FLATNESS OF SEMILINEAR PARABOLIC PDES — A GENERALIZED CAUCHY–KOWALEVSKI APPROACH

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Abstract. A generalized Cauchy-Kowalevski approach is proposed for flatness-based trajectory planning for boundary controlled semilinear systems of PDEs in a one-dimensional spatial domain. For this, the ansatz presented in [16] using formal integration is generalized towards a unified design framework, which covers linear and semilinear PDEs including rather broad classes of nonlinearities arising in applications. In addition, an efficient semi-numerical solution of the implicit state and input parametrizations is developed and evaluated in simulation scenarios. Simulation results for various types of nonlinearities and a tubular reactor model described by a system of semilinear reaction-diffusion-convection equations illustrate the applicability of the proposed method.

1. INTRODUCTION

One fundamental problem in control theory is the systematic determination of open-loop controls that realize finite time transitions between operating points along predefined paths. Examples include start up, shutdown, and operation of chemical reactors, reheating of metal slabs in steel processing, adaptive mechatronic structures, or multi-agent deployment. Since modeling of these systems leads to semilinear partial differential equations (PDEs), the solution of this trajectory planning problem is severely complicated by the corresponding infinite-dimensional system dynamics.

For finite-dimensional linear and nonlinear control systems, differential flatness [3] has evolved into a well established inversion-based tool for trajectory planning and tracking control [24, 30]. A differentially flat system is endogenously equivalent to a system without dynamics described by a collection of independent variables, namely the flat or basic output, respectively, having the same number of components as the number of system inputs [3, 4]. In other words, any system variable can be differentially parametrized in terms of the basic output and its time derivatives up to a certain problem-dependent order. Assigning a suitably differentiable reference trajectory for the basic output directly provides the respective state and input trajectory. In the nominal case, the latter can be utilized as an open-loop control to realize the corresponding state trajectories. In addition, the idea of equivalence and flatness can in principle be directly adapted to systems of PDEs (see, e.g., the treatise in [25]).

Thereby, given parabolic PDEs with boundary control operational calculus and formal power series have been applied for the state and input parametrization in terms of the basic output by means of fractional differentiation operators or infinite power series representations. In order to achieve convergence of the parametrizations, basic output trajectories have to be restricted to a certain Gevrey class. Besides PDEs in a single spatial coordinate [10, 12, 23, 2, 26, 18, 19, 15], certain extensions to PDEs defined on higher-dimensional domains are available [16, 23, 26, 14]. The experimental validation of flatness-based trajectory planning for PDE systems is addressed, e.g., in [33, 17, 28, 29]. Whereas there exists a rather broad catalog of applications, flatness-based trajectory planning for semilinear PDEs is still restricted to polynomial nonlinearities [12, 2, 18].

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In the following, we will overcome this constraint by considering a generalized Cauchy-Kowalevski approach for the analysis of boundary controlled semilinear systems of PDEs in a one-dimensional spatial domain. For this, the initial-boundary-value problem is reformulated as a Cauchy problem in the spatial variable. Hence, an additional degree of freedom can be introduced, which enables us to parametrize the system state and the boundary input in terms of a basic output. The abstract framework of scales of Banach spaces in Gevrey classes turns out to be an appropriate functional analytic set-up for the rigorous study of the properties of the parametrized Cauchy problem (cf. also [34] and [31] for comprehensive introductions).

For scalar semilinear PDEs we prove that a local solution can be obtained via successive approximation under certain assumptions on the basic output and the nonlinearity by applying methods mainly developed in [20, 21] and [9]. Furthermore, estimates for the interval of existence are provided. We note that our approach is inspired by the work of Guo and Littman [7] who investigated the null controllability for the semilinear heat equation with techniques based on [9] and [1]. However, while the considerations in [7] are on a pure abstract level, we investigate the concrete applicability of these ideas to various kinds of trajectory planning problems for broad classes of nonlinearities including polynomials, analytic functions and nonlinearities satisfying a Gevrey class condition.

The presented approach is moreover applied to systems of semilinear PDEs. In addition, a semi-numerical algorithm based on the discretized iteration scheme induced by the successive approximation method is proposed, which provides an efficient tool to evaluate the control input and the respective state parametrization. Finally, the developed techniques are applied to a tubular reactor model described by a system of coupled semilinear reaction-diffusion-convection equations.

The paper is organized as follows. In Section 2 we present our main idea underlying flatness-based trajectory planning for semilinear parabolic PDEs. Mathematical preliminaries and tools are summarized in Section 3 towards the analysis of the scalar case in Section 4. Systems of semilinear PDEs and a semi-numerical realization of the proposed design approach are considered in Section 5. Final remarks conclude the paper.

2. Flatness-based trajectory planning

In the following, a general systematics for flatness-based trajectory planning is presented, which is based on the reformulation of the governing distributed-parameter system as a Cauchy problem in the spatial coordinate.

2.1. **Boundary control problem.** We consider systems of semilinear PDEs

$$\partial_t \boldsymbol{u}(x,t) = \partial_x^2 \boldsymbol{u}(x,t) + \partial_x \boldsymbol{u}(x,t) - \boldsymbol{f}(\boldsymbol{u}(x,t),x) \tag{1a}$$

where $\boldsymbol{u} : (0,1) \times (0,\tau) \to \mathbb{R}^m$, for $\tau > 0$ and $m \in \mathbb{N}$. Boundary conditions are imposed according to

$$P_1 \partial_x \boldsymbol{u}(0,t) + P_0 \boldsymbol{u}(0,t) = 0, \tag{1b}$$

$$\boldsymbol{g}(\partial_x \boldsymbol{u}(1,t), \boldsymbol{u}(1,t), \boldsymbol{h}(t)) = 0, \tag{1c}$$

for $t \in [0,\tau)$, where P_0, $P_1 \in \mathbb{R}^{m \times m}$ and $\boldsymbol{h}$ denotes the control input. The system is initially in steady state

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \tag{1d}$$

for $x \in [0,1]$ where $\boldsymbol{u}_0 = \boldsymbol{u}_s(\boldsymbol{h}_0;\cdot)$ for $\boldsymbol{u}_s(\boldsymbol{h}_0;\cdot)$ a solution of the boundary-value problem associated to (1a)-(1c) at $t = 0$ given by

$$\partial_x^2 \boldsymbol{u}_s(x) + \partial_x \boldsymbol{u}_s(x) - \boldsymbol{f}(\boldsymbol{u}_s(x), x) = 0, \tag{2a}$$

$$P_1 \partial_x \boldsymbol{u}_s(0) + P_0 \boldsymbol{u}_s(0) = 0, \tag{2b}$$

$$\boldsymbol{g}(\partial_x \boldsymbol{u}_s(1), \boldsymbol{u}_s(1), \boldsymbol{h}_0) = 0. \tag{2c}$$

We require that the control input $\boldsymbol{h}$ can be (at least locally) expressed in terms of the boundary values, i.e.,

$$\boldsymbol{h}(t) = [h_1(t), \dots, h_m(t)]^T = \tilde{\boldsymbol{g}}(\partial_x \boldsymbol{u}(1,t), \boldsymbol{u}(1,t)). \tag{3}$$

The considered trajectory planning problem consists in the design of a feedforward control $\boldsymbol{h}^*$ to realize the transition from the initial steady state $\boldsymbol{u}_0$ to a final steady state $\boldsymbol{u}_T$ within the finite time interval $t \in [0,T]$ along a predefined spatial-temporal profile $\boldsymbol{u}^*$.

2.2. **Implicit formal state and input parametrization.** The basic idea underlying flatness-based trajectory planning for parabolic PDEs is to reformulate the initial-boundary-value problem (1) as a Cauchy problem in the spatial variable x. The boundary condition (1b) is interpreted as initial data at $x = 0$. However, since the differential operator is of second order in x another set of m initial conditions has to be imposed. We introduce a new variable $t \mapsto \boldsymbol{y}(t) = [y_1(t), \dots, y_m(t)]^T$, serving as an additional degree of freedom, where

$$\boldsymbol{y}(t) = Q_1 \partial_x \boldsymbol{u}(0,t) + Q_0 \boldsymbol{u}(0,t) \tag{4}$$

for $Q_0, Q_1 \in \mathbb{R}^{m\times m}$. The linear system of $2m$ equations defined by (4) and (1b) allows for a unique solution for $\boldsymbol{u}(0,t)$ and $\partial_x \boldsymbol{u}(0,t)$ provided the coefficient matrix

$$M := \begin{bmatrix} Q_1 & Q_0 \\ P_1 & P_0 \end{bmatrix} \in \mathbb{R}^{2m\times 2m} \tag{5}$$

is non-singular. In this case, there exist $C_0, C_1 \in \mathbb{R}^{m\times m}$ such that

$$\boldsymbol{u}(0,t) = C_0 \boldsymbol{y}(t), \quad \partial_x \boldsymbol{u}(0,t) = C_1 \boldsymbol{y}(t).$$

With this, a system of m equations is obtained, i.e.

$$\partial_x^2 \boldsymbol{u}(x,t) = \partial_t \boldsymbol{u}(x,t) - \partial_x \boldsymbol{u}(x,t) + \boldsymbol{f}(\boldsymbol{u}(x,t), x) \tag{6a}$$

$$\boldsymbol{u}(0,t) = C_0 \boldsymbol{y}(t), \quad \partial_x \boldsymbol{u}(0,t) = C_1 \boldsymbol{y}(t). \tag{6b}$$

If a solution $\boldsymbol{u}(\boldsymbol{y};\cdot)$ of (6) exists at $x = 1$ for given $\boldsymbol{y}$, then the input can be parametrized in terms of $\boldsymbol{y}$ using (3), i.e.

$$\boldsymbol{h}(t) = \tilde{\boldsymbol{g}}(\partial_x \boldsymbol{u}(\boldsymbol{y};1,t), \boldsymbol{u}(\boldsymbol{y};1,t)). \tag{7}$$

Thus, in accordance with common practice, $\boldsymbol{y}$ is subsequently called a flat or basic output. In particular, by prescribing a desired path $\boldsymbol{y}^*$ the solution of (6) yields the feedforward control $\boldsymbol{h}^*$ by evaluating (7), which is required to track the corresponding spatial-temporal path $\boldsymbol{u}^*$ in open-loop. For this, note that the parametrization similarly holds in steady state such that desired profiles are governed by the boundary-value problem

$$\partial_x^2 \boldsymbol{u}_s(x) + \partial_x \boldsymbol{u}_s(x) - \boldsymbol{f}(\boldsymbol{u}_s(x), x) = 0, \tag{8a}$$

$$\boldsymbol{u}_s(0) = C_0 \boldsymbol{y}_s, \quad \partial_x \boldsymbol{u}_s(0) = C_1 \boldsymbol{y}_s \tag{8b}$$

in terms of stationary values $\boldsymbol{y}_s$ of the basic output. Hence, different values of $\boldsymbol{y}_s$ result in different steady state profiles $\boldsymbol{u}_s(\boldsymbol{y}_s;\cdot)$. This enables the realization of the transition from $\boldsymbol{u}_0(\boldsymbol{y}_0^*;\cdot)$ to $\boldsymbol{u}_T(\boldsymbol{y}_T^*;\cdot)$ provided that $\boldsymbol{y}^*(0) = \boldsymbol{y}_0^*$, $\boldsymbol{y}^*(T) = \boldsymbol{y}_T^*$ and $\boldsymbol{y}^{*(n)} = \boldsymbol{0}$ at $t = 0, T$ for $n \geq 1$. In particular, the latter conditions imply that $\boldsymbol{y}^*$ has to be locally non-analytic.

