HARDY SPACE INFINITE ELEMENTS FOR EXTERIOR MAXWELL PROBLEMS

L. Nannen[†,*], **T. Hohage**[†], **A. Schädle**[‡], **J. Schöberl**[§]
[†]Institute for Numerical and Applied Mathematics, University of Göttingen, Germany
[‡]Mathematical Institute, Heinrich-Heine-Universität Düsseldorf, Germany
[§]Institute for Analysis and Scientific Computing, TU Wien, Austria
[*]Email: nannen@math.uni-goettingen.de

Talk Abstract

We present a construction of prismatic Hardy space infinite elements for exterior Maxwell problems. They fit into the discrete de Rham diagram and are well suited to solve resonance problems. Numerical tests indicate exponential convergence in the number of unknowns for the infinite elements.

Introduction

The time harmonic second order Maxwell system for the electric field $\mathbf{u} \in H_{\mathrm{loc}}(\mathbf{curl};\Omega)$ is in variational form given by

$$\int_\Omega \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v} - \varepsilon\kappa^2\, \mathbf{u} \cdot \mathbf{v} dx = g(\mathbf{v}) \qquad (1)$$

for test functions $\mathbf{v} \in H_{\mathrm{comp}}(\mathbf{curl};\Omega)$. ε is the local permittivity, $\Omega \subset \mathbb{R}^3$ an unbounded domain and $H_{\mathrm{loc}}(\mathbf{curl};\Omega)$ $(H_{\mathrm{comp}}(\mathbf{curl};\Omega))$ the space of vector fields $\mathbf{v}$ which are together with the curl $\nabla \times \mathbf{v}$ locally (compactly supported) in $(L^2(\Omega))^3$.

We consider two types of problems: The scattering and the resonance problem. The scattering problem consists of finding a solution $\mathbf{u}$ to (1) for a given wavenumber $\kappa > 0$ and a given functional g. In the resonance problem we are looking for eigenpairs $(\kappa^2, \mathbf{u})$ to (1) with $g \equiv 0$ where κ is now a complex resonance with $\Re(\kappa) > 0$ and $\mathbf{u}$ the resonance function.

Since the domain Ω is unbounded, both problems have to be completed by a suitable radiation condition. Whereas for the scattering problem the Silver-Müller radiation condition leads to a well-posed problem, the resonance functions $\mathbf{u}$ of the resonance problem are exponentially growing at infinity, so that the Silver-Müller condition is not a valid characterization of outgoing waves. A series representation in terms of Hankel functions of the first kind and their derivatives (see [1]) or equivalently the pole condition as discussed below remain valid for resonance problems.

Pole Condition

We split the domain Ω into a bounded interior domain $\Omega_{\mathrm{i}} := \Omega \cap P$, P being a convex polyhedron, and an unbounded exterior domain $\Omega_{\mathrm{e}} := \mathbb{R}^3 \backslash P$, which is assumed to be homogeneous with no sources. We use generalized radial coordinates $(\xi, \hat{x}) \in [0,\infty) \times \Gamma$

$$F(\xi,\hat{x}) := \hat{x} + \xi(\hat{x} - V_0) \qquad (2)$$

with $\Gamma := \overline{\Omega_{\mathrm{i}}} \cap \overline{\Omega_{\mathrm{e}}}$ and $V_0 \in P$. The pole condition in the form of [2, Def. 3.1] states roughly speaking that a function $u \in H^1_{\mathrm{loc}}(\Omega_{\mathrm{e}})$ is outgoing, if the Laplace transform in generalized radial direction

$$\hat{u}(s,\hat{x}) := \int_0^\infty e^{-s\xi} u(F(\xi,\hat{x})) d\xi \qquad (3)$$

belongs for each $\hat{x} \in \Gamma$ to the Hardy space $H^-(\mathbb{R})$ of L^2 boundary values of holomorphic functions in $\mathbb{C}^- := \{s \in \mathbb{C} \mid \Im(s) < 0\}$. Note, that the Laplace transform (3) is initially defined for $\{s \in \mathbb{C} \mid \Re(s) > 0\}$, but for solutions to the Helmholtz equation

$$\int_{\Omega_{\mathrm{e}}} \nabla u \cdot \nabla v - \kappa^2\, uv dx = g(v)\,, \qquad (4)$$

