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SEMICLASSICAL LIMIT IN A SIMPLIFIED QUANTUM ENERGY-TRANSPORT MODEL FOR SEMICONDUCTORS

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Abstract. The semiclassical limit in a quantum energy-transport model for semiconductors is proved. The system consists of a nonlinear parabolic fourth-order equation for the electron density, including temperature gradients; a degenerate elliptic heat equation for the electron temperature; and the Poisson equation for the electric potential. The equations are solved in a bounded domain with periodic boundary conditions. The asymptotic limit is based on a priori estimates independent of the scaled Planck constant, obtained from entropy functionals, on the use of Gagliardo-Nirenberg inequalities, and weak compactness methods.

1. **Introduction.** Quantum fluid equations may be employed to model and simulate quantum diffusive effects in nanoscale semiconductor devices [4, 18]. These models can be derived by applying a moment method to the relaxation-time Wigner equation and by performing a Chapman-Enskog expansion around the quantum equilibrium [11]. They are alternatives to dissipative Schrödinger equations [12, 22] whose numerical solution is generally very time-consuming. The simplest quantum

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fluid model are the quantum drift-diffusion or density-gradient equations, which are popular in engineering applications since they are capable of describing quantum confinement and tunneling effects and they can be solved numerically in an efficient way [1, 2]. For mathematical results, we refer to, e.g., [8, 5, 6, 14, 19] and references therein.

Quantum drift-diffusion models do not take into account heating phenomena, which may be important even in quantum devices. Temperature effects can be included by computing more moments of the Wigner equation, e.g. the energy density, which leads to quantum energy-transport equations. Grubin and Kreskovsky seem to be the first who have proposed a quantum energy-balance system [15]. Later, a nonlocal quantum energy-transport model has been derived by Degond et al. from a Wigner equation [9]. In the $O(\hbar^4)$ approximation (where $\hbar$ is the reduced Planck constant), the equations become local, but their mathematical structure is still unclear. Another quantum energy-transport model has been studied by Chen and Liu [7]. Their model consists of a quantum drift-diffusion-type equation for the particle density, coupled to an energy equation.

In [20], a simplified quantum energy-transport model has been formally derived in the large-time and small-velocity limit from the quantum hydrodynamic equations [17], and the existence of global-in-time weak solutions has been proved. Compared to previous models, the proposed system contains temperature gradients in the continuity equation for the particle density, which do not allow for the use of standard tools developed for the classical drift-diffusion equations [23]. In this paper, we continue the analysis initiated in [20] by performing the semiclassical limit. This limit has been performed in the quantum drift-diffusion equations [5, 6] but it is open in the simplified quantum energy-transport model.

More precisely, we consider the following scaled equations for the electron density n, electron temperature θ, and electric potential V,

$$n_t + \operatorname{div}\left(\frac{\varepsilon^2}{6} n \nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big) - \nabla(n\theta) + n\nabla V\right) = 0, \tag{1}$$

$$-\operatorname{div}(n\nabla\theta) = \frac{n}{\tau_e}(\theta_L(x) - \theta), \tag{2}$$

$$\lambda^2 \Delta V = n - C(x) \quad \text{in } \mathbb{T}^d,\ t>0, \qquad \int_{\mathbb{T}^d} V\,dx = 0, \tag{3}$$

with the initial conditions

$$n(\cdot,0) = n_0 \quad \text{in } \mathbb{T}^d, \tag{4}$$

where $\mathbb{T}^d \subset \mathbb{R}^d$ is the d-dimensional torus. In the above equations, $\varepsilon > 0$ is the scaled Planck constant, $\tau_e > 0$ is the energy relaxation time, and $\lambda > 0$ is the scaled Debye length. The given functions $0 < m_L \le \theta_L(x) \le M_L$ and $C(x)$ model the space-dependent lattice temperature and the semiconductor doping profile, respectively.

Equation (1) contains the diffusive part $-\theta\nabla n$, the drift part $n\nabla(V-\theta)$, and the quantum correction involving the Bohm potential $\Delta\sqrt{n}/\sqrt{n}$. Equation (2) is derived from the energy balance equation of the quantum hydrodynamic model [20]. It contains the heat conductivity $\kappa = n$. In applications, κ usually depends on the temperature, for instance, $\kappa = n\theta$. In [20], the simplification $\kappa = n$ has been proposed since the expression $\kappa\nabla\theta = n\theta\nabla\theta$ cannot be easily handled in the analysis, due to the quadratic structure in θ. We remark that the case $\kappa = n\theta_L$ has been considered in [7]. When the temperature is constant, $\theta = 1$, we recover the quantum drift-diffusion equations. If, additionally, the quantum term vanishes, $\varepsilon = 0$, we obtain the semiclassical drift-diffusion equations.

