FLATNESS-BASED TRAJECTORY PLANNING FOR SEMILINEAR PARABOLIC PDES

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ABSTRACT. Trajectory planning is considered for semilinear parabolic partial differential equations (PDEs) with boundary control. For this, a novel flatness-based technique is proposed, which is based on the reformulation of the boundary control problem as a Cauchy problem followed by a formal integration. Solution existence is analyzed using a generalized Cauchy-Kowalevskaja Theorem in suitable Gevrey classes and formal integration. This moreover enables to deduce efficient semi-numerical design techniques, which are illustrated in a simulation scenario.

1. INTRODUCTION

Trajectory planning, i.e., the determination of a feedforward control to realize desired state or output trajectories, is of fundamental importance in many applications arising, e.g., in chemical engineering, mechatronics, and robotics. For this, flatness-based techniques have evolved into a rather systematic design approach for finite-dimensional and certain classes of infinite-dimensional systems governed by PDEs [3, 7, 16, 9, 12]. While there exists a rather broad catalog of applications, given semilinear PDEs flatness is however still restricted to polynomial nonlinearities [9, 2, 12].

In order to address this limitation, subsequently a novel approach is presented for flatness-based trajectory planning for boundary controlled semilinear parabolic PDEs in a one-dimensional spatial domain. Herein, the boundary control problem is transformed into an initial value problem for a semilinear second order PDE parametrized in terms of the flat or basic output. We investigate the resulting Cauchy problem on a rigorous level and analyze the existence of a local solution for nonlinearities satisfying a Gevrey class 2 condition. The approach relies on the reformulation of the Cauchy problem as a first order system in scales of Banach spaces of Gevrey class functions, which turns out to be an appropriate problem set-up (cf. also [19] for a comprehensive introduction). The system is studied in integral form and a solution is obtained with the method of successive approximation. This in addition enables to deduce efficient semi-numerical trajectory planning techniques. For this, two algorithms, i.e., a discrete iteration scheme induced by the successive approximation and an explicit algorithm using quadrature formulas, are proposed for the numerical evaluation of the control input in terms of the basic output. Both algorithms yield highly accurate results with the latter being computationally much less expensive. The applicability of our method is confirmed by simulation results for a semilinear parabolic PDE with trigonometric nonlinearity.

The paper is organized as follows. Section 2 introduces the formal state and input parametrization in terms of a Cauchy problem. Mathematical preliminaries and the solution existence are addressed in Section 3 followed by the semi-numerical realization in Section 4. Simulation results are presented in Section 5. Final remarks conclude the paper.

2. FLATNESS-BASED TRAJECTORY PLANNING

We study boundary control problems for semilinear reaction-diffusion equations with a Neumann boundary condition at $x = 0$ and a Dirichlet input h at $x = 1$, i.e.

$$\partial_t u(x,t) = \partial_x^2 u(x,t) - f(u(x,t),x) \tag{1a}$$

$$\partial_x u(0,t) = 0, \quad u(1,t) = h(t), \tag{1b}$$

where u is real valued and considered on the domain $(0,1) \times (0,\tau)$ for some $\tau > 0$. The system is assumed to be initially in steady state

$$u(x,0) = u_0(x), \quad x \in [0,1] \tag{1c}$$

with u_0 a solution $u_s(h_s; \cdot)$ of the boundary-value problem associated to (1a), (1b) for some fixed $h_s = h(0)$, i.e.

$$\partial_x^2 u_s(x) - f(u_s(x),x) = 0,$$
$$\partial_x u_s(0) = 0, \quad u_s(1) = h_s. \tag{2}$$

Based on (1), the considered trajectory planning problem concerns the design of a feedforward control h^* to realize the transition from the initial steady state u_0 to a final steady state u_1 within the finite time interval $t \in [0,T]$ along a predefined spatial-temporal profile u^*.

For this, we proceed with a flatness-based approach by reformulating the initial-boundary-value problem (1) as a Cauchy problem in the spatial variable x (see also [11] for the linear case). The boundary condition at $x = 0$ is thereby interpreted as initial data, i.e.

$$\partial_x^2 u(x,t) = \partial_t u(x,t) + f(u(x,t),x)$$
$$\partial_x u(0,t) = 0. \tag{3a}$$

Since (3a) is of 2nd order in x an additional linear independent boundary condition at $x = 0$ is required. Therefore, impose

$$u(0,t) = y(t) \tag{3b}$$

with the function $t \mapsto y(t)$ serving a as degree of freedom. If a solution $u(y; \cdot)$ of (3) exists at $x = 1$ for given y, then the input can be parametrized in terms of y by

$$h(t) = u(y; 1, t). \tag{4}$$

Subsequently, y is called a flat or basic output formally parametrizing state u and input h. In particular, by prescribing a desired path y^* the solution of (3) yields the feedforward control h^*, which is required to track the corresponding spatial-temporal path u^* in open-loop.