$v \in H^1_{\mathrm{comp}}(\Omega_{\mathrm{e}})$, there exist a holomorphic extension of $\hat{u}(\cdot,\hat{x})$ to $\mathbb{C}^-$ and the pole condition is equivalent to a series representation in terms of Hankel functions of the first kind and for $\kappa > 0$ to the Sommerfeld radiation condition.

Since a solution $\mathbf{u} \in H_{\mathrm{loc}}(\mathbf{curl};\Omega_{\mathrm{e}})$ to (1) satisfies the vector valued Helmholtz equation and since the Silver-Müller condition is equivalent to the Sommerfeld radiation condition for the Cartesian components of $\mathbf{u}$ (see [1]), we can formulate the pole condition as follows: A function $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^T \in H_{\mathrm{loc}}(\mathbf{curl};\Omega_{\mathrm{e}})$ is outgoing, if

$$\hat{\mathbf{u}}_j(s,\hat{x}) := \int_0^\infty e^{-s\xi} \mathbf{u}_j(F(\xi,\hat{x})) d\xi \qquad (5)$$

belongs for $\hat{x} \in \Gamma$ to $H^-(\mathbb{R})$ for each component $j = 1, 2, 3$.

Scalar Hardy Space Infinite Elements

In this section we give a short summary of the Hardy space infinite elements for Helmholtz problems of [2],

[3]. In the interior domain $\Omega_{\rm i}$ we use a standard finite element method with a tetrahedral mesh and a surface triangulation $\mathcal{T}$ of Γ. Hence, we have a segmentation of $\Omega_{\rm e}$ into pyramidal frustums

$$K := \{F(\xi, \hat{x}) \in \mathbb{R}^3 \mid \xi \geq 0,\ \hat{x} \in T\} \tag{6}$$

with a possibly curved surface triangle $T \in \mathcal{T}$.

The local basis functions in K are tensor products of "radial" basis functions in W_ξ and the surface basis functions in W_T, that are the traces of the non-vanishing finite element basis functions of the tetrahedron corresponding to the surface triangle T:

$$W_K := W_\xi \otimes W_T. \tag{7}$$

For the "radial" basis functions we use a Möbius transformation $\varphi_{\kappa_0}(z) = i\kappa_0 \frac{z+1}{z-1}$ with $\Re(\kappa_0) > 0$ to construct a family of unitary mappings

$$\mathcal{M}_{\kappa_0}\, \hat{v} := (\hat{v} \circ \varphi_{\kappa_0}) \cdot \sqrt{\varphi'_{\kappa_0}}, \quad \hat{v} \in H^-(\kappa_0\mathbb{R}), \tag{8}$$

of the rotated Hardy space $H^-(\kappa_0\mathbb{R}) := \{\hat{v} \mid \hat{v}(\kappa_0 \bullet) \in H^-(\mathbb{R})\}$ into the Hardy space $H^+(S^1)$ of L^2 boundary functions of holomorphic functions in the complex unit disk $D := \{z \in \mathbb{C} \mid |z| < 1\}$. Using the Galerkin basis $W_\xi := \mathrm{span}\{\Psi_{-1}, ..., \Psi_N\}$ with

$$\Psi_{-1}(z) := \frac{1}{2i\kappa_0},\ \Psi_j(z) := \frac{z-1}{2i\kappa_0} z^j,\ z \in S^1, \tag{9}$$

$j = 0, ..., N$, for $\hat{V} := \mathcal{M}_{\kappa_0}\, \hat{v} \in H^+(S^1)$ leads automatically to an outgoing discrete solution. In this way the radiation condition is incorporated in the function space.