We consider periodic boundary conditions in order to avoid technical problems with boundary integrals occuring in the derivation of the a priori estimates. Compared to the quantum drift-diffusion equations, the quantum energy-transport system is of degenerate type due to the term $\mathrm{div}(n\nabla\theta)$ in (2). Together with the nonlinear fourth-order quantum term, their treatment is the main difficulty in the analysis. The simplified model (1)-(4) serves as a first step to understand analytically the interplay between diffusive effects, induced by the electron temperature, and quantum phenomena, modeled by the Bohm potential.

Formally, in the limit $\varepsilon \to 0$, system (1)-(3) reduces to the energy-transport model

$$n_t = \mathrm{div}(\nabla(n\theta) - n\nabla V), \tag{5}$$

$$-\,\mathrm{div}(n\nabla\theta) = \frac{n}{\tau_e}(\theta_L(x) - \theta), \tag{6}$$

$$\lambda^2\Delta V = n - C(x) \quad \text{in } \mathbb{T}^d,\ t > 0, \qquad \int_{\mathbb{T}^d} V\,dx = 0, \tag{7}$$

Energy-transport models have been derived from the semiconductor Boltzmann equation by Ben Abdallah and Degond [3], and they are analytically studied in, for instance, [10]. The above system is a simplified version of the class of energy-transport models derived in [3] since only the Fourier term contributes to the heat flux. In this paper, we make the limit $\varepsilon \to 0$ rigorous and prove, as a by-product, the existence of global weak solutions to (5)-(7). Before we explain and state our main results, we recall the existence result of [20] in order to make precise the regularity of the solutions.

Theorem 1.1 (Theorem 1 in [20])**.** *Let $d \le 3$, ε, λ, $\tau_e > 0$, C, $\theta_L \in L^\infty(\mathbb{T}^d)$ such that $0 < m_L \le \theta_L(x) \le M_L$ for $x \in \mathbb{T}^d$. Let the initial datum $n_0 \in L^1(\mathbb{T}^d)$ satisfy $n_0 \ge 0$ in $\mathbb{T}^d$, $\int_{\mathbb{T}^d} n_0 \log n_0 dx < \infty$, and $\int_{\mathbb{T}^d}(n_0 - C)dx = 0$. Then there exists a global weak solution $(n_\varepsilon, \theta_\varepsilon, V_\varepsilon)$ satisfying $n_\varepsilon \ge 0$ and $0 < m_L \le \theta \le M_L$ in $\mathbb{T}^d \times (0,\infty)$ and*

$$\sqrt{n_\varepsilon} \in L^2_{\mathrm{loc}}(0,\infty; H^2(\mathbb{T}^d)) \cap L^\infty_{\mathrm{loc}}(0,\infty; L^2(\mathbb{T}^d)), \quad n_\varepsilon \in W^{1,11/10}_{\mathrm{loc}}(0,\infty; H^{-2}(\mathbb{T}^d)),$$
$$\sqrt{n_\varepsilon}\theta_\varepsilon \in L^2_{\mathrm{loc}}(0,\infty; H^1(\mathbb{T}^d)), \quad n_\varepsilon\theta_\varepsilon \in L^{8/7}_{\mathrm{loc}}(0,\infty; W^{1,4/3}(\mathbb{T}^d)),$$
$$V_\varepsilon \in L^2_{\mathrm{loc}}(0,\infty; H^2(\mathbb{T}^d)).$$

The solution satisfies the equations

$$\partial_t n_\varepsilon + \frac{\varepsilon^2}{6}\nabla^2 : \left(\sqrt{n_\varepsilon}\nabla^2\sqrt{n_\varepsilon} - \nabla\sqrt{n_\varepsilon} \otimes \nabla\sqrt{n_\varepsilon}\right) = \mathrm{div}(\nabla(n_\varepsilon\theta_\varepsilon) - n_\varepsilon\nabla V_\varepsilon), \tag{8}$$

$$-\,\mathrm{div}\left(\sqrt{n_\varepsilon}\nabla(\sqrt{n_\varepsilon}\theta_\varepsilon) - \sqrt{n_\varepsilon}\theta_\varepsilon\nabla\sqrt{n_\varepsilon}\right) = \frac{n_\varepsilon}{2\tau_e}(\theta_L(x) - \theta_\varepsilon), \tag{9}$$

$$\lambda^2\Delta V_\varepsilon = n_\varepsilon - C(x) \quad \text{in } \mathbb{T}^d \tag{10}$$

in the sense of distributions.