During the past century, nonlinear equations of type (6) have been studied extensively, see, e.g., [6, 22, 32] and the references therein. It is well-known that local solutions exist in Gevrey classes under certain assumptions on the initial data and the arising nonlinearities. However, to apply the introduced flatness-based trajectory planning approach it is important to guarantee not only the existence of solutions on a pure abstract level but also to provide methods for the explicit evaluation of the control input. The approach presented in this paper meets both requirements. On the one hand, we discuss the conditions that have to be imposed on the nonlinearities and the basic output to guarantee a local solution of (6). The method of proof relies on the reformulation of the PDE as an abstract Volterra-type integral equation, which is solved via successive approximation within a suitable functional analytic set-up. On the other hand, the iteration scheme defined by the method of successive approximation provides an efficient algorithm for the numerical evaluation.

3. Mathematical Preliminaries

In the following, essential results on Gevrey class functions and scales of Banach spaces are provided, which are required for the analysis of (6). We abbreviate $\frac{d^n}{dt^n}y(t)$ by $y^{(n)}(t)$ and for $n = 1$ we write $y'(t)$ instead of $y^{(1)}(t)$.

3.1. Function spaces in Gevrey classes.

Definition 1. Let $\Omega \subset \mathbb{R}$ be an open set. A function $v : \Omega \to \mathbb{R}$ is of Gevrey class $d > 0$ if $v \in C^\infty(\Omega)$ and for every compact subset $I \subset \Omega$, there exist two positive constants γ, M such that

$$\max_{t\in I} |v^{(n)}(t)| \leq M\frac{n!^d}{\gamma^n} \tag{9}$$

for all $n \in \mathbb{N}_0$.

It is well-known that a Banach space consisting of Gevrey class d functions can be constructed by fixing the set I and the constant γ in the above estimate [11]. This motivates the next definition.

Definition 2. Let $d \geq 1$, $I \subset \mathbb{R}$ compact and $\gamma > 0$ be fixed. We say that $v \in \mathbb{G}^d(I,\gamma)$ if there exists an open set $\Omega \supset I$ such that $v \in C^\infty(\Omega)$ and (9) holds for some constant $M > 0$.

There are various possibilities to define a norm on $\mathbb{G}^d(I,\gamma)$ (see [13] or [11]). However, it is impossible to formulate equation (6) on one *single* space since the differential operator ∂_t on the right-hand side does not map $\mathbb{G}^d(I,\gamma)$ into itself. This is elaborated in the following Lemma.

Lemma 1. Suppose that $v \in \mathbb{G}^d(I,\gamma)$ satisfies estimate (9) for some constant $M > 0$. Then for $l \in \mathbb{N}$ fixed and every $\sigma \in \mathbb{R}$ with $0 < \sigma < \gamma$ there exists a constant $M' > 0$ such that

$$\max_{t\in I} |v^{(l+n)}(t)| \leq M'\frac{n!^d}{\sigma^n} \tag{10}$$

for all $n \in \mathbb{N}_0$.

Proof. For $\sigma, \gamma \in \mathbb{R}^+$ with $\sigma < \gamma$ and $d \geq 1$

$$\sup_{x\geq 0}\left\{\left(\frac{\sigma}{\gamma}\right)^x x^d\right\} = \left(\frac{d}{e\ln(\gamma/\sigma)}\right)^d$$

holds. Hence, we obtain

$$\frac{\sigma^n}{n!^d}|v^{(l+n)}(t)| = \frac{1}{\sigma^l}\left(\frac{\sigma}{\gamma}\right)^{n+l}\left[\frac{(n+l)!}{n!}\right]^d\frac{\gamma^{n+l}}{(n+l)!^d}|v^{(l+n)}(t)| \leq \frac{1}{\sigma^l}\left(\frac{\sigma}{\gamma}\right)^{n+l}(n+l)^{dl}\frac{\gamma^{n+l}}{(n+l)!^d}|v^{(l+n)}(t)|$$

$$\leq \frac{1}{\sigma^l}\left[\frac{dl}{e(\ln\gamma - \ln\sigma)}\right]^{dl}\frac{\gamma^{n+l}}{(n+l)!^d}|v^{(l+n)}(t)| \leq \frac{M}{\sigma^l}\left[\frac{dl}{e(\ln\gamma - \ln\sigma)}\right]^{dl} =: M'.$$

□

The above result shows that a differential operator ∂_t^l maps $\mathbb{G}^d(I,\gamma)$ into the larger space $\mathbb{G}^d(I,\sigma)$. This suggests the formulation of (6) as an abstract equation in a one-parameter family of Banach spaces, which can be obtained by varying the constant γ in estimate (9).

3.2. **Scales of Banach spaces in Gevrey classes.** From now on we restrict ourselves to functions of Gevrey class $d = 2$ and consider a single fixed compact set $I \subset \mathbb{R}$.

Definition 3. For fixed constants $0 < \sigma_0 < \sigma_1$ define a scale function by

$$\sigma(s) = (1-s)\sigma_0 + s\sigma_1, \tag{11}$$

where $s \in [0,1]$. We say that $v \in \mathcal{G}_s$ (where the dependence on the interval I is dropped for notational convenience) if $v \in C^\infty(\Omega)$ for $I \subset \Omega$ and

$$||v||_s := \sum_{n=0}^{\infty} \frac{\sigma^n(s)}{n!^2} \max_{t\in I} |v^{(n)}(t)| < \infty.$$

According to [13], $\mathcal{G}_s$ with the norm $||\cdot||_s$ is a Banach space and a scale of Banach spaces can be defined by $\{\mathcal{G}_s\}_{s\in[0,1]}$, where $\mathcal{G}_s \subseteq \mathcal{G}_{s'}$ and

$$||v||_{s'} \leq ||v||_s$$

for $0 \leq s' \leq s \leq 1$.

Some important properties of $\mathcal{G}_s$-spaces follow below.

Corollary 1. $\mathcal{G}_s \subset \mathbb{G}^2(I,\sigma(s))$ and conversely $\mathbb{G}^2(I,\gamma) \subset \mathcal{G}_s$ if and only if $\gamma > \sigma(s)$.

Proof. The first assertion is obvious since for all $n \in \mathbb{N}_0$,

$$\frac{\sigma^n(s)}{n!^2} \max_{t\in I} |v^{(n)}(t)| \leq ||v||_s =: M.$$

For $v \in \mathbb{G}^2(I,\gamma)$ we insert (9) to obtain

$$||v||_s = \sum_{n=0}^{\infty} \frac{\sigma^n(s)}{n!^2} \max_{t\in I} |v^{(n)}(t)| \leq M \sum_{n=0}^{\infty} \left(\frac{\sigma(s)}{\gamma}\right)^n, \tag{12}$$

which is finite if and only if $\gamma > \sigma(s)$. □

Lemma 2. The Banach space $\mathcal{G}_s$ is a Banach algebra. In particular, for $v, w \in \mathcal{G}_s$ it holds that $vw \in \mathcal{G}_s$ and

$$||vw||_s \leq ||v||_s ||w||_s.$$

Proof. By the Leibniz rule we have

$$|(vw)^{(n)}(t)| \leq \sum_{j=0}^{n} \left(\frac{n!}{j!(n-j)!}\right)^2 |v^{(j)}(t)||w^{(n-j)}(t)|.$$

Multiplying by $\sigma^n(s)/n!^2$, taking the maximum and summing over n yields

$$\sum_{n=0}^{\infty} \frac{\sigma^n(s)}{n!^2} \max_{t\in I} |(vw)^{(n)}(t)| \leq \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{\sigma^j(s)}{j!^2} \max_{t\in I} |v^{(j)}(t)| \frac{\sigma^{n-j}(s)}{(n-j)!^2} \max_{t\in I} |w^{(n-j)}(t)| = ||v||_s ||w||_s.$$

□

Finally, we prove what we anticipated in Lemma 1 and investigate the properties of a differential operator A acting on the scale, where we define

$$Av(t) := \partial_t v(t).$$

Lemma 3. The operator A is bounded from $\mathcal{G}_s$ to $\mathcal{G}_{s'}$ for $0 \leq s' < s \leq 1$ and

$$||Av||_{s'} \leq \frac{C_A}{(s-s')^2}||v||_s$$

for all $v \in \mathcal{G}_s$, where $C_A := (2/e)^2(1/\sigma_0)(\sigma_1/(\sigma_1 - \sigma_0))^2$.

Proof. Let $v \in \mathcal{G}_s$. For $n \in \mathbb{N}$ we have (see also the proof of Lemma 1)

$$\begin{aligned}\frac{\sigma^n(s')}{n!^2}\max_{t\in I}|v^{(n+1)}(t)| &\leq \frac{1}{\sigma_0}\left(\frac{\sigma(s')}{\sigma(s)}\right)^{n+1}(n+1)^2\frac{\sigma^{n+1}(s)}{(n+1)!^2}\max_{t\in I}|v^{(n+1)}(t)| \\ &\leq \frac{1}{\sigma_0}\left(\frac{2}{e}\right)^2\frac{\sigma^{n+1}(s)\max_{t\in I}|v^{(n+1)}(t)|}{(\ln\sigma(s) - \ln\sigma(s'))^2(n+1)!^2}.\end{aligned}$$

The concavity of the logarithm implies that

$$\ln\sigma(s) - \ln\sigma(s') \geq \frac{\sigma(s)-\sigma(s')}{\sigma(s)} \geq (s-s')\frac{\sigma_1-\sigma_0}{\sigma_1}$$

and hence

$$||Av||_{s'} = \sum_{n=0}^{\infty}\frac{\sigma^n(s')}{n!^2}\max_{t\in I}|v^{(n+1)}(t)| \leq \frac{C_A}{(s-s')^2}\sum_{m=1}^{\infty}\frac{\sigma^m(s)}{m!^2}\max_{t\in I}|v^{(m)}(t)| \leq \frac{C_A}{(s-s')^2}||v||_s.$$

□

3.3. **Trajectory assignment for the basic output.** For the appropriate explicit assignment of basic output trajectories recall from Section 2.2 that $\boldsymbol{y}$ has to be locally non-analytic to solve the trajectory planning problem. In order to address this, we consider the following function introduced in [5, 12] and analyze its properties.

Lemma 4. Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\psi(t) = \begin{cases} e^{-1/t(1-t)} & t \in (0,1) \\ 0 & \text{else.}\end{cases}$$

For $T > 0$ the function $\Psi_T : \mathbb{R} \to \mathbb{R}$ with

$$\Psi_T(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{N_\Psi}\int_0^{t/T}\psi(\tau)d\tau & t \in (0,T) \\ 1 & t \geq T\end{cases} \tag{13}$$

is of Gevrey class 2, where $N_\Psi = \int_0^1 \psi(\tau)d\tau \approx 0.007$. In particular, for all $n \in \mathbb{N}_0$ it holds that

$$\sup_{t\in\mathbb{R}}|\Psi_T^{(n)}(t)| \leq M_\psi\frac{n!^2}{\gamma^n}$$

with $\gamma = T/3$ and $M_\psi = 1/(3eN_\Psi) \approx 17.44$.