Since $\left(\mathcal{L}^{-1}\, \mathcal{M}^{-1}_{\kappa_0}\, \Psi_j\right)(0) = 0$ for $j = 0, ..., N$ and $\left(\mathcal{L}^{-1}\, \mathcal{M}^{-1}_{\kappa_0}\, \Psi_{-1}\right)(0) = 1$, the basis function Ψ_{-1} is used to couple the Hardy space infinite elements of $\Omega_{\rm e}$ to the finite elements of $\Omega_{\rm i}$. The parameter κ_0 should be adapted to the wavenumber of the sought solution u.

Fig. 1 shows a scheme of the basis functions. The tensor products with Ψ_{-1} are vertex ($\bullet$), edge ($\blacksquare$) and surface ($\blacktriangledown$) basis functions on the surface triangle T, while the tensor products with Ψ_j, $j = 0, ..., N$, are infinite ray ($\circ$), face ($\square$) and segment ($\triangledown$) basis functions.

It remains to derive a variational formulation in a tensor product space involving $H^+(S^1)$, i.e. in the scalar case we have to transform the radial direction of the integrals in (4) for each segment K of $\Omega_{\rm e}$ into the Hardy space $H^+(S^1)$. For this we first need to transform the pyramidal frustum $K = F(\hat{K})$ into the right prism $\hat{K} := [0, \infty) \times T$ using the canonical transformations (see e.g. [4, Sec. 3.9]):

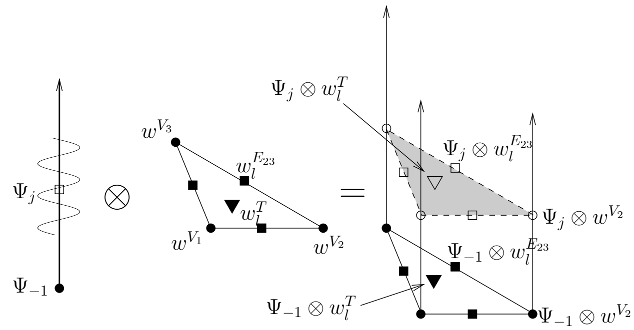

Figure 1: basis functions in $W_K = W_\xi \otimes W_T$

Lemma 1 *Let $K \subset \mathbb{R}^3$ such that $K = F(\hat{K})$ with Jacobian Matrix $J = J_F$, $|J|$ the determinant of J, $\nabla_{\hat{x}}$ the surface gradient and $\nabla_{\xi,\hat{x}} := (\partial_\xi, \nabla_{\hat{x}})^T$.*

1. *For $\hat{v} \in H^1(\hat{K})$ let $v \circ F := \hat{v}$. Then*

$$(\nabla v) \circ F = J^{-T} \nabla_{\xi,\hat{x}} \hat{v}.$$

2. *For $\hat{\mathbf{v}} \in H(\mathbf{curl}, \hat{K})$ let $\mathbf{v} \circ F := J^{-T}\hat{\mathbf{v}}$. Then*

$$(\nabla \times \mathbf{v}) \circ F = \frac{1}{|J|} J\, \nabla_{\xi,\hat{x}} \times \hat{\mathbf{v}}.$$

3. *For $\hat{\mathbf{v}} \in H(\mathrm{div}, \hat{K})$ let $\mathbf{v} \circ F := \frac{1}{|J|} J\, \hat{\mathbf{v}}$. Then*

$$(\nabla \cdot \mathbf{v}) \circ F = \frac{1}{|J|} \nabla_{\xi,\hat{x}} \cdot \hat{\mathbf{v}}.$$

Then we use for the infinite $\xi-$direction the following identity from ([2, Lemma A.1]), which holds for suitable functions u and v:

$$\int_0^\infty u(\xi) v(\xi)\, d\xi = a(U, V) \tag{10}$$

with $U := \mathcal{M}_{\kappa_0}\, \mathcal{L}\, u$, $V := \mathcal{M}_{\kappa_0}\, \mathcal{L}\, v$ in $H^+(S^1)$ and

$$a(U, V) := \frac{-2i\kappa_0}{2\pi} \int_{S^1} U(z) V(\overline{z}) |dz|. \tag{11}$$

There are some more details of the method mainly concerning the treatment of the Jacobian, which cannot be presented here. They can be found in [2], [3].