3. ANALYTIC RESULTS

It is well known that (3) even in the linear case admits a unique solution if and only if y is of Gevrey class $d \leq 2$. For semilinear problems existence of solutions is verified in [4] on a small spatial interval for nonlinearities of Gevrey class $d \leq 2$ in t and analytic in the remaining variables. In [18] this was further generalized requiring only continuity in x instead of analyticity. Subsequently, we pursue a different ansatz and investigate (3) within the framework of the abstract Cauchy-Kowalevskaja Theorem mainly developed in [15, 13, 14, 6, 5]. We show that a local solution can be obtained if both the basic output and the nonlinearity satisfy certain Gevrey class conditions.

3.1. Scales of Banach spaces.

In the following, essential results on Gevrey class functions and scales of Banach spaces are provided, which are required for the analysis of (3), where we restrict ourselves to functions of Gevrey class $d = 2$. We abbreviate $\frac{d^n}{dt^n}u(t)$ by $u^{(n)}(t)$ and for $n = 1$ we write $u'(t)$ instead of $u^{(1)}(t)$.

Definition 1. Let $\Omega \subset \mathbb{R}$ be an open set. A function $u : \Omega \to \mathbb{R}$ is of Gevrey class 2 if $u \in C^\infty(\Omega)$ and for every compact subset $I \subset \Omega$ there exist positive constants γ, M such that

$$\max_{t \in I} |u^{(n)}(t)| \leq M \frac{n!^2}{\gamma^n}, \quad \forall n \in \mathbb{N}_0. \tag{5}$$

To obtain a Banach space, Gevrey class functions have to be restricted to one single subset I and the constant γ in (5) has to be fixed [8]. However, it is impossible to formulate (3) on one *single* Banach space due to the properties of the differential operator in t. This is illustrated in the following Lemma, whose proof is given in Appendix A.

Lemma 1. *Suppose $u : I \to \mathbb{R}$ satisfies estimate (5) for fixed $\gamma > 0$ and some constant $M > 0$. Then u' satisfies (5) only for $\sigma < \gamma$. In particular, for any $\sigma \in \mathbb{R}$ with $0 < \sigma < \gamma$ there exists a constant $M' > 0$ such that*

$$\max_{t \in I} |u^{(n+1)}(t)| \leq M' \frac{n!^2}{\sigma^n}, \quad n \in \mathbb{N}_0. \tag{6}$$

This problem can be overcome by introducing a one-parameter family of function spaces allowing the respective constant in (5) to vary. On each single space a norm is defined according to [10].

Definition 2. For fixed constants $0 < \sigma_0 < \sigma_1$ we define a scale function by $\sigma(s) = (1-s)\sigma_0 + s\sigma_1$, where $s \in [0, 1]$. We say that $u \in \mathcal{G}_s$ (where we drop the dependence on the interval I for notational convenience) if $u \in C^\infty(\Omega)$ for some $I \subset \Omega$ and

$$||u||_s := \sum_{n=0}^\infty \frac{\sigma^n(s)}{n!^2} \max_{t \in I} |u^{(n)}(t)| < \infty. \tag{7}$$

According to [10], $\mathcal{G}_s$ with norm $||\cdot||_s$ is a Banach space and a scale of Banach spaces can be defined by $\{\mathcal{G}_s\}_{s \in [0,1]}$, where $\mathcal{G}_s \subseteq \mathcal{G}_{s'}$ and $||u||_{s'} \leq ||u||_s$, $0 \leq s' \leq s \leq 1$. Furthermore, each space $\mathcal{G}_s$ is a Banach algebra. In particular, for $u, v \in \mathcal{G}_s$, it holds that $uv \in \mathcal{G}_s$ and

$$||uv||_s \leq ||u||_s ||v||_s.$$

Note that the norm (7) differs from those in [1] and [5] and simplifies most of the proofs. The Banach algebra property can be easily verified using the Leibniz rule. The next result is an abstract version of Lemma 1. On $\mathcal{G}_s$ we define an operator D by $(Du)(t) := u'(t)$.

Lemma 2. *For $0 \leq s' < s \leq 1$ the operator $D : \mathcal{G}_s \to \mathcal{G}_{s'}$ is bounded and*

$$||Du||_{s'} \leq \frac{C_D}{(s - s')^2} ||u||_s$$

for all $u \in \mathcal{G}_s$, where $C_D := (2/e)^2 (1/\sigma_0)(\sigma_1/(\sigma_1 - \sigma_0))^2$.

The proof of this result is omitted but is available in [17].