The proof of this lemma is provided in Appendix B. Note that differing from [12], where implicit estimates for (13) are obtained depending on an abstract parameter, our results are explicit. After these technical preparations, we turn to the original problem, where we first restrict ourselves to a single semilinear PDE.

4. Scalar semilinear PDEs

For the scalar case let $m = 1$ in (1) or (6), respectively. Note that making use of a suitable change of variables enables to eliminate the convective term $\partial_x u$. It is hence sufficient to study second order Cauchy problems of the form

$$\begin{aligned} \partial_x^2 u(x,t) &= \partial_t u(x,t) + f(u(x,t),x) \\ u(0,t) &= c_0 y(t), \quad \partial_x u(0,t) = c_1 y(t). \end{aligned} \tag{14}$$

for c_0, $c_1 \in \mathbb{R}$. We consider (14) on an extended spatial interval $[0,L]$ with $L > 1$ so that $u : [0,L] \times [0,T] \to \mathbb{R}$ for $T > 0$, where $[0,T]$ is the transition interval. Hence, a formal (implicit) solution to (14) is obtained as

$$u(x,t) = (c_0 + xc_1)y(t) + \int_0^x \int_0^\eta \partial_t u(\xi,t) + f(u(\xi,t),\xi) d\xi d\eta. \tag{15}$$

This formal solution is the core of the subsequent analysis.

4.1. **Main result.** Let $\{\mathcal{G}_s\}_{s\in[0,1]}$ denote the scale of Banach spaces, where we set $I := [0,T]$ and fix the constants σ_0, σ_1 in Definition 3. To obtain an abstract formulation of (15) the state variable is considered as a function of x with values in $\mathcal{G}_s$, i.e., we define $U : [0,L] \to \mathcal{G}_s$ such that

$$U(x)(t) := u(x,t).$$

The integral equation (15) can then be formally rewritten as

$$U(x) = (c_0 + xc_1)y + \int_0^x \int_0^\eta AU(\xi) + F(U(\xi),\xi) d\xi d\eta \tag{16}$$

where $F(U(x),x)(t) := f(u(x,t),x)$ and

$$AU(x)(t) = \partial_t u(x,t).$$

We define a sequence of functions $(U^{[k]}(x))_{k\in\mathbb{N}_0}$ by

$$\begin{aligned} U^{[0]}(x) &= (c_0 + xc_1)y, \\ U^{[k+1]}(x) &= U^{[0]}(x) + \int_0^x \int_0^\eta AU^{[k]}(\xi) + F(U^{[k]}(\xi),\xi) d\xi d\eta. \end{aligned} \tag{17}$$

Assumption 1.

(A1) Assume that $y \in \mathbb{G}^2(I,\gamma)$ for $\gamma > \sigma_1$. By Corollary 1, this implies that $y \in \mathcal{G}_1$ and for $L > 0$ there exists a constant $R_0 > 0$ such that $||U^{[0]}(x)||_s \le R_0 < \infty$ for all $s \in [0,1]$ and $0 \le x \le L$.

(A2) There exists a constant $R > R_0$ such that the nonlinearity defines a continuous map $F : \mathcal{B}_s(R) \times [0,L] \to \mathcal{G}_s$ with

$$\mathcal{B}_s(R) := \{v \in \mathcal{G}_s : ||v||_s < R\}, \quad R > 0.$$

In addition, the (local) Lipschitz-type estimate

$$||F(v,x) - F(w,x)||_s \le C_F ||v - w||_s \tag{18}$$

holds for $v, w \in \mathcal{B}_s(R)$, $0 \le x \le L$ and a constant $C_F > 0$ independent of x, s.

(A3) $||F(0,x)||_1 \le K$ for all $x \in [0,L]$.

With these assumptions, the main result follows below.

Theorem 1. Let $C := C_F + C_A$ (cf. Lemma 3). There exists a constant $r > 0$ with

$$r < \min\left\{\frac{L}{2}, \frac{1}{2\sqrt{12C}}, \frac{1}{2}\sqrt{\frac{R-R_0}{96C(R_0+K)}}\right\} \tag{19}$$

such that for every $0 \le s < 1$ and $0 \le x < r(1-s)$ the sequence $(U^{[k]}(x))_{k\in\mathbb{N}_0}$ converges to a limit function $U(x)$ in $\mathcal{G}_s$ with convergence being uniform with respect to x on compact subsets of $[0, r(1-s))$. Furthermore, $U(x)$ is twice continuously differentiable with respect to x and $u(x,t) = U(x)(t)$ solves the Cauchy problem (14).

The proof of Theorem 1 can be found in Appendix A.

Remark 1. It can be shown that the solution is in fact unique in a given scale of Banach spaces. However, for our purposes it is sufficient to guarantee the convergence of the iteration scheme and the existence of at least one solution of (15).

Remark 2. The above theorem is a modification of a result by Kano and Nishida [9, Theorem A]. Note that instead of solving a second-order problem, (14) can be also rewritten as a system of first-order equations, cf. Section 5. In this case, [7, Theorem 2.1] can be applied to ensure convergence of the successive approximation under similar assumptions. Similar to Theorem 1 the result in [7] is a generalization of [9]. However, our proof is less technical than that of [7] and it allows for simpler estimates on the radius of convergence (19) which can also be verified more easily.

4.2. **Catalog of scalar examples.** Subsequently, semilinear scalar reaction-diffusion equations with a Neumann boundary condition at $x = 0$ and a Dirichlet input at $x = 1$ are considered, i.e.

$$\begin{aligned}\partial_t u(x,t) &= \partial_x^2 u(x,t) - f(u(x,t)),\\ \partial_x u(0,t) &= 0, \quad u(1,t) = h(t).\end{aligned} \tag{20}$$

Proceeding as in Section 2.2 yields (14) for $c_0 = 1$ and $c_1 = 0$. In view of (8) steady state profiles can be defined in terms of constant $y = y_s$, i.e.

$$\begin{aligned}&\partial_x^2 u_s(x) - f(u_s(x)) = 0,\\ &u_s(0) = y_s, \quad \partial_x u_s(0) = 0.\end{aligned} \tag{21}$$

Corollary 2. Let $(u_s^k)_{k=0,\dots,n}$ denote a sequence of steady states (21) to be attained at successive time instances $(T_{2k})_{k=0,\dots,n}$. We define

$$y^*(t) := y_0^* + \sum_{k=1}^{n}(y_k^* - y_{k-1}^*)\Psi_{T_{2k}-T_{2k-1}}(t - T_{2k-1}) \tag{22}$$

with $y_k^* := u_s^k(0)$ and $0 \le T_0 \le T_1 < T_2 \le T_3 < T_4 \dots < T_{2n}$. Lemma 4 implies that $y^* : \mathbb{R} \to \mathbb{R}$ is of Gevrey class 2 such that

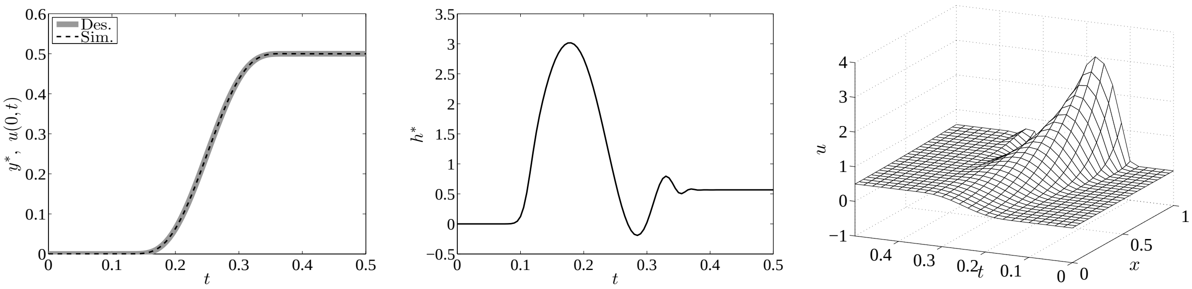
$$\max_{t\ge 0} |y^{*(n)}(t)| \le M_y \frac{n!^2}{\gamma_y^n} \tag{23}$$

for $\gamma_y = \frac{1}{3}\min_{k=1,\dots,n}\{T_{2k} - T_{2k-1}\}$ and

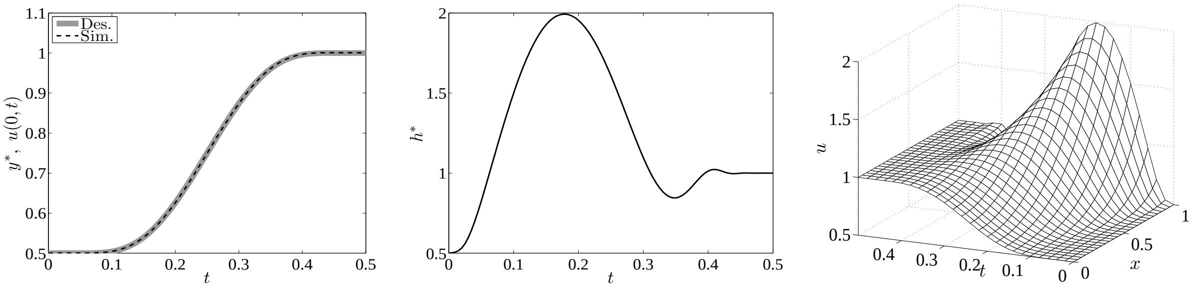
$$M_y = n \max\Big\{\max_{k=0,\dots,n} |y_k^*|, \max_{k=1,\dots,n} M_\psi |y_k^* - y_{k-1}^*|\Big\}.$$

Due to the local non-analyticity of $\Psi_T(t)$ it follows that the derivatives $y^{*(j)}$, $j \ge 1$, vanish at $t = T_{2k}$ and $t = T_{2k+1}$ for $k = 0, \dots, n$ with $u_s^k(0) = y^*(T_{2k+1}) = y^*(T_{2k})$, i.e., in view of (21) the steady state u_s^k is reached at $t = T_{2k}$ and is held for $t \in [T_{2k}, T_{2k+1}]$.

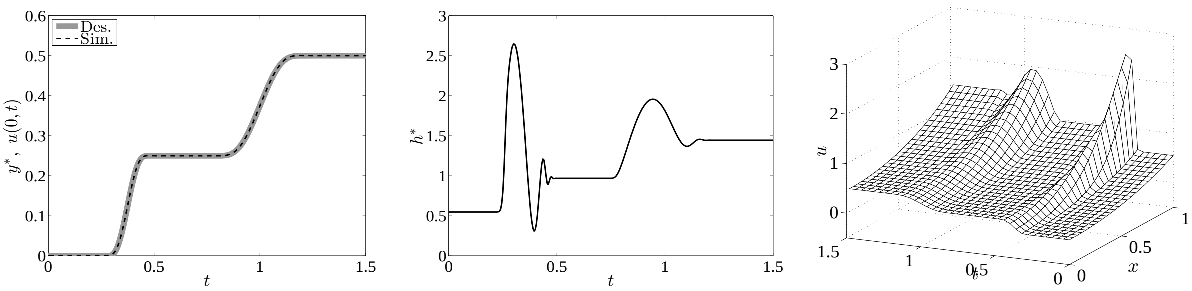
The application of the proposed method is subsequently discussed for three different types of nonlinearities f: polynomials, functions satisfying a Gevrey class 2 condition, and real analytic functions. Note that presently available results for flatness-based trajectory planning are inherently restricted to polynomial nonlinearities, see, e.g., [12, 2, 18] and the references therein.