Here, "∇^2" denotes the Hessian, the double points ":" signify summation over both matrix indices, and $a \otimes b$ is the matrix with components $a_i b_j$. Notice that the lack of regularity makes it necessary to write (1) and (2) in the form (8) and (9), respectively.

Our main tools to prove the asymptotic limit $\varepsilon \to 0$ are entropy estimates independent of ε and Gagliardo-Nirenberg inequalities. Indeed, introduce the logarithmic entropy

$$E_1(n) = \int_{\mathbb{T}^d} \phi_1(n)dx = \int_{\mathbb{T}^d} \big(n(\log n - 1) + 1\big)dx.$$

A formal computation, which will be made rigorous in the proof of Proposition 1, shows that

$$\begin{aligned} &\frac{dE_1}{dt} + \frac{\varepsilon^2}{12}\int_{\mathbb{T}^d} n|\nabla^2 \log n|^2 dx + 4\int_{\mathbb{T}^d} \theta|\nabla\sqrt{n}|^2 dx \\ &\quad = -2\int_{\mathbb{T}^d} \sqrt{n}\nabla\sqrt{n}\cdot\nabla\theta dx - \frac{1}{\lambda^2}\int_{\mathbb{T}^d}(n - C(x))ndx \\ &\quad \le \int_{\mathbb{T}^d} \theta|\nabla\sqrt{n}|^2 dx + \int_{\mathbb{T}^d} \frac{n}{\theta}|\nabla\theta|^2 dx + \frac{1}{4\lambda^2}\int_{\mathbb{T}^d} C(x)^2 dx, \end{aligned} \tag{11}$$

using Young's inequality. The first integral on the right-hand side can be absorbed by the last integral on the left-hand side. By the maximum principle, θ is bounded from below, and the second integral on the right-hand side can be estimated from above by

$$\frac{1}{m_L}\int_{\mathbb{T}^d} n|\nabla\theta|^2 dx,$$

where $m_L = \min_{\mathbb{T}^d}\theta > 0$ is independent of ε. In order to estimate this integral, we take θ as a test function in the weak formulation of (2):

$$\int_{\mathbb{T}^d} n|\nabla\theta|^2 dx = \frac{1}{\tau_e}\int_{\mathbb{T}^d} n(\theta_L(x) - \theta)\theta dx \le -\frac{1}{2\tau_e}\int_{\mathbb{T}^d} n\theta^2 dx + \frac{\|\theta_L\|_{L^\infty(\mathbb{T}^d)}}{2\tau_e}\int_{\mathbb{T}^d} ndx.$$

The last integral is bounded since the total mass $\int_{\mathbb{T}^d} ndx$ is constant in time. Putting the above estimates together shows that

$$\frac{dE_1}{dt} + \frac{\varepsilon^2}{12}\int_{\mathbb{T}^d} n|\nabla^2\log n|^2 dx + 4m_L\int_{\mathbb{T}^d} |\nabla\sqrt{n}|^2 dx \le K_1,$$

where $K_1 > 0$ depends on n_0, θ_L etc. but not on ε. This provides an ε-uniform H^1 bound for $\sqrt{n}$ which is the starting point for further estimates derived from Gagliardo-Nirenberg inequalities. Since no gradient bounds for θ are available, equation (2) has to be interpreted in the sense of (9).

Our first main result is as follows.