3.2. Abstract integral equation.

Eqn. (3) is subsequently considered on an extended spatial interval $[0, L]$ for $L > 1$. By introducing new variables $[u_1, u_2, u_3] := [u, \partial_x u, \partial_t u]$ we reformulate

(3) as a first order system and formally integrate, which yields

$$u_1(x,t) = y(t) + \int_0^x u_2(\xi, t)d\xi$$

$$u_2(x,t) = \int_0^x \big(u_3(\xi, t) + f(u_1, \xi)\big)d\xi \tag{8}$$

$$u_3(x,t) = y'(t) + \int_0^x \partial_t u_2(\xi, t)d\xi$$

To obtain an abstract formulation the state variable is considered as a function of x with values in a function space, i.e., we define $U_i : [0,L] \to \mathcal{G}_s$ by $U_i(x)(t) := u_i(x,t)$. The above system of integral equations can then be written as

$$U_i(x) = U_{i,0} + \int_0^x G_i(U_1(\xi), U_2(\xi), U_3(\xi))d\xi \tag{9a}$$

with $U_{1,0} = y$, $U_{2,0} = 0$, $U_{3,0} = y'$ and

$$G_1(U_1(x), U_2(x), U_3(x)) = U_2(x)$$
$$G_2(U_1(x), U_2(x), U_3(x)) = U_3(x) + F(U_1(x), x) \tag{9b}$$
$$G_3(U_1(x), U_2(x), U_3(x)) = DU_2(x)$$

for $F(U_1(x), x)(t) := f(u_1(x,t), x)$ and $(DU_2(x))(t) := \partial_t u_2(x,t)$.

3.3. **Local existence of solutions.** In order to construct a solution of (9) we define a sequence of functions $(U_i^{[k]}(x))_{k \in \mathbb{N}_0}$ by $U_i^{[0]}(x) = U_{i,0}$ and

$$U_i^{[k+1]}(x) = U_{i,0} + \int_0^x G_i\big(U_1^{[k]}(\xi), U_2^{[k]}(\xi), U_3^{[k]}(\xi)\big)d\xi.$$

We consider a fixed scale of Banach spaces $\{\mathcal{G}_s\}_{s \in [0,1]}$ and for the sake of simplicity we assume that $F(0,x) = 0$.

Assumptions.

(A1) Assume that $y \in \mathcal{G}_1$, $y' \in \mathcal{G}_1$. In particular, this implies $||y||_1 \leq R_0 < \infty$ for some constant R_0.

(A2) The nonlinear function $F : \mathcal{B}_s(R) \times [0,L] \to \mathcal{G}_s$ defines a continuous map, where $\mathcal{B}_s(R) := \{u \in \mathcal{G}_s : ||u||_s < R\}$ for fixed $R > R_0$ and $s \in [0,1]$. In addition, the (local) Lipschitz estimate

$$||F(u,x) - F(v,x)||_s \leq C_F ||u - v||_s \tag{10}$$

holds for $u, v \in \mathcal{B}_s(R)$, $0 \leq x \leq L$ and a constant $C_F > 0$ independent of u, v, x, s.

Lemma 2 and the preceding assumptions allow for an application of [5, Theorem 2.1].

Theorem 1. *There exits a constant $r > 0$ such that for every $0 \leq s < 1/2$ and $0 \leq x < r(1-s)$ the sequence $(U_i^{[k]}(x))_{k \in \mathbb{N}_0}$ converges to a limit function $U_i(x)$ in $\mathcal{G}_s$ with convergence being uniform with respect to x on compact subsets of $[0, r(1-s))$. The functions $U_i(x)$ are continuously differentiable with respect to x and $u(x,t) := U_1(x)(t)$ solves the Cauchy problem (3).*

For applications it is crucial to decide whether a given nonlinear function f satisfies Assumption (A2). To address this, we provide a classification.

Lemma 3. *Let $R > 0$ be fixed and let $f : \mathbb{R} \to \mathbb{R}$ be a function of Gevrey class 2 satisfying*

$$\max_{x \in [-R,R]} |f^{(n)}(x)| \leq M_f \frac{n!^2}{\gamma_f^n}, \quad \forall n \in \mathbb{N}_0 \tag{11}$$

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for constants $M_f > 0$ and $\gamma_f > R$. Let F be defined by

$$F(u)(t) := (f \circ u)(t), \quad t \in I = [0, T].$$

The function F maps $\mathcal{B}_s(R)$ into $\mathcal{G}_s$, is differentiable (in the sense of Fréchet) at any $u \in \mathcal{B}_s(R)$, and for any $u, v \in \mathcal{B}_s(R)$ there exists a positive constant C_F such that

$$||F(u) - F(v)||_s \leq C_F ||u - v||_s.$$

The proof of Lemma 3 is sketched in Appendix B. Having guaranteed the existence of a solution to the Cauchy problem (3) in terms of a basic output the theoretical results are exploited to develop efficient semi-numerical techniques to solve the trajectory planning problem.

4. SEMI-NUMERICAL REALIZATION

Two different algorithmic realizations are proposed to determine the control input h from (3) for given y. In the following, the equation is studied on $[0, 1] \times [0, T]$, where $T < \tau$ is determined by the considered steady state to steady state transition. We define an equidistant grid with spacings Δx, Δt such that $n_x \Delta x = 1$ and $n_t \Delta t = T$ for some integers n_x, n_t and $x_l := l \Delta x$, $t_j := j \Delta t$.