(a) Polynomial nonlinearity $f(u) = u^3$ and desired basic output trajectory $y^*(t) = 0.5\Psi_{0.4-0.1}(t-0.1)$.



(b) Gevrey class 2 nonlinearity $f(u) = \exp(-90/u)$ for $u \geq 0$ and desired basic output trajectory $y^*(t) = 0.5 + 0.5\Psi_{0.5}(t)$.



(c) Real analytic nonlinearity $f(u) = \exp(u/(1+u/20))$ for $u > -20$ and desired basic output trajectory $y^*(t) = 0.25\Psi_{0.5-0.25}(t-0.25) + (0.5-0.25)\Psi_{1.25-0.75}(t-0.75)$.

Figure 1. Numerical results for the scalar semilinear PDE (20) with nonlinearities $f(u)$ as indicated above. Left column: comparison of $u(0,t)$ and $y^*(t)$; middle column: feedforward control $h^*(t)$; right column: evolving profile $u(x,t)$ for $h^*(t)$.

In order to illustrate this we perform numerical simulations which are based on the iteration scheme imposed by the method of successive approximation. We choose three different nonlinearities representing the aforementioned categories (see Fig. 1). For details on the discretization, we refer to Section 5. In all three examples the iteration converged on $[0,1] \times [0,T]$ and was stopped after a certain number of iteration steps. The control input was determined as $h^*(t) := \hat{u}(1,t)$, where $\hat{u}$ denotes the approximate numerical solution of (15). With $h(t) = h^*(t)$ the Matlab routine *pdepe* was used to solve the original initial-boundary-value problem (20). For comparison purposes, the solution $u(0,t)$ is compared to the desired trajectory $y^*(t)$, cf. Fig. 1. The good accordance between obtained and desired values suggests that the method performs very well on domains of reasonable extent.

In the following, we compare these observations to the analytic results by making use of Theorem 1. In the subsequent examples the basic output trajectory y^* is given according to (22), which is a $\mathbb{G}^2(I,\gamma_y)$-function by Corollary 2. Hence, the scale of Banach spaces $\{\mathcal{G}_s\}_{s\in[0,1]}$ has to be defined in such a way that Assumption (A1) holds. To this end the constant σ_1 in the scale function (11) is fixed with $\sigma_1 < \gamma_y$. Choosing $\sigma_0 < \sigma_1$ in (11) determines $\{\mathcal{G}_s\}_{s\in[0,1]}$ as well as the constant C_A, see Lemma 3.

4.2.1. *Polynomial nonlinearities.* The next result shows that the (local) Lipschitz condition required in Assumption (A2) can be fulfilled for monomials (and hence for all polynomial nonlinearities).

Lemma 5. Let $R > 0$ and $p \in \mathbb{N}$. The function $v \mapsto v^p$ is continuous from $\mathcal{B}_s(R)$ into $\mathcal{G}_s$. Furthermore, for any $v, w \in \mathcal{B}_s(R)$,

$$||v^p - w^p||_s \leq pR^{p-1}||v-w||_s.$$

Proof. Using the triangle inequality and the Banach algebra property of $\mathcal{G}_s$ we obtain

$$\begin{aligned} ||v^p - w^p||_s &\leq ||v-w||_s \sum_{j=0}^{p-1} \|v\|_s^{p-1-j}\|w\|_s^j \\ &\leq ||v-w||_s pR^{p-1}. \end{aligned}$$

□

As an explicit example, consider the cubic nonlinearity $f(u) = u^3$ and the desired basic output trajectory $y^*(t) = 0.5\Psi_{0.3}(t-0.1)$ for which (23) holds with $\gamma_y = 0.1$ and $M_y = 0.5M_\psi \approx 8.72$. We fix $I := [0, 0.5]$ and set $\sigma_1 := 1/30$ and $\sigma_0 := 1/40$ which yields $C_A \approx 346.46$. Applying (12) implies $||y^*||_1 \leq R_0 \approx 13.08$. Lemma 5 shows that (A2) can be satisfied for any fixed constant $R > R_0$. Hence, Theorem 1 applies and proves the convergence of the successive approximation to a solution of (14), (15) for $c_0 = 1$ and $c_1 = 0$ with $u(x,\cdot) \in \mathcal{G}_s$ for $0 \leq x < r(1-s)$. Setting $R := 30$, we obtain $C_F = 3R^2 \approx 2700$ so that insertion into (19) yields $r \approx 0.001$.

Remark 3. Obviously, the analytically obtained interval of existence in x is significantly smaller than the numerical results indicate. As an explanation note first that the choice of the parameters σ_1, σ_2 and R might not be optimal. Moreover, the theoretic results are achieved for a rather generic set-up, which provides general results at the price of a small range of predictability only.

4.2.2. *Nonlinearities of Gevrey class* 2. We now turn to more general nonlinearities. It is well-known that Gevrey classes are closed under composition of functions restricting the analysis to admissible nonlinearities of at most Gevrey class 2. However, to obtain a mapping $\mathcal{B}_s(R) \to \mathcal{G}_s$, as it is required in Assumption (A2), additional conditions on the constants in the Gevrey estimates have to be imposed. Note that our subsequent analysis is different to [7] since it is based on a version of the formula of Faá di Bruno (see [13]) and it also includes the required Lipschitz estimates.

Lemma 6. Let $R > 0$ be fixed and let $f : \mathbb{R} \to \mathbb{R}$ be a function of Gevrey class 2 satisfying

$$\max_{x\in[-R,R]} |f^{(n)}(x)| \leq M\frac{n!^2}{\gamma^n} \quad \forall n \in \mathbb{N}_0 \tag{24}$$

for constants $M > 0$ and $\gamma > R$. Define

$$F(v)(t) := (f \circ v)(t), \quad t \in I = [0,T].$$

The function F maps $\mathcal{B}_s(R)$ into $\mathcal{G}_s$, is differentiable (in the sense of Fréchet) at any $v \in \mathcal{B}_s(R)$, and for any $v, w \in \mathcal{B}_s(R)$,

$$||F(v) - F(w)||_s \leq C_F ||v - w||_s,$$

where $C_F := \frac{M}{\delta - R}\left(\frac{2}{e\ln(\gamma/\delta)}\right)^2$ for $R < \delta < \gamma$ fixed.

The proof of Lemma 6 is given in Appendix B.2. As an explicit example we consider the function

$$f(x) = \begin{cases} e^{-a/x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{25}$$

for fixed $a > 0$ and define $F(u) := f \circ u$. Such nonlinearities appear, e.g., in chemical engineering describing Arrhenius type reaction terms. Similar to the proof of Lemma 4 it can be shown that

$$\max_{x\in\mathbb{R}} |f^{(n)}(x)| \leq \frac{3^n n!^2}{a^n}. \tag{26}$$

The numerical simulation in Fig. 1 was performed for $a = 90$ and a basic output $y^*(t) = 0.5 + 0.5\Psi_{0.5}(t)$ such that (23) holds with $\gamma_y = 1/6$ and $M_y = 1/2M_\psi \approx 8.72$. Setting $I := [0, 0.5]$, $\sigma_1 := 1/20$ and $\sigma_2 := 1/50$ yields $C_A \approx 75.19$ and $||y^*||_1 \leq R_0 \approx 12.46$. With $R := 26$ and $\delta := 28$ in Lemma 6 we obtain $C_F \approx 44.43$. Hence, Theorem 1 implies $r \approx 0.005$. As in the first example, Remark 3 has to be taken into account for the interpretation of the result.

4.2.3. *Analytic nonlinearities.* Nonlinearities arising in applications are often described by real analytic functions. These are studied below.

Corollary 3. Let $R > 0$ be fixed. Let $f : \mathbb{R} \to \mathbb{R}$ be real analytic at $x = 0$ such that the Taylor series of f possesses the radius of convergence $\rho > R$. Then f satisfies the assumptions of Lemma 6, in particular for the nonlinearity which is defined by $F(u) := f \circ u$, Assumption (A2) in Theorem 1 holds.

For the proof, the reader is referred to Appendix B.3. As an explicit example we define $F(u) := f \circ u$, where

$$f(x) = \exp\left(\frac{x}{1+x/b}\right) \text{ for } |x| < b,$$

and $b > 0$. The function f is real analytic at $x = 0$ and its Taylor series possesses the radius of convergence $\rho = b$. In order to satisfy the conditions of Corollary 3 as well the assumptions of Theorem 1, the basic output y^*, the constant σ_1 in the scale function, and the constant R in Assumption (A2) have to be chosen to satisfy $R_0 < R < b$, where $||y^*||_1 \leq R_0$. As an example, for the numerical simulation we set $b = 20$ and assign $y^*(t) = 0.25\Psi_{0.5-0.25}(t - 0.25) + (0.5 - 0.25)\Psi_{1.25-0.75}(t - 0.75)$ for which $\gamma_y = 1/12$ and $M_y = 1/2M_\psi$. We set $I := [0, 1.5]$ in Definition 3 and choose $\sigma_1 := 1/60$, $\sigma_0 := \sigma_1/2$ so that (12) yields $R_0 \approx 10.9$. Hence, the constant R can be chosen to fulfill $R_0 < R < b$, which implies that Corollary 3 and consequently Theorem 1 apply. A numerical value for C_F can be obtained by inspecting the proof of Corollary 3. However, we will not detail this calculation for the sake of readability.

5. Systems of semilinear PDEs

In the following, coupled systems of semilinear parabolic PDEs are considered with the control located at the boundary $x = 1$. Herein, we focus on the second order Cauchy problem given by (6) with

$$\boldsymbol{f}(\boldsymbol{u}(x,t), x) = [f_1(\boldsymbol{u}(x,t), x), \ldots, f_m(\boldsymbol{u}(x,t), x)]^T.$$

and $\boldsymbol{u} = [u_1, \ldots, u_m]^T$. This system is considered on an extended interval $[0, L] \times [0, T]$ for $L > 1$. As for scalar PDEs, the aim is to formally integrate the equations and to apply the method of successive approximation to prove the existence of a local solution. We subsequently apply [7, Theorem 2.1] to equations of type (6) within the introduced flatness-based design systematics to infer convergence of the iteration sequence similar to Theorem 1. To this end (6) has to be rewritten as a system of abstract integral equations based on a first order formulation of the problem.

