n+1}, \ldots, r_d) \) such that

\[
\bar{a}^\varepsilon = \bar{a}^0 + O(\varepsilon),
\]

and \( \bar{a}^0 \) constitute the unique solution of the continuous Goursat problem for the system (134), (135) with the Goursat data on (136). If, in addition,
• the convergence (137) is locally uniform in $C^{K+1}$,

• the difference quotients of order $\leq K + 1$ of the discrete Goursat data are bounded independently of $\epsilon$:

$$\left| \left( \prod_{i=1}^{K+1} \delta_{x_i}^\epsilon \right) a_k^\epsilon \right| \leq M \text{ on } \mathcal{G}_k^\epsilon,$$

where all $j_i$ in the product have to belong to $\mathcal{D}_k \cap \{1, \ldots, n\}$,

• and all limit functions $a_{k0}^0$ belong to $C^K$,

then the convergence (139) is in the sense of $C^K$, i.e.

$$\sup \left| \left( \prod_{i=1}^{K} \delta_{x_i}^\epsilon \right) a_k^\epsilon(x) - \left( \prod_{i=1}^{K} \partial_{x_i} \right) a_k^0(x) \right| \to 0 \quad \text{as} \quad \epsilon \to 0,$$

where all $j_i$ in the products are in $\{1, \ldots, n\}$, and the supremum is taken over $\Omega^\epsilon(\bar{r}_1 - K\epsilon, \ldots, \bar{r}_n - K\epsilon, r_{n+1}, \ldots, r_d)$.

Proof of this theorem is completely analogous to the proofs of the two- and three-dimensional results given above.

7 Appendix

Proof of Lemma 2. Set $p = p(x) = \sum_{j=1}^{d} (x_j/\epsilon_j)$. Suppose that the statement is valid for all $x$ with $p(x) < p_0$, and consider some $x$ with $p(x) = p_0$. Notice that $p(x - \epsilon_i e_i) = p(x) - 1$. Therefore,

$$\Delta(x) \leq \max \left( \Delta(0), \frac{K}{\mathcal{K}} \right) (1 + \epsilon_i \mathcal{K}) \exp \left( 2\mathcal{K} \sum_{j=1}^{d} x_j - 2\mathcal{K} \epsilon_i \right) + \epsilon_i \mathcal{K}.$$

It remains to estimate the last term on the right-hand side by

$$\epsilon_i \mathcal{K} \leq \frac{K}{\mathcal{K}} \left( e^{2\mathcal{K} \epsilon_i} - (1 + \epsilon_i \mathcal{K}) \right)$$

$$\leq \max \left( \Delta(0), \frac{K}{\mathcal{K}} \right) \left( e^{2\mathcal{K} \epsilon_i} - (1 + \epsilon_i \mathcal{K}) \right) \exp \left( 2\mathcal{K} \sum_{j=1}^{d} x_j - 2\mathcal{K} \epsilon_i \right).$$

Induction with respect to $p_0$ proves the Lemma.
Proof of Lemma 4. We prove first the case $Q = 0$. Since $\Delta$ is continuous and hence bounded, there exists some $M \geq 0$ such that
\[
\Delta(x_1, \ldots, x_d) \leq M \exp \left(2dL \sum_{i=1}^{d} x_i \right) \text{ on } \Omega(r).
\tag{140}
\]
We show that if this inequality holds for some $M > 0$, then it holds also with $M$ replaced by $M/2$. Indeed, (140) yields:
\[
\Delta(x_1, \ldots, x_d) \leq LM \exp \left(2dL(x_1 + \ldots + x_d) \right) \sum_{j=1}^{d} \int_{x_j}^{x_j} e^{2dL(\xi_j-x_j)}d\xi_j
\]
\[
= LM \exp \left(2dL(x_1 + \ldots + x_d) \right) \sum_{j=1}^{d} \frac{1 - \exp(-2dLx_j)}{2dL}
\]
\[
\leq \left(M/2\right) \exp \left(2dL(x_1 + \ldots + x_d) \right).
\]
We conclude that (140) holds with any $M > 0$, and so $\Delta \leq 0$. Next, consider the case $Q > 0$. Introduce the auxiliary function
\[
q(x_1, \ldots, x_d) := 2Q \exp \left(2dL \sum_{i=1}^{d} x_i \right).
\]
Exactly as above, we show:
\[
L \sum_{j=1}^{d} \int_{x_j}^{x_j} q(x_1, \ldots, x_{j-1}, \xi_j, x_{j+1}, \ldots, x_d)d\xi_j \leq \frac{1}{2} q(x_1, \ldots, x_d)
\]
\[
\leq q(x_1, \ldots, x_d) - Q.
\]
Therefore, the function $\Delta - q$ satisfies (140) with $Q = 0$, so that $\Delta - q \leq 0$. 