4.1. Discrete successive approximation.
The analytic results presented in the previous section suggest the implementation of a discrete analogue of the iteration sequence defined by

$$u_i^{[k+1]}(x, t) = u_{i,0}(t) = \int_0^x g_i(u_1^{[k]}(\xi, t), u_2^{[k]}(\xi, t), u_3^{[k]}(\xi, t)) d\xi \tag{12}$$

for $u_i^{[0]}(x, t) = u_{i,0}(t)$, $i = 1, 2, 3$, $k \geq 0$, and functions g_i formally defined according to the integrands in (8). Let $y(t_j)$ and $y'(t_j)$ denote the numerical evaluations of the basic output and its time derivative. The first iteration step is trivial and given by

$$u_1^{[0]}(x_l, t_j) = y(t_j), \; u_2^{[0]}(x_l, t_j) = 0, \; u_3^{[0]}(x_l, t_j) = y'(t_j).$$

To compute the numerical value $u_i^{[k+1]}(x_l, t_j)$ given $u_i^{[k]}(x_l, t_j)$ the arising time derivative of u_2 in $(0, T)$ is approximated by finite differences of second order accuracy. For $t = 0$ and $t = T$ derivatives are set to zero, which is justified since we only consider functions satisfying $y^{(n)}(0) = y^{(n)}(T) = 0$ for all $n \in \mathbb{N}_0$. It can easily be checked that these boundary conditions are propagated in the iteration. To abbreviate notation, formally define $\mathcal{D}$ for the approximation of the time derivative by

$$\mathcal{D}u_i^{[k]}(x_l, t_j) := \begin{cases} 0 & j = 0, n_t \\ \frac{u_i^{[k]}(x_l, t_{j+1}) - u_i^{[k]}(x_l, t_{j-1})}{2\Delta t} & j = 1, ..., n_t - 1. \end{cases} \tag{13}$$

For the evaluation of the integrals (12), approximated integrands are defined by $g_1^{[k]}(x_l, t_j) := u_2^{[k]}(x_l, t_j)$, $g_2^{[k]}(x_l, t_j) := u_3^{[k]}(x_l, t_j) + f(u_1^{[k]}(x_l, t_j), x_l)$, and $g_3^{[k]}(x_l, t_j) := \mathcal{D}u_2^{[k]}(x_l, t_j)$. In the following, $\mathcal{I}_{t_j}^{x_l}[\cdot]$ denotes an approximation of the spatial integral with lower bound $x_0 = 0$ and upper bound x_l for a fixed time t_j, which is obtained by standard quadrature formulas as described below. For $l = 1$ the trapezoidal rule yields

$$\mathcal{I}_{t_j}^{x_1}[g_i^{[k]}] := \frac{\Delta x}{2}(g_i^{[k]}(x_0, t_j) + g_i^{[k]}(x_1, t_j))$$

For l even we apply Simpson's rule

$$\mathcal{I}_{t_j}^{x_l}[g_i^{[k]}] := \frac{\Delta x}{3}\left(g_i^{[k]}(x_0, t_j) + 2\sum_{m=1}^{l/2-1} g_i^{[k]}(x_{2m}, t_j) + 4\sum_{m=1}^{l/2} g_i^{[k]}(x_{2m-1}, t_j) + g_i^{[k]}(x_l, t_j) \right). \tag{14}$$

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For l odd an additional trapezoidal step is necessary, i.e.

$$\mathcal{I}_{t_j}^{x_l}[g_i^{[k]}] := \frac{\Delta x}{3}\left(g_i^{[k]}(x_0, t_j) + 2\sum_{m=1}^{(l-1)/2-1} g_i^{[k]}(x_{2m}, t_j) + 4\sum_{l=1}^{(l-1)/2} g_i^{[k]}(x_{2m-1}, t_j) + g_i^{[k]}(x_{l-1}, t_j)\right)$$
$$+ \frac{\Delta x}{2}\left(g_i^{[k]}(x_{l-1}, t_j) + g_i^{[k]}(x_l, t_j))\right). \tag{15}$$

Thus, the iteration reduces to

$$u_i^{[k+1]}(x_l, t_j) := u_i^{[0]}(x_l, t_j) + \mathcal{I}_{t_j}^{x_l}[g_i^{[k]}].$$

The iteration is stopped once

$$\max_i \max_j \left|u_i^{[k+1]}(1, t_j) - u_i^{[k]}(1, t_j)\right| < \varepsilon \tag{16}$$

for user-defined $\varepsilon > 0$. With $k^* := k+1$ an approximation of the control input is obtained according to (4) by

$$h_{\mathrm{sa}}(t_j) = u_1^{[k^*]}(1, t_j), \quad j = 0, \ldots, n_t. \tag{17}$$

4.2. **Explicit integration.** The above presented algorithm is strongly connected to the analytic method of proof and hence is a promising candidate to obtain reliable results. However, one can ask for other methods to solve (3) numerically for a given basic output y. To this end we consider an integral formulation based on the second order form of the equation, which reads

$$u(x, t) = y(t) + \int_0^x K(x, \xi)(\partial_t u(\xi, t) + f(u(\xi, t), \xi))d\xi, \tag{18}$$

where $K(x, \xi) := x - \xi$. The numerical solution of nonlinear Volterra-type integral equations by means of quadrature methods is in general complicated by the fact that for each grid point x_l a system of nonlinear equations has to be solved. However, the integral kernel in (18) allows for an *explicit* scheme described in the following.