5.1. **First order formulation – abstract integral equation.** For the reformulation as a first order system we introduce new variables according to

$$[\bar{u}_1, \bar{u}_2, \bar{u}_3, \ldots, \bar{u}_{3m-2}, \bar{u}_{3m-1}, \bar{u}_{3m}] := [u_1, \partial_x u_1, \partial_t u_1, \ldots, u_m, \partial_x u_m, \partial_t u_m].$$

With this, (6) is equivalent to

$$\begin{aligned} \partial_x \bar{u}_{3j-2}(x,t) &= \bar{u}_{3j-1}(x,t), \\ \partial_x \bar{u}_{3j-1}(x,t) &= \bar{u}_{3j}(x,t) - \bar{u}_{3j-1}(x,t) + f_j\big(\bar{u}_1(x,t), \bar{u}_4(x,t), \ldots, \bar{u}_{3m-2}(x,t), x\big), \\ \partial_x \bar{u}_{3j}(x,t) &= \partial_t \bar{u}_{3j-1}(x,t) \end{aligned} \tag{27}$$

for $j = 1, \ldots, m$ with initial conditions $\bar{u}_{3j-2}(0,t) = \sum_{k=1}^m (C_0)_{j,k} y_k(t)$, $\bar{u}_{3j-1}(0,t) = \sum_{k=1}^m (C_1)_{j,k} y_k(t)$, and $\bar{u}_{3j}(0,t) = \sum_{k=1}^m (C_0)_{j,k} y_k'(t)$. Let $\{\mathcal{G}_s\}_{s\in[0,1]}$ denote the scale of Banach spaces introduced in Section 3, where we fix the interval $I := [0,T]$ and the constants σ_0, σ_1 in Definition 3. As in the scalar case we formally integrate the $3m$ equations (27) and introduce the Banach space valued variables $U_i : [0,L] \to \mathcal{G}_s$ defined by $U_i(x)(t) := \bar{u}_i(x,t)$ to obtain

$$\begin{aligned} U_{3j-2}(x) &= U_{3j-2}(0) + \int_0^x U_{3j-1}(\xi) d\xi, \\ U_{3j-1}(x) &= U_{3j-1}(0) + \int_0^x \Big(U_{3j}(\xi) - U_{3j-1}(\xi) + F_j\big(U_1(\xi), U_4(\xi), \ldots, U_{3m-2}(\xi), \xi\big)\Big) d\xi, \\ U_{3j}(x) &= U_{3j}(0) + \int_0^x A U_{3j-1}(\xi) d\xi \end{aligned} \tag{28}$$

for $j = 1, \ldots, m$, where $AU_{3j-1}(x)(t) := \partial_t \bar{u}_{3j-1}(x,t)$ and

$$F_j\big(U_1(x), U_4(x), \ldots, U_{3m-2}(x), x\big)(t) := f_j\big(\bar{u}_1(x,t), \bar{u}_4(x,t), \ldots, \bar{u}_{3m-2}(x,t), x\big).$$

5.2. **Successive approximation.** To obtain a more compact formulation of (28) we write

$$U_i(x) = U_i(0) + \int_0^x G_i(U_1(\xi), \ldots, U_{3m}(\xi), \xi) d\xi \tag{29}$$

for $i = 1, \ldots, 3m$, where the functions G_i are defined by the respective integrands. The method of successive approximation yields the iteration scheme

$$\begin{aligned} U_i^{[0]}(x) &= U_i(0), \\ U_i^{[k+1]}(x) &= U_i^{[0]}(x) + \int_0^x G_i\big(U_1^{[k]}(\xi), \ldots, U_{3m}^{[k]}(\xi), \xi\big) d\xi. \end{aligned} \tag{30}$$

for $k \in \mathbb{N}_0$.

Assumption 2. Let $i = 1, 2, \ldots, 3m$, $j = 1, 2, \ldots, m$ and $k = 1, 4, \ldots, 3m-2$. We assume:

- (B1) The functions y_j as well as the derivatives y_j' are elements of $\mathcal{G}_1$. This implies the existence of constants $R_{i,0} > 0$ such that $||U_i(0)||_1 \le R_{i,0}$.
- (B2) There exist constants $R_k > R_{k,0}$ such that the functions $F_j : \mathcal{B}_s(R_1) \times \mathcal{B}_s(R_4) \times \cdots \times \mathcal{B}_s(R_{3m-2}) \times [0,L] \to \mathcal{G}_s$ are continuous. In addition, there exists a constant $C_F > 0$ such that

$$||F_j\big(v_1, \ldots, v_{3m-2}, x\big) - F_j\big(w_1, \ldots, w_{3m-2}, x\big)||_s \le C_F\big(||v_1 - w_1||_s + \cdots + ||v_{3m-2} - w_{3m-2}||_s\big)$$

holds for all $||v_k||_s, ||w_k||_s < R_k$ and all $0 \le x \le L$.

(B3) The functions $F_j(0,\dots,0,x)$ are continuous with values in $\mathcal{G}_1$ and satisfy $||F_j(0,\dots,0,x)||_1 \leq K_j$ for all $x \in [0,L]$ and certain constants $K_j \geq 0$.

One can verify for equations of type (28) that Assumptions (B1)-(B3) and Lemma 3 imply the fulfillment of the conditions (H1)-(H4) in [7]. Thus, the application of [7, Theorem 2.1] enables to deduce the following convergence result.

Theorem 2. There exists a constant $r > 0$ such that for every $i = 1,\dots,3m$, $0 \leq s < 1/2$, and $0 \leq x < r(1-s)$, the sequence $(U_i^{[k]}(x))_{k\in\mathbb{N}_0}$ defined by (30) converges to a limit function $U_i(x) \in \mathcal{G}_s$ with convergence being uniform on compact subsets of $[0, r(1-s))$. The functions $U_i : [0, r(1-s)) \to \mathcal{G}_s$ are continuously differentiable with respect to x and $u_i(x,t) = U_i(x)(t)$ solve (27).

Note that the restriction $s < 1/2$ is only for technical reasons and we refer to [7] for details.

5.3. **Semi-numerical realization.** Theoretical predictions for the interval of existence in Theorem 2 can be obtained in principle by inspecting the proof of [7, Theorem 2.1]. Since Theorem 1 and [7, Theorem 2.1] are based on the same methods, we expect that the the arising bounds on the constant r in Theorem 2 are similarly restrictive for problems arising in applications (cf. also Section 4). However, in the previous section it is illustrated that a numerical algorithm based on a discrete analogue of the iteration scheme induced by the method of successive approximation enables the computations also on significantly larger domains depending on the properties of the basic output and the arising nonlinearities. In the following, we outline the main steps to evaluate numerically the control input $\boldsymbol{h}^*$ for a given basic output $\boldsymbol{y}^*$, with components y_j^* defined according to (22). For details we refer the reader to [27].

Consider the iteration scheme (30) on $[0,1] \times [0,T]$, where the value of $T > 0$ is determined by the steady state to steady state transition. A uniform grid is defined on the domain with spacings Δx, Δt such that $n_x \Delta x = 1$ and $n_t \Delta t = T$ for some integers n_x, n_t. The basic output $\boldsymbol{y}^*$ is evaluated using the definitions of Lemma 4, where an adaptive Lobatto quadrature is applied for the calculation of the integrals. On $[0,1] \times (0,T)$, first order time derivatives occurring on the right hand side of (30) are approximated using central finite differences of second order accuracy. At the boundaries $t \in \{0, T\}$, time derivatives are set equal to zero, which is justified as the derivatives of *all* orders of the basic output vanish at these points. The spatial integrals in (30) are approximated by standard quadrature formulas. Depending on the upper integral bound, in particular on the number of subintervals (even or odd), we use either the composite Simpson rule or combine it with an additional trapezoidal step. The successive approximation is stopped once the maximum difference between two iterations is below a certain user defined value. An approximate solution of (28) (and hence (6)) is obtained and the control input can be determined using (7).

5.4. **Tubular reactor example.** In the following, numerical results are presented for the example of a tubular reactor governed by

$$\begin{aligned} \partial_t u_1(x,t) &= \tfrac{Le}{Pe_1} \partial_x^2 u_1(x,t) + Le\partial_x u_1(x,t) + LeD(1-u_1(x,t)) f(u_2(x,t)), \\ \partial_t u_2(x,t) &= \tfrac{1}{Pe_2} \partial_x^2 u_2(x,t) + \partial_x u_2(x,t) - \beta u_2(x,t) + BDa(1-u_1(x,t)) f(u_2(x,t)), \end{aligned} \tag{31a}$$

where $u_1, u_2 : [0,1] \times [0,\tau) \to \mathbb{R}$ for $\tau > 0$. The nonlinearity is given by

$$f(u_2) = \exp\left(\frac{u_2}{1+u_2/b}\right). \tag{31b}$$

Boundary conditions are imposed according to

$$\begin{aligned} \partial_x u_1(0,t) &= 0, \quad \tfrac{1}{Pe_1}\partial_x u_1(1,t) + u_1(1,t) = h_1(t), \\ \partial_x u_2(0,t) &= 0, \quad \tfrac{1}{Pe_2}\partial_x u_2(1,t) + u_2(1,t) = h_2(t). \end{aligned} \tag{31c}$$

The system is supposed to be initially in a steady state $[u_{1,0}(x), u_{2,0}(x)]$ such that $u_{1,0}(0) = u_{2,0}(0) = 0$. Here, u_1 and u_2 correspond to the conversion and normalized temperature, Pe_i, Le, and Da denote the Peclet, Lewis, and Damkoehler numbers, and B, β, b represent dimensionless parameters, respectively. The reader is referred to, e.g., [8] for model details and to [18] for the considered normalization process.

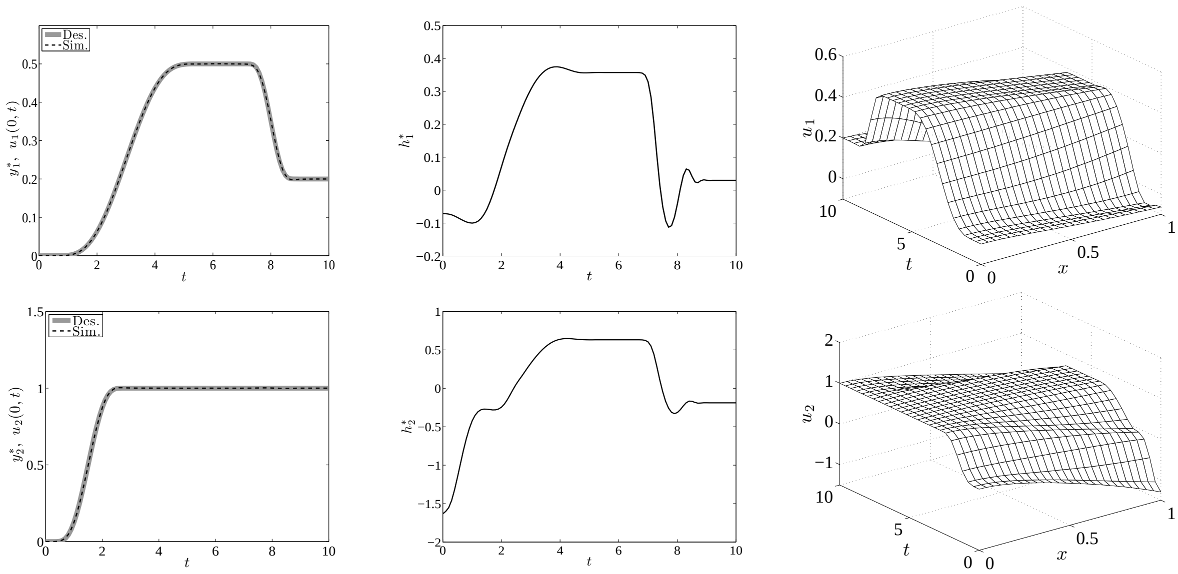


FIGURE 2. Numerical results for the semilinear PDE system (31) with $b = 20$, $Le = 1$, $Pe_1 = Pe_2 = 6$, $\beta = 1.5$, $B = 12$, $Da = 0.11$. Left column: comparison of $\boldsymbol{u}(0,t)$ and $\boldsymbol{y}^*(t)$; middle column: feedforward control $\boldsymbol{h}^*(t)$; right column: evolving profile $\boldsymbol{u}(x,t)$ for $\boldsymbol{h}^*(t)$.