Proof of Lemma 5. Expand $f$ with respect to $a$ and $b$ around $a_0 = a(x_0, y_0)$ and $b_0 = b(x_0, y_0)$:
\[
f(a, b) = \sum_{k+\ell \leq K} \mathcal{L}_{k\ell}[(a - a_0)^{\otimes k}, (b - b_0)^{\otimes \ell}] + \rho,
\tag{141}
\]
where $\mathcal{L}_{k\ell} := (k!\ell!)^{-1}D_a^k D_b^{\ell} f(a_0, b_0)$ is a $(k + \ell)$-linear symmetric map. By Taylor’s theorem,
\[
|\rho| \leq \sum_{k+\ell = K+1} \| (D_a^k D_b^{\ell} f)[a, b] \|_0 \cdot (|a - a_0| + |b - b_0|)^{K+1}
\]
\[
\leq \sum_{k+\ell = K+1} \| (D_a^k D_b^{\ell} f)[a, b] \|_0 \cdot (L(|x - x_0| + |y - y_0|))^{K+1},
\tag{142}
\]

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where $L$ is a Lipschitz constant of $a$ and $b$. To apply $(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n$ to $\rho$ means to evaluate a weighted sum (with $\epsilon$–independent weights) of $\rho(x, y)$ at lattice sites $(x, y)$ no more than $K$ steps away from $(x_0, y_0)$, and then to divide by $\epsilon^K$. Using the estimate (142), we find with some $C_K > 0$:

$$|(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n \rho(x_0, y_0)| \leq \epsilon^{-K} C_K \sum_{|x, y - (x_0, y_0)| \leq K} |\rho(x, y)| \leq \epsilon^{-K} C_K K^2 \sum_{k+\ell = K+1} \|(D_a^k D_b^\ell f)[a, b]\|_0 \cdot (LK\epsilon)^{K+1} \leq \epsilon \cdot C_K K^{K+3} L^{K+1} \sum_{k+\ell = K+1} \|(D_a^k D_b^\ell f)[a, b]\|_0 := B\epsilon.$$  

This is the $B$-term in Lemma 3.

Next, we apply $(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n$ to the sum on the right–hand side of (141). We have to estimate expressions like $(\delta_x^\epsilon)^m(\delta_y^\epsilon)^n L[c_1, \ldots, c_k]$, where $L[c_1, \ldots, c_k]$ denotes some $k$-linear map from $X^k$ to $X$. Let $L^*(C)$ with $C = \{c_1, \ldots, c_P\}$ symbolically stand for any linear combination of such expressions

$$\sum_{j=1}^J \lambda_j \cdot L[c_{j,1}, \ldots, c_{j,k}]$$  

with arguments $c_{j,i}$ arbitrarily chosen from $C$.

Assume that $c_i$ are actually functions on $\Omega^s(r)$, so that $L[c_1, \ldots, c_k]$ is $(x, y)$-dependent. We start with the observation that

$$(\delta_x^\epsilon) L[c_1, \ldots, c_k] = \sum_{\kappa=1}^k L[\tilde{c}_1, \ldots, \tilde{c}_{\kappa-1}, \delta_x^\epsilon c_\kappa, \tilde{c}_{\kappa+1}, \ldots, \tilde{c}_k]$$  