For this, assume that the time derivative vanishes at $t = 0$ and $t = T$, which is justified as we consider transitions between steady states such that (13) applies for the discretized time derivative. Set $u(x_0, t_j) = y(t_j)$ and apply the trapezoidal rule to (18), which yields

$$u(x_1, t_j) = y(t_j) + \tfrac{(\Delta x)^2}{2}\big(\mathcal{D}y(t_j) + f(y(t_j), x_0)\big).$$

Suppose that the values of $u(x_m, t_j)$ are available for $m = 0, \ldots, l - 1$ and $j = 0, \ldots, n_t$. Hence, introducing $g_1(x_l, x_m, t_j) := K(x_l, x_m)u(x_m, t_j)$ and $g_2(x_l, x_m, t_j) := K(x_l, x_m)f(u(x_m, t_j), x_m)$ the approximated solution to (18) follows as

$$u(x_l, t_j) = y(t_j) + \mathcal{D}\mathcal{I}_{t_j}^{x_l}[g_1(x_l, \cdot, \cdot)] + \mathcal{I}_{t_j}^{x_l}[g_2(x_l, \cdot, \cdot)]$$

with the discrete integral operator $\mathcal{I}_{t_j}^{x_l}[\cdot]$ defined above and $g_i(x_l, \cdot, \cdot)$ indicating that x_l is fixed. Here, the order of integration and differentiation was interchanged with the benefit that the time derivative has to be evaluated only once and does not have to be stored for further evaluations. Due to the fact that $K(x_l, x_l) = 0$ an explicit expression for $u(x_l, t_j)$ is obtained depending on the values of $u(x_m, t_j)$ for $m = 0, \ldots, l - 1$ and $j = 0, \ldots, n_t$. The control input hence follows from

$$h_{\mathrm{ei}}(t_j) = u(1, t_j), \quad j = 0, \ldots, n_t. \tag{19}$$

5. TRAJECTORY ASSIGNMENT AND EVALUATION

The semi-numerical methods proposed above directly enable an efficient realization of the theoretical results and hence admit the explicit computation of the feedforward control h^* given a desired basic output trajectory y^* to achieve finite time transitions between steady states.

5.1. **Trajectory assignment for the basic output.** For the appropriate explicit assignment of the basic output trajectory consider the following lemma.

Lemma 4. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined by $\psi(t) = \exp(-1/[t(1-t)])$ for $t \in (0,1)$ and $\psi(t) = 0$ for $t \neq (0,1)$. For $T > 0$ the function $\Psi_T : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Psi_T(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{\Psi_0} \int_0^{t/T} \psi(\tau) d\tau & t \in (0,T) \\ 1 & t \geq T \end{cases} \tag{20}$$

is of Gevrey class 2, where $\Psi_0 := \int_0^1 \psi(\tau) d\tau$. In particular

$$\sup_{t \in \mathbb{R}} |\Psi_T^{(n)}(t)| \leq M_\psi \frac{n!^2}{\gamma^n}, \quad \forall n \in \mathbb{N}_0$$

with $\gamma = T/3$ and $M_\psi = 1/(3e\Psi_0)$.

The proof is provided in Appendix B.1. Note that differing from [9], where implicit estimates for (20) are obtained depending on an abstract parameter, our results are explicit. Moreover, note that $\Psi_T(t)$ is locally non-analytic at $t \in \{0, T\}$. This property can be exploited to solve the trajectory planning problem. In view of (3), steady state profiles (2) can be equivalently defined in terms of y, i.e.

$$\partial_x^2 u_s(x) - f(u_s(x), x) = 0,$$
$$u_s(0) = y_s, \quad \partial_x u_s(0) = 0 \tag{21}$$

for constant y_s. With this, the following result applies.

Corollary 1. Let $(u_k)_{k=0,...,n}$ denote a sequence of steady states (21) to be attained at successive time instances $(T_{2k})_{k=0,...,n}$. We define

$$y^*(t) := y_0^* + \sum_{k=1}^n (y_k^* - y_{k-1}^*) \Psi_{T_{2k} - T_{2k-1}}(t - T_{2k-1}) \tag{22}$$

with $y_k^* := u_k(0)$ and $0 \leq T_0 \leq T_1 < T_2 \leq T_3 < T_4 \ldots < T_{2n}$. Lemma 4 implies that $y^* : \mathbb{R} \to \mathbb{R}$ is of Gevrey class 2 such that $\max_{t \geq 0} |y^{*(n)}(t)| \leq M_y n!^2/\gamma_y^n$ for $\gamma_y = \frac{1}{3} \min_{k=1,...,n} \{T_{2k} - T_{2k-1}\}$ and

$$M_y = n \max \left\{ \max_{k=0,...,n} |y_k^*|, \max_{k=1,...,n} M_\psi |y_k^* - y_{k-1}^*| \right\}.$$

Due to the local non-analyticity of $\Psi_T(t)$ it follows that derivatives $y^{*(j)}$, $j \geq 1$, vanish at $t = T_{2k}$ and $t = T_{2k+1}$ for $k = 0, ..., n$ with $u_s^k(0) = y^*(T_{2k+1}) = y^*(T_{2k})$, i.e., in view of (21) the steady state u_s^k is reached at $t = T_{2k}$ and is held for $t \in [T_{2k}, T_{2k+1}]$..