5.4.1. *Flatness-based trajectory planning.* Proceeding as in Section 2 provides a formal parametrization according to (6) with $m = 2$ and the basic output components $y_1 = u_1(0,t)$ and $y_2 = u_2(0,t)$, i.e. $C_0 = \mathbb{I}$, $C_1 = 0$ with $\mathbb{I}$ being the identity matrix.

For the convergence analysis, Theorem 2 is applied, which is based on the reformulation of the spatial Cauchy problem as a first order system of equations. Hence, let

$$[\bar{u}_1(x,t), \bar{u}_2(x,t), \bar{u}_3(x,t), \bar{u}_4(x,t), \bar{u}_5(x,t), \bar{u}_6(x,t)] :=$$
$$[u_1(x,t), \partial_x u_1(x,t), \partial_t u_1(x,t), u_2(x,t), \partial_x u_2(x,t), \partial_t u_2(x,t)],$$

which yields

$$\begin{aligned}
\partial_x \bar{u}_1(x,t) =& \bar{u}_2(x,t),\\
\partial_x \bar{u}_2(x,t) =& a_1 \bar{u}_3(x,t) - a_2 \bar{u}_2(x,t) - a_3(1 - \bar{u}_1(x,t)) f(\bar{u}_4(x,t)),\\
\partial_x \bar{u}_3(x,t) =& \partial_t \bar{u}_2(x,t),\\
\partial_x \bar{u}_4(x,t) =& \bar{u}_5(x,t),\\
\partial_x \bar{u}_5(x,t) =& a_4 \bar{u}_6(x,t) - a_4 \bar{u}_5(x,t) + a_5 \bar{u}_4(x,t) - a_6(1 - \bar{u}_1(x,t)) f(\bar{u}_4(x,t)),\\
\partial_x \bar{u}_6(x,t) =& \partial_t \bar{u}_5(x,t),
\end{aligned} \tag{32}$$

with the constants

$$\{a_1, a_2, a_3, a_4, a_5, a_6\} := \{Pe_1/Le, Pe_1, Pe_1 Da, Pe_2, Pe_2\beta, Pe_2 B Da\}$$

and the initial conditions

$$\bar{u}_1(0,t) = y_1^*(t), \quad \bar{u}_2(0,t) = 0, \quad \bar{u}_3(0,t) = \partial_t y_1^*(t),$$
$$\bar{u}_4(0,t) = y_2^*(t), \quad \bar{u}_5(0,t) = 0, \quad \bar{u}_6(0,t) = \partial_t y_2^*(t).$$

Formally integrating (32) allows for an abstract formulation according to (29). Note that the coefficients in the formal parametrization are not normalized to unity. This, however, does not influence the fulfillment of Assumptions (B1)-(B3) and Theorem 2 still applies.

As an example, the desired basic output trajectory $\boldsymbol{y}^* = [y_1^*, y_2^*]$ is assigned according to (22) with

$$\begin{aligned} y_1^*(t) &= 0.5\Psi_6(t) + (0.2 - 0.5)\Psi_2(t-7), \\ y_2^*(t) &= \Psi_3(t), \end{aligned} \tag{33}$$

and $\Psi_T(\cdot)$ as introduced in (13). Subsequently, the scale of Banach spaces $\{\mathcal{G}_s\}_{s\in[0,1]}$ is fixed by considering $I := [0,10]$, $\sigma_0 := 1/40$, and $\sigma_1 := 1/20$. Corollaries 1 and 2 thereby imply (B1). The nonlinear function (31b) was already discussed in Section 4.2.3, and for a suitable choice of parameters, Corollary 6 applies. A short calculation, which is left to the reader, shows that the nonlinear term in (32) satisfies (B2). Furthermore, it is easy to see that Assumption (B3) holds. As a result, Theorem 2 implies the convergence of the iteration scheme defined by the method of successive approximation on a certain (small) spatial interval and in final consequence the existence of a local solution of the original initial value problem in the spatial variable, which is twice continuously differentiable with respect to x, where $u_1(x,\cdot) \in \mathcal{G}_s$ and $u_2(x,\cdot) \in \mathcal{G}_s$.

5.4.2. *Simulation results.* Numerical results are obtained for the tubular reactor example (31) on the domain $[0,1] \times [0,10]$ and the desired basic output $\boldsymbol{y}^*$ given by (33). For the determination of the formal state and input parametrization, the successive approximation is evaluated according to Section 5.3 using $\Delta x = 0.01$, $\Delta t = 0.1$. Here, 25 iterations are utilized to approximate the numerical solution $\hat{u}_i(x,t)$ of (32). The feedforward control $\boldsymbol{h}^* = [h_1^*, h_2^*]$ follows from the evaluation of the inhomogeneous boundary conditions (31c) at $x = 1$, i.e.

$$\begin{aligned} h_1^*(t) &= \tfrac{1}{Pe_1}\hat{u}_2(1,t) + \hat{u}_1(1,t), \\ h_2^*(t) &= \tfrac{1}{Pe_2}\hat{u}_5(1,t) + \hat{u}_4(1,t). \end{aligned}$$

Numerical results by making use of the Matlab routine *pdepe* for the solution of (31) with the feedforward control $\boldsymbol{h} = \boldsymbol{h}^*$ are shown in Fig. 2. Here, a comparison of the obtained trajectories $\boldsymbol{u}(0,t)$ and the desired paths $\boldsymbol{y}^*$ is provided (left column), which illustrates the high tracking accuracy by means of the flatness-based feedforward control $\boldsymbol{h}^*$ (middle column). Moreover, the desired finite time transition starting at the zero initial state to the final steady state prescribed in terms of the stationary values of $\boldsymbol{y}^*$ is precisely realized as is shown in Fig. 2 (right column).

Appendix A. Proof of Theorem 1

The proof of Theorem 1 is based on [9, Theorem A] with certain modifications due to the second-order nature of our problem. Let $\mathcal{B}_s(R) := \{v \in \mathcal{G}_s : ||v||_s < R\}$ for $R > R_0 > 0$. In the following, we prove that $(U^{[k]})_{k\in\mathbb{N}_0}$, defined by

$$\begin{aligned} U^{[0]}(x) &= (c_0 + xc_1)y, \\ U^{[k+1]}(x) &= U^{[0]}(x) + \int_0^x \int_0^\eta G(U^{[k]}(\xi),\xi)d\xi d\eta, \end{aligned}$$

with $G(U(x),x) := AU(x) + F(U(x),x)$ converges to a solution of (16). Lemma 3, (A2) and the embedding $\mathcal{G}_s \subset \mathcal{G}_{s'}$ for $0 \leq s' < s \leq 1$ imply that

$$||G(v,x) - G(w,x)||_{s'} \leq \frac{C}{(s-s')^2}||v-w||_s \tag{34}$$

for $w, v \in \mathcal{B}_s(R)$ and $C := C_A + C_F$. For some constant $a > 0$ let $\mathcal{X}_a$ denote the space of continuous functions defined on $[0, a(1-s))$ with values in $\mathcal{G}_s$ for every $s \in [0,1)$. On $\mathcal{X}_a$ define a norm by

$$|U|_{\mathcal{X}_a} := \sup_{\substack{0\leq s<1 \\ 0\leq x<a(1-s)}} ||U(x)||_s \left(1 - \frac{x}{a(1-s)}\right). \tag{35}$$

In the following we operate on function spaces of type $\mathcal{X}_a$. The properties of the operator A imply that the function G maps $\mathcal{G}_s$ only into $\mathcal{G}_{s'}$ for $s' < s$. It is hence necessary to ensure that the k-th approximation is in $\mathcal{G}_s$ for every $s \in [0,1)$. The aim is to proceed inductively. Assumption (A1) implies that $U^{[0]}(x) \in \mathcal{G}_s$ for $s \in [0,1]$ and $x \in [0,L]$. Thus, $U^{[1]}(x) \in \mathcal{G}_{s'}$ for $s' \in [0,1)$ and the same x-interval. The major problem then arises from the fact that the nonlinear function F is defined only on $\mathcal{B}_s(R)$. Therefore, it is necessary to ensure that $||U^{[k]}(x)||_s < R$ for $s \in [0,1)$ and $k \in \mathbb{N}_0$ on some common x-interval. Note that

$$U^{[k+1]}(x) = U^{[0]}(x) + \sum_{j=0}^{k}\big(U^{[j+1]}(x) - U^{[j]}(x)\big),$$

which implies

$$||U^{[k+1]}(x)||_s \leq ||U^{[0]}(x)||_s + \sum_{j=0}^{k} ||U^{[j+1]}(x) - U^{[j]}(x)||_s.$$

Since $||U^{[0]}(x)||_s \leq R_0$ by assumption, we require that

$$\sum_{j=0}^{\infty} ||U^{[j+1]}(x) - U^{[j]}(x)||_s \leq \frac{R-R_0}{2}. \tag{36}$$

The major difficulty in the proof of Theorem 1 is to guarantee (36). This can be achieved by diminishing the interval of existence in every iteration step. To this end, we recursively define a sequence of real numbers $(a_k)_{k\in\mathbb{N}_0}$ by

$$a_{k+1} = a_k - \frac{a_0}{2^{k+2}}.$$

Note that (a_k) is decreasing and $\lim_{k\to\infty} a_k = a_0/2$. For the moment we only require that $0 < a_0 \leq L$ but further smallness conditions for a_0 are imposed later.

Consider now the spaces $\mathcal{X}_{a_k}$, $k \in \mathbb{N}_0$ with norm (35). Since (a_k) is decreasing $|U|_{\mathcal{X}_{a_{k+1}}} \leq |U|_{\mathcal{X}_{a_k}}$ for $U \in \mathcal{X}_{a_k}$. We show by induction that $||U^{[k]}(x)||_s < R$ for $x \in [0, a_k(1-s))$ and every $s \in [0,1)$. For $k = 0$ this is true by assumption. Assume that the statement holds up to some $k \in \mathbb{N}$. Then the next approximation $U^{[k+1]}$ is well defined and

$$\mu_i := |U^{[i+1]} - U^{[i]}|_{\mathcal{X}_{a_i}} < \infty$$

for $i = 0, \ldots, k$, where $\mu_i \in \mathbb{R}$ is introduced for notational convenience. For $x \in [0, a_{k+1}(1-s))$ we infer that

$$||U^{[k+1]}(x) - U^{[k]}(x)||_s \leq \frac{\mu_k}{1 - \frac{x}{a_k(1-s)}} \leq \frac{\mu_k}{1 - \frac{a_{k+1}}{a_k}}.$$