Tensor Product Sequence

If the bounded interior domain is simply connected, the standard sequence

$$H^1(\Omega_{\rm i})/\mathbb{R} \xrightarrow{\nabla} H(\mathbf{curl}, \Omega_{\rm i}) \xrightarrow{\nabla\times} H(\mathrm{div}, \Omega_{\rm i}) \xrightarrow{\nabla\cdot} L^2(\Omega_{\rm i}) \tag{12}$$

is exact. Since we are interested in resonance problems, it is necessary to carry over this property to the discrete spaces in order to avoid spurious resonances (see [4]). The Hardy space infinite elements of the exterior domain are built by using the tensor product of chain complexes (see [5] for all details).

As presented in the previous section the local discretization W_K of the transformed space corresponding to $H^1_{\text{loc}}(K)$ is a tensor product of W_ξ (discrete transformed $H^1_{\text{loc}}([0,\infty))$) and W_T (discrete $H^1(T)$). The space of transformed derivatives of W_ξ is by direct calculations the space $W'_\xi := \text{span}\{\psi_{-1}, ..., \psi_N\}$ with

$$\psi_{-1}(z) := \frac{1}{2},\ \psi_j(z) := \frac{z+1}{2} z^j,\ z \in S^1, j = 0, ..., N. \tag{13}$$

Thus, we have a one-dimensional "radial" sequence

$$W_\xi \xrightarrow{\hat{\partial}_\xi} W'_\xi. \tag{14}$$

Additionally, there exist two surface sequences

$$H^1(T) \xrightarrow{\nabla_{\hat{x}}} H(\text{Curl}, T) \xrightarrow{\nu \times \nabla_{\hat{x}} \cdot} L^2(T), \tag{15a}$$

$$H^1(T) \xrightarrow{\nu \times \nabla_{\hat{x}}} H(\text{Div}, T) \xrightarrow{\nabla_{\hat{x}} \cdot} L^2(T). \tag{15b}$$

Using the discretization $V_T \subset H(\text{Curl}, T)$ and $X_T \subset L^2(T)$ of the first surface sequence, we get a discrete block sequence

$$\begin{array}{ccccc} W_\xi \otimes W_T & \longrightarrow & W_\xi \otimes V_T & \longrightarrow & W_\xi \otimes X_T \\ \downarrow & & \downarrow & & \downarrow \\ W'_\xi \otimes W_T & \longrightarrow & W'_\xi \otimes V_T & \longrightarrow & W'_\xi \otimes X_T. \end{array} \tag{16}$$

The tensor product sequence is given by the direct sums over the diagonals:

$$\begin{aligned} W_\xi \otimes W_T &\to W_\xi \otimes V_T \oplus W'_\xi \otimes W_T \\ &\to W_\xi \otimes X_T \oplus W'_\xi \otimes V_T \to W'_\xi \otimes X_T. \end{aligned} \tag{17}$$

Curl-Conforming Elements

As in the scalar case (1) is splitted into an interior part Ω_{i} handled by a standard finite element method, while the exterior part Ω_{e} is transformed segment by segment using Lemma 1 and the identity (10) for the infinite ξ-direction. The local Hardy space infinite element is given by $V_K := W_\xi \otimes V_T \oplus W'_\xi \otimes W_T$. Fig. 2 and Fig. 3 show the arrangement of the basis functions specified below. Note, that for curl-conforming elements only the tangential directions indicated by the arrows are continuous over the segment boundaries.