5.2. **Trigonometric nonlinearity – analytical results.** With these preliminaries, we consider the boundary control problem (1) for a trigonometric nonlinearity defined by

$$f(u) := \sin(u).$$

The system is assumed to be initially in rest, i.e., $u_0 = y_0^* = 0$. Let $u_1(x)$ and $u_2(x)$ denote two steady states governed by (21) with $y_1^* = 0.25$ and $y_2^* = 0.5$. Flatness-based trajectory planning is applied to determine the feedforward control to realize the smooth transition sequence $u_0 \to u_1 \to u_2$ within prescribed time intervals. We consider this example for

$$T_0 = 0.25, \ T_1 = 0.5, \ T_2 = 0.75, \ T_3 = 1.25 \tag{23}$$

and assign the desired basic output trajectory y^* according to (22). Corollary 1 yields $\gamma_y = 1/12$ and $M_y = 1/2 M_\psi$. We set $I := [0, T_3]$ in Definition 2 and choose the constants in the scale function as $\sigma_0 < \sigma_1 < \gamma_y$ such that

$$\|y\|_1 = \sum_{n=0}^{\infty} \frac{\sigma_1^n}{n!^2} \max_{t \in I} |y^{*(n)}(t)| \leq \frac{M_y}{1 - \sigma_1/\gamma_y} =: R_0.$$

Application of Lemma 1 shows $\|y'\|_1 < \infty$, which verifies Assumption (A1). Choosing $\sigma_1 := 1/60$, $\sigma_0 := \sigma_1/2$ yields $R_0 \approx 10.9$. It is left to show that the nonlinearity is admissible by analyzing the assumptions of Lemma 3. Since all derivatives of the sine function are bounded it is particularly simple to obtain (11). For $R := 12$, $\gamma_f := 15$ and $M_f := 100$ we obtain

$$\max_{x \in [-R, R]} |\sin^{(n)}(x)| \leq 1 \leq M_f \frac{n!^2}{\gamma_f^n}, \quad \forall n \in \mathbb{N}_0.$$

We conclude that Theorem 1 applies, which (setting $s = 0$) proves the existence of a local solution u of (3) for $x \in [0, r)$, $t \in I$, where u is a twice continuously differentiable function of x and $u(x, \cdot) \in \mathcal{G}_0$.

5.3. **Trigonometric nonlinearity – numerical results.** We perform numerical simulations for the domain $[0, 1] \times [0, 1.5]$ with $\Delta x = \Delta t = 0.01$ to compare the two algorithms presented in Section 4. At first, discrete successive approximation is considered as described in Section 4.1, where $\varepsilon = 10^{-4}$ is assigned in (16). For the considered problem the resulting $k^* = 28$ iterations were evaluated in a normalized CPU time of $t_{\text{cpu}}^{\text{sa}} = 1$. It is evident that the iterative discrete successive approximation is computationally much more expensive than the explicit integration proposed in Section 4.2. For the latter, the evaluation of the parametrization requires a normalized CPU time $t_{\text{cpu}}^{\text{ei}} = t_{\text{cpu}}^{\text{sa}}/14$ on the same machine.

The corresponding steady states u_k and the numerically determined feedforward controls $h_{\text{sa}}^*(t)$ and $h_{\text{ei}}^*(t)$ according to (17) and (19) for y replaced by y^* are shown in Figure 1 (top). Thereby, $\max_t |h_{\text{sa}}^* - h_{\text{ei}}^*(t)| < 10^{-4}$ is obtained, which confirms both the convergence and the accuracy of the schemes. The original problem (1) was solved numerically by making use of the MATLAB routine *pdepe* for $h = h_{\text{ei}}^*(t)$. The resulting spatial-temporal evolution is depicted in Figure 1 (bottom, left) in the (x, t)-plane. The profile confirms the precise realization of the desired finite time transition starting at the zero initial state u_0 to the final steady state u_2 via the intermediate state u_1. A further comparison of the obtained trajectory $u(0, t)$ and the desired path y^* is provided in Figure 1 (bottom, right), which illustrates the high tracking accuracy by means of the flatness-based feedforward control.