In order to obtain (36) we consider

$$\sum_{j=0}^{k}||U^{[j+1]}(x)-U^{[j]}(x)||_s \leq \sum_{j=0}^{k}\frac{\mu_j}{1-\frac{a_{j+1}}{a_j}} \leq 4\sum_{j=0}^{k}2^j\mu_j, \tag{37}$$

where the definition of the sequence (a_k) was inserted. The next step is to estimate μ_j. By definition

$$||U^{[k+1]}(x)-U^{[k]}(x)||_s \leq \int_0^x\int_0^\eta ||G(U^{[k]}(\xi),\xi)-G(U^{(k-1)}(\xi),\xi)||_s d\xi d\eta,$$

which is considered for $x \in [0, a_k(1-s))$. In order to apply (34) we construct a larger index $s(\xi)$ depending on the integration variables (the index s in the above inequality now corresponds to the smaller one of the index pair in (34)). Observing that $\xi < a_k(1-s(\xi))$ must hold, which implies that $s < s(\xi) < 1-\xi/a_k$, we choose $s(\xi) = \frac{1}{2}(1+s-\xi/a_k)$. Thus,

$$\begin{aligned}
&\int_0^x\int_0^\eta ||G(U^{[k]}(\xi),\xi)-G(U^{(k-1)}(\xi),\xi)||_s d\xi d\eta \leq C\int_0^x\int_0^\eta \frac{||U^{[k]}(\xi)-U^{(k-1)}(\xi)||_{s(\xi)}}{(s(\xi)-s)^2}d\xi d\eta \\
&\leq C\int_0^x\int_0^\eta \frac{\mu_{k-1}}{1-\frac{\xi}{a_{k-1}(1-s(\xi))}}\frac{1}{(s(\xi)-s)^2}d\xi d\eta \leq C\int_0^x\int_0^\eta \frac{\mu_{k-1}}{1-\frac{\xi}{a_k(1-s(\xi))}}\frac{1}{(s(\xi)-s)^2}d\xi d\eta \\
&\leq C\mu_{k-1}\int_0^x\int_0^\eta \frac{a_k(1-s(\xi))}{(a_k(1-s(\xi))-\xi)\,(s(\xi)-s)^2}d\xi d\eta \leq 4C\mu_{k-1}a_k^2\int_0^x\int_0^\eta \frac{a_k(1-s)+\xi}{(a_k(1-s)-\xi)^3}d\xi d\eta \\
&\leq 4C\mu_{k-1}a_k^2\,(a_k(1-s)+x)\int_0^x\int_0^\eta \frac{d\xi d\eta}{(a_k(1-s)-\xi)^3}.
\end{aligned}$$

Integration yields

$$||U^{[k+1]}(x)-U^{[k]}(x)||_s \leq \frac{2C\mu_{k-1}a_k^2}{1-\frac{x}{a_k(1-s)}}\frac{x^2}{(a_k(1-s))^2}\Big(1+\frac{x}{a_k(1-s)}\Big).$$

Hence,

$$\mu_k = \sup_{\substack{0\leq s<1\\ 0\leq x<a_k(1-s)}} ||U^{[k+1]}(x)-U^{[k]}(x)||_s\left(1-\frac{x}{a_k(1-s)}\right) \leq 4C\mu_{k-1}a_k^2 \leq 4Ca_0^2\mu_{k-1}.$$

Subsequently, let a_0 be such that $a_0^2 < \frac{1}{12C}$, which is equivalent to $4Ca_0^2 < 1/3$. We infer that $\mu_j < \mu_{j-1}/3$ for $j=1,\ldots,k$ and $\mu_j < (1/3)^j\mu_0$. To obtain an estimate for μ_0, Assumption (A3) is used to calculate

$$\begin{aligned}
||U^{[1]}(x)-U^{[0]}(x)||_s &\leq \int_0^x\int_0^\eta ||G(U^{[0]}(\xi),\xi)||_s d\xi d\eta \leq C\int_0^x\int_0^\eta \frac{||U^{[0]}(\xi)||_{s(\xi)}+K}{(s(\xi)-s)^2}d\xi d\eta \\
&\leq C(R_0+K)\int_0^x\int_0^\eta \frac{1}{(s(\xi)-s)^2\big(1-\frac{\xi}{a_0(1-s(\xi))}\big)}d\xi d\eta.
\end{aligned}$$

With the choice $s(\xi) = \frac{1}{2}(1+s-\xi/a_0)$ and essentially the same calculation as above, we conclude that

$$\mu_0 = \sup_{\substack{0\leq s<1\\ 0\leq x<a_0(1-s)}} ||U^{[1]}(x)-U^{[0]}(x)||_s\left(1-\tfrac{x}{a_0(1-s)}\right) \leq 4C(R_0+K)a_0^2.$$

With these estimates at hand we return to the original question and obtain

$$\sum_{j=0}^{k}2^j\mu_j \leq \mu_0\sum_{j=0}^{\infty}\left(\tfrac{2}{3}\right)^j = 3\mu_0 \leq 12C(R_0+K)a_0^2.$$

To deduce (36) let

$$a_0^2 < \frac{R-R_0}{96C(R_0+K)} \tag{38}$$

so that (37) yields

$$\sum_{j=0}^{k} ||U^{[j+1]}(x) - U^{[j]}(x)||_s \leq 48C(R_0+K)a_0^2 < \frac{R-R_0}{2}$$

and hence

$$||U^{[k+1]}(x)||_s < \frac{R+R_0}{2} < R$$

for $x \in [0, a_{k+1}(1-s))$ and $s \in [0,1)$. We conclude that the statement holds for all $k \in \mathbb{N}_0$.

These preliminaries enable us to prove the convergence of the successive approximation for $x \in [0, r(1-s))$ with

$$r := \frac{a_0}{2} = \lim_{k\to\infty} a_k.$$

The above considerations yield

$$||U^{[k+1]}(x) - U^{[k]}(x)||_s \leq \frac{\mu_k}{1 - \frac{x}{a_k(1-s)}} \leq \frac{\mu_k}{1 - \frac{x}{r(1-s)}}$$

and

$$|U^{[k+1]} - U^{[k]}|_{\mathcal{X}_r} \leq \mu_k.$$

For every x in $[0, r(1-s))$ and $k, j \in \mathbb{N}$ with $k > j$ this implies that

$$||U^{[k]}(x) - U^{[j]}(x)||_s \leq \frac{\sum_{i=j}^{k-1} \mu_i}{1 - \frac{x}{r(1-s)}} \leq \frac{\mu_0 \sum_{i=j}^{k-1} \left(\frac{1}{3}\right)^i}{1 - \frac{x}{r(1-s)}} \leq \frac{\mu_0}{1 - \frac{x}{r(1-s)}} \frac{3}{2} \left(\frac{1}{3}\right)^j.$$

Thus, for every $s \in [0,1)$ and $x \in [0, r(1-s))$ the sequence $(U^{[k]}(x))_{k\in\mathbb{N}_0}$ converges to a limit function $U(x)$ in $\mathcal{G}_s$ with $||U(x)||_s \leq \frac{R+R_0}{2} < R$. Convergence is thereby uniform with respect to x on every compact subset of $[0, r(1-s))$. It remains to show that $U(x)$ solves (16). For $0 \leq s' < s < 1$ we obtain

$$\left\| (c_0 + xc_1)y + \int_0^x \int_0^\eta G(U(\xi), \xi) d\xi d\eta - U(x) \right\|_{s'} \leq \int_0^x \int_0^\eta ||G(U(\xi), \xi) - G(U^{[k]}(\xi), \xi)||_{s'} d\xi d\eta$$

$$+ ||U^{[k+1]}(x) - U(x)||_s \leq \frac{C}{(s-s')^2} \int_0^x \int_0^\eta ||U(\xi) - U^{[k]}(\xi)||_s d\xi d\eta + ||U^{[k+1]}(x) - U(x)||_s.$$

The right hand side converges to zero as $k \to \infty$, which implies the claim.

Appendix B. Proof of Lemma 4, Lemma 6 and Corollary 3

B.1. **Proof of Lemma 4.** We study the properties of ψ and restrict ourselves to $t \in (0, 1/2]$ for symmetry reasons. The function ψ is real analytic on $(0,1)$ and can be analytically extended to a complex function in a small neighbourhood of t for every $t \in (0, 1/2]$. For $n \in \mathbb{N}_0$, Cauchy's integral formula is applied to obtain

$$\psi^{(n)}(t) = \frac{n!}{2\pi i} \int_\Gamma \frac{\psi(z)}{(z-t)^{n+1}} dz$$

where we set

$$\Gamma := \{z \in \mathbb{C} : z = t + \tfrac{t}{2} e^{i\varphi}, t \in (0, \tfrac{1}{2}], \varphi \in [0, 2\pi)\}.$$

Thus,

$$\psi^{(n)}(t) = \frac{n!}{2\pi} \left(\frac{2}{t}\right)^n \int_0^{2\pi} \psi(t, \varphi) e^{-in\varphi} d\varphi$$

and

$$|\psi^{(n)}(t)| \le \frac{n!}{2\pi}\left(\frac{2}{t}\right)^n \int_0^{2\pi} |\psi(t,\varphi)| d\varphi = \frac{n!}{2\pi}\left(\frac{2}{t}\right)^n \int_0^{2\pi} \exp\left(-\mathrm{Re}\tfrac{1}{z(t,\varphi)}\right) \exp\left(-\mathrm{Re}\tfrac{1}{1-z(t,\varphi)}\right) d\varphi,$$

where $\mathrm{Re}(1/z(1-z)) = \mathrm{Re}(1/z) + \mathrm{Re}(1/1-z)$ is used. Note that for $z \in \Gamma$, the individual terms can be estimated by

$$\mathrm{Re}\frac{1}{1-z} \ge 1, \quad \mathrm{Re}\frac{1}{z} \ge \frac{2}{3t}$$

such that

$$|\psi^{(n)}(t)| \le \frac{n!}{e}\left(\frac{2}{t}\right)^n e^{-2/3t} \le \frac{n!}{e}\left(\frac{3n}{e}\right)^n \le \frac{n!^2 3^n}{e},$$

where we use the fact that $x^a e^{-bx} \le (a/eb)^a$ for $a \ge 0$ and $b > 0$ as well as the estimate $n^n \le n! e^n$. For $n \ge 1$ this implies that

$$|\Psi_T^{(n)}(t)| = \frac{|\psi^{(n-1)}(t/T)|}{N_\Psi T^n} \le \frac{n!^2}{3eN_\Psi}\left(\frac{3}{T}\right)^n.$$

Note that $1/(3eN_\Psi) > 1$ and since $|\Psi_T(t)| \le 1$ we conclude that the above estimate holds for all $n \ge 0$.