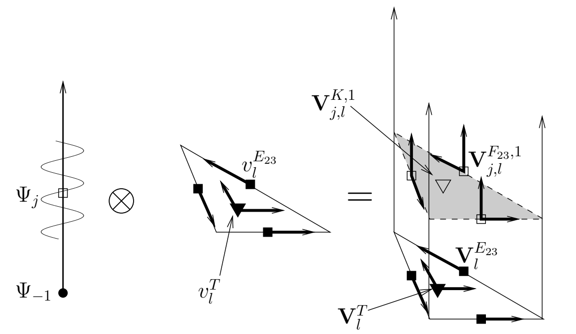

Figure 2: basis functions in $W_\xi \otimes V_T$

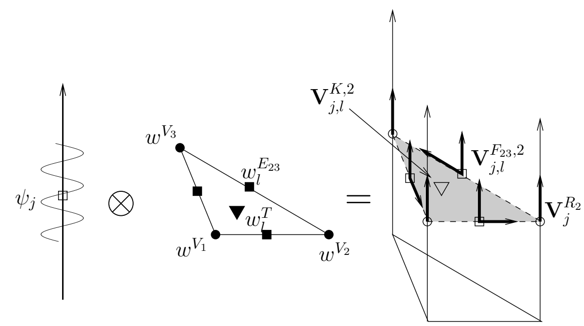

Figure 3: basis functions in $W'_\xi \otimes W_T$

If $\mathbf{v}_l^{E_{ik}} \in V_T$ denotes the edge basis functions of the $H(\text{Curl}, T)$ surface triangular element and $\mathbf{v}_l^T \in V_T$ the surface triangle basis functions, the basis functions in V_K are

1. edge functions $\mathbf{V}_l^{E_{ik}} := \begin{pmatrix} 0 \\ \Psi_{-1} \otimes \mathbf{v}_l^{E_{ik}} \end{pmatrix}$,
2. surface functions $\mathbf{V}_l^T := \begin{pmatrix} 0 \\ \Psi_{-1} \otimes \mathbf{v}_l^T \end{pmatrix}$,
3. ray functions $\mathbf{V}_j^{R_i} := \begin{pmatrix} \psi_j \otimes w^{V_i} \\ \mathbf{0} \end{pmatrix}$,
4. two types of infinite face functions
   (a) $\mathbf{V}_{j,l}^{F_{ik},1} := \begin{pmatrix} 0 \\ \Psi_j \otimes \mathbf{v}_l^{E_{ik}} \end{pmatrix}$,
   (b) $\mathbf{V}_{j,l}^{F_{ik},2} := \begin{pmatrix} \psi_j \otimes w_l^{E_{ik}} \\ \mathbf{0} \end{pmatrix}$
5. and two types of segment functions
   (a) $\mathbf{V}_{j,l}^{K,1} := \begin{pmatrix} 0 \\ \Psi_j \otimes \mathbf{v}_l^T \end{pmatrix}$ and
   (b) $\mathbf{V}_{j,l}^{K,2} := \begin{pmatrix} \psi_j \otimes w_l^T \\ \mathbf{0} \end{pmatrix}$.

Remember that the first component of the basis functions is the infinite ξ-direction, while the second component is the two-dimensional surface $\hat{x}$.

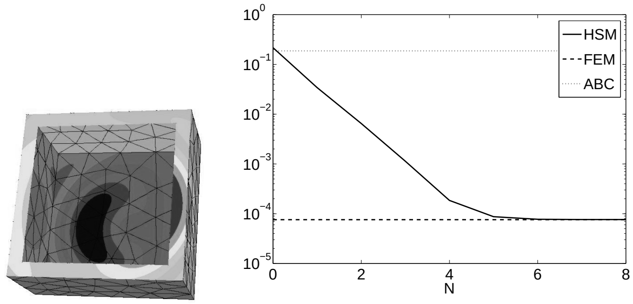

Figure 4: Right: $H(\mathbf{curl}, \Omega_\mathrm{i})$-error of the HSM w.r.t. the number N of degrees of freedom in radial direction compared to the finite element error and the error of a first order absorbing boundary condition for the two different domains Ω_i on the left; Left: Cross-section of one Cartesian component of a magnetic dipole

Numerical Examples

In the first example we resolve a magnetic dipole located at a point $y := (1, 2, -1)^T$

$$E_y(x) = \nabla_x \times \left(\frac{e^{i\kappa|x-y|}}{|x-y|} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right),$$

which is a radiating solution of (1) with given wavenumber $\kappa = 1$ and $\Omega := \mathbb{R}^3 \setminus [-5, 5]^3$. Fig. 4 shows a fast convergence of the Hardy space method, such that setting $N = 5$ suffices to reach the finite element error.