6. OUTLOOK

In this contribution, a novel flatness-based technique is proposed to solve trajectory planning problems for semilinear parabolic distributed-parameter systems with boundary control avoiding presently existing restrictions to polynomial nonlinearities. The approach is based on the interpretation of the governing equations as an (abstract) Cauchy problem in the spatial coordinate, which enables the introduction of a basic output to formally parametrize the system state and input. Existence of solutions is analyzed for Gevrey class functions in scales of Banach spaces by making use of a generalized Cauchy-Kowalevskaja Theorem and formal integration. These results moreover directly enable to develop efficient semi-numerical techniques to solve the trajectory planning problem. Simulation results confirm the theoretical results and illustrate both the applicability and the achievable high tracking performance. Future work includes explicit estimates on the interval of existence of solutions to the parametrized Cauchy problem and addresses the case of coupled systems of PDEs.
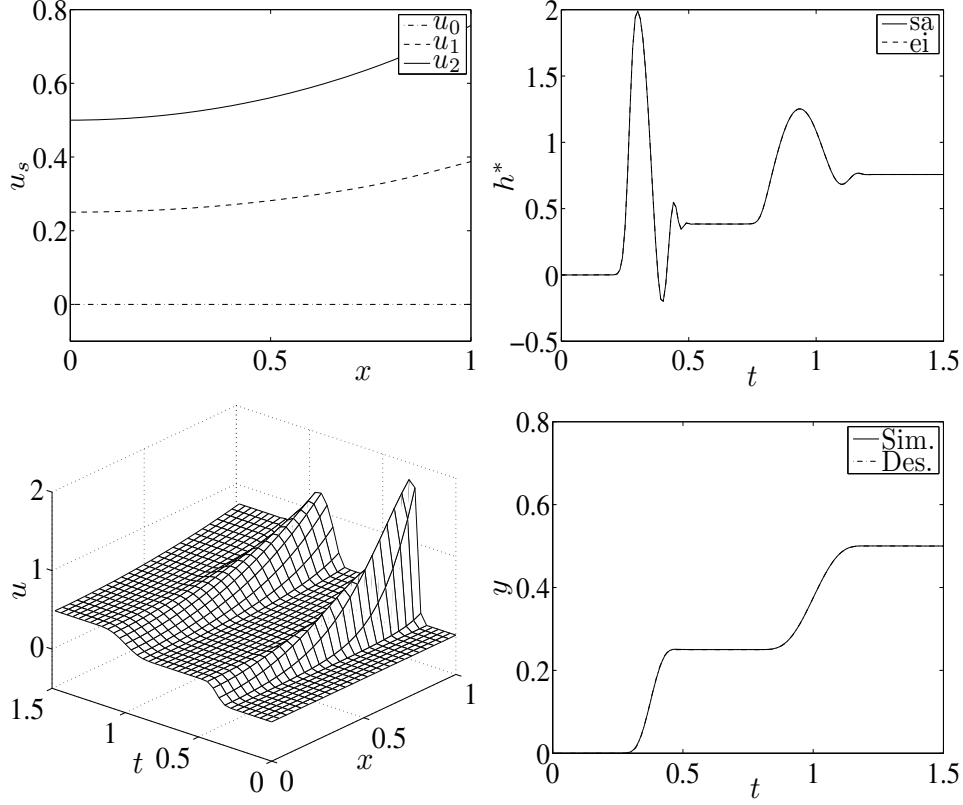
FIGURE 1. Numerical results for (1) with nonlinearity $f(u) = \sin(u)$ and desired basic output trajectory (22) with (23). Steady state profiles (top, left); comparison of feedforward controls (top, right); state evolution (bottom, left); comparison of obtained $y = u(0, t)$ and desired trajectory y^* (bottom, right).

APPENDIX A. PROOF OF LEMMA 1

Noting $\sup_{x \geq 0}\{(\sigma/\gamma)^x x^2\} = (2/e \ln(\gamma/\sigma))^2$ for $\sigma, \gamma \in \mathbb{R}^+$ with $\sigma < \gamma$, we obtain

$$\frac{\sigma^n}{n!^2}|u^{(n+1)}(t)| = \frac{1}{\sigma}\left(\frac{\sigma}{\gamma}\right)^{n+1}(n+1)^2\frac{\gamma^{n+1}}{(n+1)!^2}|u^{(n+1)}(t)|$$

$$\leq \frac{1}{\sigma}\left[\frac{2}{e(\ln\gamma - \ln\sigma)}\right]^2\frac{\gamma^{n+1}}{(n+1)!^2}|u^{(n+1)}(t)| \leq \frac{M}{\sigma}\left[\frac{2}{e(\ln\gamma - \ln\sigma)}\right]^2.$$

APPENDIX B. LEMMA 3: SKETCH OF PROOF

The first part of the proof is a one-dimensional version of a result in [10]. The assumption on f implies that

$$\sum_{j=0}^{\infty}\frac{R^j}{j!^2}\max_{x\in J}|f^{(j)}(x)| \leq \frac{M_f}{1 - (R/\gamma_f)} := C_0. \tag{24}$$