B.2. **Proof of Lemma 6.** The first part of the proof is a one-dimensional version of a result in [13]. First, note that any $v \in \mathcal{B}_s(R)$ satisfies $\max_{t\in I} |v(t)| \le ||v||_s < R$, hence it suffices to consider f on $J := [-R, R]$. Here, f is assumed to be a $\mathbb{G}^2(J,\gamma)$-function satisfying

$$\max_{x\in J} |f^{(n)}(x)| \le M\frac{n!^2}{\gamma^n}$$

for $\gamma > R$. We fix another constant δ such that $R < \delta < \gamma$. By Lemma 1, the m-th derivative of f for $m \in \mathbb{N}_0$ belongs to $\mathbb{G}^2(J,\delta)$ and

$$\max_{x\in J} |f^{(m+n)}(x)| \le M'(m)\frac{n!^2}{\delta^n}$$

for $M'(m) := \frac{M}{\delta^m}\left[\frac{2m}{e(\ln\gamma - \ln\delta)}\right]^{2m}$. This yields

$$\sum_{j=0}^{\infty} \frac{R^j}{j!^2} \max_{x\in J} |f^{(m+j)}(x)| \le \frac{M'(m)}{1-(R/\delta)} := C_m. \tag{39}$$

To obtain an estimate for the n-th derivative of the composition $f \circ v$, the following version of the formula of Faà di Bruno, cf. [13], is used

$$(f\circ v)^{(n)} = \sum_{j=1}^{n} \frac{f^{(j)}\circ v}{j!} \sum_{\substack{k_1,\ldots,k_j\in\mathbb{N}\\ \sum k_i = n}} \frac{n!}{k_1!\cdots k_j!} \prod_{i=1}^{j} v^{(k_i)} \tag{40}$$

for $n \ge 1$. Furthermore, it can easily verified by induction that $j! \le \frac{n!}{k_1!\cdots k_j!}$ for $j \ge 1$ and $k_1,\ldots,k_j \in \mathbb{N}$ with $\sum k_i = n$. This implies that $1 \le \frac{n!}{j!k_1!\cdots k_j!} \le \frac{n!^2}{(j!k_1!\cdots k_j!)^2}$ and hence

$$\frac{1}{n!j!k_1!\cdots k_j!} \le \frac{1}{(j!k_1!\cdots k_j!)^2}.$$

For $n \geq 1$ it follows that

$$\frac{\sigma^n(s)}{n!^2} \max_{t\in I} |(f\circ v)^{(n)}(t)| \leq \sum_{j=1}^{n} \frac{\sigma^n(s)}{n!j!} \max_{t\in I} |(f^{(j)}\circ v)(t)| \sum_{\substack{k_1,\dots,k_j\in\mathbb{N} \\ \sum k_i = n}} \frac{1}{k_1!\cdots k_j!} \prod_{i=1}^{j} \max_{t\in I} |v^{(k_i)}(t)|$$

$$\leq \sum_{j=1}^{n} \frac{1}{j!^2} \max_{t\in I} |(f^{(j)}\circ v)(t)| \sum_{\substack{k_1,\dots,k_j\in\mathbb{N} \\ \sum k_i = n}} \prod_{i=1}^{j} \frac{\sigma^{k_i}(s)}{k_i!^2} \max_{t\in I} |v^{(k_i)}(t)|.$$

Taking the sum over n yields

$$\sum_{n=1}^{\infty} \frac{\sigma^n(s)}{n!^2} \max_{t\in I} |(f\circ v)^{(n)}(t)| \leq \sum_{j=1}^{\infty} \frac{1}{j!^2} \max_{t\in I} |(f^{(j)}\circ v)(t)| \sum_{k_1,\dots,k_j\in\mathbb{N}} \prod_{i=1}^{j} \frac{\sigma^{k_i}(s)}{k_i!^2} \max_{t\in I} |v^{(k_i)}(t)|$$

$$\leq \sum_{j=1}^{\infty} \frac{1}{j!^2} \max_{t\in I} |(f^{(j)}\circ v)(t)| \left(\sum_{k=1}^{\infty} \frac{\sigma^k(s)}{k!^2} \max_{t\in I} |v^{(k)}(t)| \right)^j \leq \sum_{j=1}^{\infty} \frac{R^j}{j!^2} \max_{t\in I} |(f^{(j)}\circ v)(t)|$$

$$\leq \sum_{j=1}^{\infty} \frac{R^j}{j!^2} \max_{x\in J} |f^{(j)}(x)| < \infty.$$

Since $\max_{t\in I} |(f\circ v)(t)| \leq \max_{x\in J} |f(x)|$ we conclude that

$$||F(v)||_s = \sum_{n=0}^{\infty} \frac{\sigma^n(s)}{n!^2} \max_{t\in I} |(f\circ v)^{(n)}(t)| \leq C_0.$$

With (39) the same argument can be used to define functions $F^{(m)} : \mathcal{B}_s(R) \to \mathcal{G}_s$ by

$$F^{(m)}(v)(t) := (f^{(m)}\circ v)(t)$$

for $m \in \mathbb{N}$ with $||F^{(m)}(v)||_s \leq C_m$.

Concerning differentiability, observe that $F : \mathcal{B}_s(R) \subset \mathcal{G}_s \to \mathcal{G}_s$ is differentiable at $v \in \mathcal{B}_s(R)$ in the sense of Fréchet with derivative $DF(v)$ if $DF(v) : \mathcal{G}_s \to \mathcal{G}_s$ is a bounded linear operator that satisfies

$$\lim_{w\to 0} \frac{||F(v+w) - F(w) - DF(v)w||_s}{||w||_s} = 0. \tag{41}$$

For fixed $v \in \mathcal{B}_s(R)$ we claim that

$$[DF(v)w](t) = [F^{(1)}(v)w](t),$$

where $[F^{(1)}(v)w](t) = (f'\circ v)(t)w(t)$. The above result shows that this defines a bounded operator since

$$||F^{(1)}(v)w||_s \leq ||F^{(1)}(v)||_s ||w||_s \leq C_1 ||w||_s < \infty$$

for $w \in G_s$. To verify (41) let w be in $\mathcal{G}_s$ satisfying $||w||_s < R - ||v||_s$. This is not a restriction, since we are interested in the limit as $w \to 0$. Thus $||v+w||_s \leq ||v||_s + ||w||_s < R$ and by Taylor's theorem

$$F(v+w)(t) - F(v)(t) - [F^{(1)}(v)w](t) = f(v(t)+w(t)) - f(v(t)) - f'(v(t))w(t)$$

$$= \int_{v(t)}^{v(t)+w(t)} f^{(2)}(\xi')(v(t)+w(t)-\xi')d\xi' = w^2(t) \int_0^1 f^{(2)}(v(t)+\xi w(t))(1-\xi)d\xi,$$

where the new integration variable ξ is defined by $\xi' = v(t) + \xi w(t)$. We show now that $K(t) := \int_0^1 f^{(2)}(v(t)+\xi w(t))(1-\xi)d\xi$ is a $\mathcal{G}_s$-function. For $\xi \in [0,1]$ set $u_\xi := v+\xi w \in \mathcal{B}_s(R)$. The function $(\xi,t) \mapsto (1-\xi)f^{(2)}(u_\xi(t))$ is C^∞ on $[0,1]\times I$ and dominated convergence implies that

$$|K^{(n)}(t)| \le \int_0^1 |(f^{(2)} \circ u_\xi)^{(n)}(t)|(1-\xi)d\xi \max_{\xi\in[0,1]} |(f^{(2)} \circ u_\xi)^{(n)}(t)|.$$

Taking into account (40) yields

$$\max_{\xi\in[0,1]} |(f^{(2)} \circ u_\xi)^{(n)}(t)| \le \sum_{j=1}^{n} \frac{1}{j!} \max_{\xi\in[0,1]} |(f^{(2+j)} \circ u_\xi)(t)| \sum_{\substack{k_1,\dots,k_j\in\mathbb{N},\\ \sum k_i=n}} n! \prod_{i=1}^{j} \frac{1}{k_i!} \max_{\xi\in[0,1]} |u_\xi^{(k_i)}(t)|$$

and hence

$$\frac{\sigma^n(s)}{n!^2} \max_{\xi\in[0,1]} |(f^{(2)} \circ u_\xi)^{(n)}(t)| \le \sum_{j=1}^{n} \frac{1}{j!^2} \max_{\xi\in[0,1]} |(f^{(2+j)} \circ u_\xi)(t)| \sum_{\substack{k_1,\dots,k_j\in\mathbb{N},\\ \sum k_i=n}} \prod_{i=1}^{j} \frac{\sigma^{k_i}(s)}{k_i!^2} \max_{\xi\in[0,1]} |u_\xi^{(k_i)}(t)|$$

for $n \ge 1$. Evaluation of the sum over n yields

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sigma^n(s)}{n!^2} \max_{t\in I} |K^{(n)}(t)| &\le \sum_{j=1}^{\infty} \frac{1}{j!^2} \max_{x\in J} |f^{(2+j)}(x)| \sum_{k_1,\dots,k_j\in\mathbb{N}} \prod_{i=1}^{j} \frac{\sigma^{k_i}(s)}{k_i!^2} \max_{\substack{\xi\in[0,1]\\ t\in I}} |u_\xi^{(k_i)}(t)| \\
&\le \sum_{j=1}^{\infty} \frac{1}{j!^2} \max_{x\in J} |f^{(2+j)}(x)| \left(\sum_{k=1}^{\infty} \frac{\sigma^k(s)}{k!^2} \max_{\substack{\xi\in[0,1]\\ t\in I}} |u_\xi^{(k)}(t)| \right)^j \\
&\le \sum_{j=1}^{\infty} \frac{1}{j!^2} \max_{x\in J} |f^{(2+j)}(x)| \left(\sum_{k=1}^{\infty} \frac{\sigma^k(s)}{k!^2} \left[\max_{t\in I} |v^{(k)}(t)| + \max_{t\in I} |w^{(k)}(t)| \right] \right)^j \\
&\le \sum_{j=1}^{\infty} \frac{1}{j!^2} \max_{x\in J} |f^{(2+j)}(x)| \left(||v||_s + ||w||_s\right)^j \le \sum_{j=1}^{\infty} \frac{R^j}{j!^2} \max_{x\in J} |f^{(2+j)}(x)| < \infty.
\end{aligned}$$

Adding the term $n=0$ yields $||K||_s \le C_2 < \infty$ with C_2 defined in (39). As a result

$$||F(v+w) - F(v) - F^{(1)}(v)w||_s = ||w^2 K||_s \le ||w||_s^2 ||K||_s \le C_2 ||w||_s^2.$$

Since C_2 is independent of w we conclude that (41) holds.

It remains to prove the Lipschitz estimate. Let $v, w \in \mathcal{B}_s(R)$. We apply the mean value theorem for the Fréchet derivative to obtain

$$||F(v) - F(w)||_s \le \sup_{h\in(0,1)} ||F^{(1)}(hv + (1-h)w)||_s ||v-w||_s,$$

where the line segment $hv + (1-h)w$ is in $\mathcal{B}_s(R)$ for $h \in [0,1]$. We set $u_h := hv + (1-h)w$ and obtain

$$||F^{(1)}(u_h)||_s \le C_1$$

for all $h \in [0,1]$. We conclude that

$$\sup_{h\in(0,1)} ||F^{(1)}(hv + (1-h)w)||_s \le C_1$$

and hence $||F(v) - F(w)||_s \le C_1 ||v-w||_s$. According to (39) the constant C_1 is given by

$$C_1 = \frac{M}{\delta - R} \left[\frac{2}{e \ln(\gamma/\delta)} \right]^2.$$

B.3. **Proof of Corollary 3.** By assumption, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-\rho, \rho)$. In particular, for all $0 < R < \gamma < \rho$ there exists a constant C depending on γ such that $|a_n| \leq \frac{C}{\gamma^n}$, which implies that

$$|f^{(n)}(x)| \leq \sum_{m=n}^{\infty} m(m-1)\ldots(m-n+1)|a_m||x|^{m-n} \leq \frac{C}{\gamma^n} \sum_{m=n}^{\infty} m(m-1)\ldots(m-n+1)\left(\frac{|x|}{\gamma}\right)^{m-n}$$
$$= \frac{C}{\gamma^n} \frac{n!}{\left(1 - \frac{|x|}{\gamma}\right)^{n+1}}.$$

Setting $a := 1/(1 - R/\gamma)$ yields

$$|f^{(n)}(x)| \quad \leq \quad \frac{C}{\gamma^n} n! a^{n+1} \leq M \frac{n!^2}{\gamma^n}$$

for $|x| \leq R$ and a suitable constant $M > 0$.

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