In the second example we solve the resonance problem for $\Omega = \mathbb{R}^3 \setminus K$ and $K = [-1.2, 1.2]^3 \setminus \left([-1, 1]^3 \cup [1, 1.2] \times [-0.2, 0.2]^2\right)$ with perfectly conducting boundary conditions $E \times \nu = 0$ at ∂K.

In Fig. 5 the absolute value of two resonance functions on a cross-section of the interior domain is shown. For a closed cavity ($\Omega = [-1, 1]^3$), the resonances are positive and analytically given by $\kappa = \sqrt{l + m + n}\frac{\pi}{2}$ for $l, m, n \in \mathbb{N}_0$ such that $lm + ln + mn > 0$.

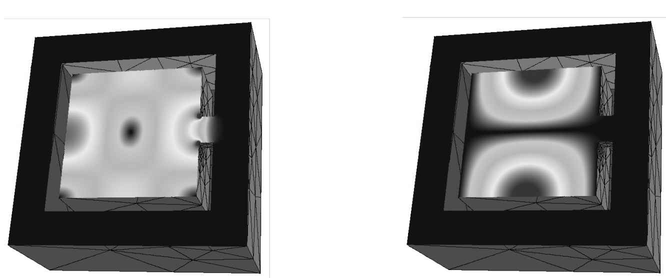

Figure 5: Cross-section of the absolute value of the two resonance functions of the resonances close to $\kappa = \sqrt{3}\frac{\pi}{2} \approx 2.72$

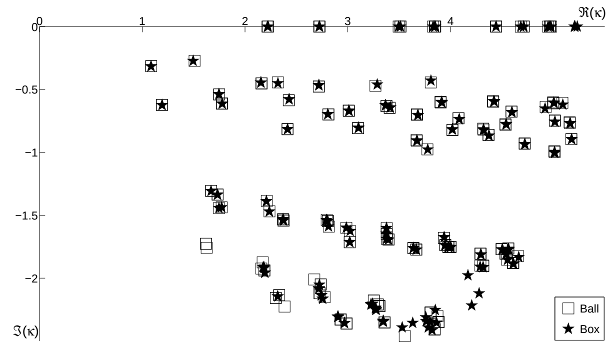

Figure 6: Resonances of an open cavity for two different discretizations: $\Omega_\mathrm{i} = \{x \in \Omega \mid |x| < 2.5\}$ and $\Omega_\mathrm{i} = [-1.7, 1.7]^3 \cap \Omega$

Fig. 6 shows the real and imaginary part of the computed resonances for two different discretizations. Both discretizations give similar results for the cavity resonances near the real axis. The exterior resonances with in absolute values larger imaginary parts are mostly identical for the two discretizations, but for the resonances at the bottom of Fig. 6 the discretizations are too coarse.

References

[1] David Colton, Rainer Kress, Inverse acoustic and electromagnetic scattering theory, Springer-Verlag, Berlin, 1998.

[2] T. Hohage, L. Nannen, “Hardy space infinite elements for scattering and resonance problems”, SIAM J. Num. Analysis, vol. 47(2), pp. 972-996, 2009.

[3] L. Nannen, A. Schädle, “Hardy space infinite elements for Helmholtz-type problems with unbounded inhomogeneities”, Wave Motion, vol. 48(2), pp. 116-129, 2011.

[4] Peter Monk, Finite element methods for Maxwell’s equations, Oxford University Press, New York, 2003.

[5] L. Nannen, T. Hohage, A. Schädle, J. Schöberl, “High order Curl-conforming Hardy space infinite elements for exterior Maxwell problems”, in preparation.