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where $J := [-R, R]$. To obtain an estimate for the n-th derivative of the composition $f \circ v$ the following version of the formula of Faà di Bruno, cf. [10], is used

$$(f \circ u)^{(n)} = \sum_{j=1}^{n} \frac{f^{(j)} \circ u}{j!} \sum_{\substack{k_1,\ldots,k_j \in \mathbb{N} \\ \sum k_i = n}} \frac{n!}{k_1! \cdots k_j!} \prod_{i=1}^{j} v^{(k_i)}$$

for $n \geq 1$. Furthermore, it can be easily verified by induction that $j! \leq n!/(k_1! \cdots k_j!)$ for $j \geq 1$ and $k_1, \ldots, k_j \in \mathbb{N}$ with $\sum k_i = n$. This implies $1/(n!j!k_1! \cdots k_j!) \leq 1/(j!k_1! \cdots k_j!)^2$. For $n \geq 1$ it hence follows that

$$\frac{\sigma^n(s)}{n!^2} \max_{t \in I} |(f \circ u)^{(n)}(t)| \leq \sum_{j=1}^{n} \frac{\max_{t \in I} |(f^{(j)} \circ u)(t)|}{j!^2} \sum_{\substack{k_1,\ldots,k_j \in \mathbb{N} \\ \sum k_i = n}} \prod_{i=1}^{j} \frac{\sigma^{k_i}(s)}{k_i!^2} \max_{t \in I} |u^{(k_i)}(t)|.$$

Taking the sum over n yields

$$\sum_{n=1}^{\infty} \frac{\sigma^n(s)}{n!^2} \max_{t \in I} |(f \circ u)^{(n)}(t)| \leq \sum_{j=1}^{\infty} \frac{\max_{t \in I} |(f^{(j)} \circ u)(t)|}{j!^2} \sum_{\substack{k_1,\ldots,k_j \in \mathbb{N} \\ \sum k_i = n}} \prod_{i=1}^{j} \frac{\sigma^{k_i}(s)}{k_i!^2} \max_{t \in I} |u^{(k_i)}(t)|$$

$$= \sum_{j=1}^{\infty} \frac{1}{j!^2} \max_{t \in I} |(f^{(j)} \circ u)(t)| \left(\sum_{k=1}^{\infty} \frac{\sigma^k(s)}{k!^2} \max_{t \in I} |u^{(k)}(t)| \right)^j$$

$$\leq \sum_{j=1}^{\infty} \frac{R^j}{j!^2} \max_{t \in I} |(f^{(j)} \circ u)(t)| \leq \sum_{j=1}^{\infty} \frac{R^j}{j!^2} \max_{x \in J} |f^{(j)}(x)| < \infty.$$

Since $\max_{t \in I} |(f \circ u)(t)| \leq \max_{x \in J} |f(x)|$ we conclude that

$$\|F(v)\|_s = \sum_{n=0}^{\infty} \frac{\sigma^n(s)}{n!^2} \max_{t \in I} |(f \circ u)^{(n)}(t)| \leq C_0.$$

Using similar arguments it can be shown that $F : \mathcal{B}_s(R) \subset \mathcal{G}_s \to \mathcal{G}_s$ is differentiable at $u \in \mathcal{B}_s(R)$ in the sense of Fréchet with derivative $F^{(1)}(u)$, where

$$[F^{(1)}(u)v](t) := (f' \circ u)(t) \cdot v(t).$$

An application of the mean value theorem for the Fréchet derivative yields the existence of a constant C_F such that

$$\|F(u) - F(v)\|_s \leq C_F \|u - v\|_s.$$

For further details, the reader is referred to [17].

B.1. **Proof of Lemma 4.** We study the properties of ψ and restrict ourselves to $t \in (0, 1/2]$ for symmetry reasons. The function ψ is real analytic on $(0, 1)$ and can thus be analytically extended to a complex function in a small neighbourhood of t for every $t \in (0, 1/2]$. For $n \in \mathbb{N}_0$ Cauchy's integral formula is applied to obtain

$$\psi^{(n)}(t) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{\psi(z)}{(z-t)^{n+1}} dz,$$

where we set $\Gamma := \{z \in \mathbb{C} : z = t + t/2 \exp(i\varphi), t \in (0, \frac{1}{2}], \varphi \in [0, 2\pi)\}$. Note that $\mathrm{Re}(1/z(1-z)) = \mathrm{Re}(1/z) + \mathrm{Re}(1/1-z)$ and for $z \in \Gamma$ the individual terms can be estimated by $\mathrm{Re}(1/1-z) \geq 1$,

$\text{Re}(1/z) \geq 2/(3t)$ such that $|\psi(z(t,\varphi)| \leq (1/e)\exp(-2/(3t))$. This estimate and the change of variables from z to φ in the above integral yields

$$|\psi^{(n)}(t)| \leq \frac{n!}{e}\left(\frac{2}{t}\right)^n e^{-2/3t} \leq \frac{n!}{e}\left(\frac{3n}{e}\right)^n \leq \frac{n!^2 3^n}{e},$$

where we use the fact that $x^a e^{-bx} \leq (a/eb)^a$ for $a \geq 0$ and $b > 0$, as well as the estimate $n^n \leq n!e^n$. For $n \geq 1$ this implies

$$|\Psi_T^{(n)}(t)| = \frac{|\psi^{(n-1)}(t/T)|}{\Psi_0 T^n} \leq \frac{n!^2}{3e\Psi_0}\left(\frac{3}{T}\right)^n.$$

Note that $1/(3e\Psi_0) > 1$ and since $|\Psi_T(t)| \leq 1$ we conclude that the above estimate holds for all $n \geq 0$.

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