where each $\tilde{c}_i$ stands either for the function $c_i(x, y)$ itself or for the shifted one $c_i(x + \epsilon, y)$. For instance, one possible choice is $\tilde{c}_i = c_i(x, y)$ for $i < \kappa$ and $\tilde{c}_i = c_i(x + \epsilon, y)$ for $i > \kappa$ in the $\kappa$th term. The equality (144) parallels the Leibnitz rule and follows immediately from the multilinearity of $L$. By induction, we conclude that

$$(\delta_x^\epsilon)^m L[c_1, \ldots, c_k] = \sum_{\kappa=1}^k L[\tilde{c}_1, \ldots, \tilde{c}_{\kappa-1}, (\delta_x^\epsilon)^m c_\kappa, \tilde{c}_{\kappa+1}, \ldots, \tilde{c}_k]$$  

$$+ L^* \left( \left\{ (\delta_x^\epsilon)^i \tilde{c}_\kappa \right\}_{i=1}^m \right)_{\kappa=1, \ldots, k}$$  

where the tilde now denotes a shift in $x$ by no more than $m\epsilon$. A further induction allows us to extend (145) to the case of mixed partial difference
quotients:

\[(\delta_x^e)^m(\delta_y^e)^nL[c_1, \ldots, c_k] = \sum_{\kappa=1}^{k} L[\bar{c}_1, \ldots, \bar{c}_{\kappa-1}, (\delta_x^e)^m(\delta_y^e)^n c_\kappa, \bar{c}_{\kappa+1}, \ldots, \bar{c}_k] + \mathcal{L}^* \left( \left\{ (\delta_x^e)^i(\delta_y^e)^j \bar{c}_\kappa \right\}_{i+j<K, \kappa=1, \ldots, k} \right) \right]. (146)

The tilde now stands for a possible shift in \(x\) and \(y\) by no more than \(m\epsilon\) and \(n\epsilon\), respectively.

Applying (146) to (141) (one has to set \(c_1 = \cdots = c_k = a - a_0\) and \(c_{k+1} = \cdots = c_{k+\ell} = b - b_0\), we find:

\[
\begin{align*}
&kL_{kl}[((\delta_x^e)^m(\delta_y^e)^n a, (\bar{a} - a_0)^{\otimes(k-1)}, (\bar{b} - b_0)^{\otimes\ell}]+ \\
&\ell L_{kl}[(\delta_x^e)^n b, (\bar{a} - a_0)^{\otimes k}, (\bar{b} - b_0)^{\otimes(\ell-1)}]+ \\
&+L^* \left( \left\{ (\delta_x^e)^i(\delta_y^e)^j \bar{a}, (\delta_x^e)^i(\delta_y^e)^j \bar{b} \right\}_{i+j<K} \right) \right]. (147)
\end{align*}
\]

Taking into account that \(|a - a_0|, |b - b_0| \leq K\epsilon\), we can estimate the first two lines of (147) as

\[
A \cdot \left( |((\delta_x^e)^m(\delta_y^e)^n a(x, y)| + |((\delta_x^e)^m(\delta_y^e)^n a(x, y)| \right),
\]

where

\[
A = \sum_{0<k+\ell\leq K} \frac{k+\ell}{k!\ell!} (K\epsilon)^{k+\ell-1} \| (D_a^k D_b^\ell f)[a, b] \|_0.
\]

This is the \(A\)-term in Lemma 3. Finally, the last line of (147) gives rise to the polynomial expression \(P\) of Lemma 3:

\[
P(s, t) = C_K \sum_{0<k+\ell\leq K} \| (D_a^k D_b^\ell f)[a, b] \|_0 s^k t^\ell
\]

after estimating \( |((\delta_x^e)^i(\delta_y^e)^j \bar{a}(x_0, y_0)| \) by \( \| a \|_{K-1} \) and similarly for \(b\).

**Remark.** An immediate consequence of the formula (146) is a product rule for discrete \(C^k\)-norms: If a product is defined between elements of \(X\), we may choose \(L[c_1, c_2] := c_1 \cdot c_2\) and so conclude

\[
\|c_1 \cdot c_2\|_K \leq C_K \|c_1\|_K \cdot \|c_2\|_K,
\]

with \(C_K > 0\) depending only on \(K\) and properties of the product.
References