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold. Let $(n_\varepsilon, \theta_\varepsilon, V_\varepsilon)$ be a weak solution to (4), (8)-(10), guaranteed by Theorem 1.1. Then there exists a subsequence (not relabeled) such that, for any $T > 0$ and $p < 3/2$,*

$$n_\varepsilon \to n \quad \text{strongly in } L^2(0,T;L^p(\mathbb{T}^d)), \tag{12}$$

$$\sqrt{n_\varepsilon} \rightharpoonup \sqrt{n} \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T}^d)), \tag{13}$$

$$\partial_t n_\varepsilon \rightharpoonup n_t \quad \text{weakly in } L^{8/7}(0,T;H^{-3}(\mathbb{T}^d)), \tag{14}$$

$$\theta_\varepsilon \rightharpoonup^* \theta \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^\infty(\mathbb{T}^d)), \tag{15}$$

$$V_\varepsilon \rightharpoonup V \quad \text{weakly in } L^2(0,T;H^2(\mathbb{T}^d)). \tag{16}$$

The limit (n,θ,V) solves the energy-transport model (4)-(7). More precisely, it possesses the regularity properties $n \in W^{1,8/7}(0,T;H^{-3}(\mathbb{T}^d))$, $\sqrt{n} \in L^2(0,T;H^1(\mathbb{T}^d))$,

$\theta \in L^\infty(0,T;\, L^\infty(\mathbb{T}^d))$, $\sqrt{n}\theta \in L^2(0,T;H^1(\mathbb{T}^d))$, $n\theta \in L^{8/7}(0,T;W^{1,4/3}(\mathbb{T}^d))$, and $V \in L^2(0,T;H^2(\mathbb{T}^d))$, and solves (5)-(7) in the following weak sense:

$$\int_0^T \langle \partial_t n, \phi\rangle_{H^{-3},H^3} dt = -\int_0^T \int_{\mathbb{T}^d} \big(\nabla(n\theta) - n\nabla V\big)\cdot\nabla\phi dx\, dt \tag{17}$$

$$\int_0^T \int_{\mathbb{T}^d} \big(\sqrt{n}\nabla(\sqrt{n}\theta) - \sqrt{n}\theta\nabla\sqrt{n}\big)\cdot\nabla\phi dx\, dt = \int_0^T \int_{\mathbb{T}^d} \frac{n}{2\tau_e}(\theta_L(x)-\theta)\phi dx\, dt, \tag{18}$$

$$\lambda^2 \int_0^T \int_{\mathbb{T}^d} \nabla V\cdot\nabla\phi dx\, dt = \int_0^T \int_{\mathbb{T}^d} (n - C(x))\phi dx\, dt \tag{19}$$

for all $\phi \in L^\infty(0,T;H^3(\mathbb{T}^d))$. The initial condition (4) is satisfied in the sense of $H^{-3}(\mathbb{T}^d)$.

We expect that the solutions of the energy-transport model (5)-(7) are smooth if $C(x)$ and $\theta_L(x)$ are smooth. Indeed, by Stampacchia truncations and the maximum principle, strict positivity of n and θ is expected. Then, by elliptic and parabolic regularity, smoothness of n, θ, and V follows. Therefore, one may expect that the limit $\varepsilon \to 0$ preserves the H^2 regularity of n_ε. Due to the highly nonlinear structure of the quantum term in (1), it is, however, not clear how to prove this. The reason is that both models require different test functions to derive H^2-type a priori estimates, namely an entropy estimate for the quantum model and classical regularity theory and maximum principle arguments for the classical model.

In one space dimension, we can improve this result since the second entropy functional

$$E_0(n) = \int_{\mathbb{T}^d} \phi_0(n)dx = \int_{\mathbb{T}^d} (n - \log n)dx$$

provides additional estimates. Indeed, after a formal calculation (see the proof of Proposition 2 for details):

$$\begin{aligned}\frac{dE_0}{dt} &+ \frac{\varepsilon^2}{12}\int_{\mathbb{T}} (\log n)_{xx}^2 dx + \int_{\mathbb{T}} \theta(\log n)_x^2 dx \\ &= -\int_{\mathbb{T}} \theta_x(\log n)_x dx - \frac{1}{\lambda^2}\int_{\mathbb{T}} n\log n dx + \int_{\mathbb{T}^d} C(x)\log n dx.\end{aligned} \tag{20}$$

The third integral on the right-hand side is estimated by $E_0(n)$; the second integral is bounded since $-x\log x \le 1/e$ for $x > 0$; and the first integral can be treated similarly as above by employing the temperature equation, which is tested with $-1/n$:

$$-\int_{\mathbb{T}} \theta_x(\log n)_x dx = -\int_{\mathbb{T}} n\theta_x\Big(-\frac{1}{n}\Big)_x dx = \frac{1}{\tau_e}\int_{\mathbb{T}} (\theta - \theta_L)dx \le K_2$$

for some constant $K_2 > 0$ which is independent of ε. Hence, by the Gronwall lemma, we derive a uniform H^1 bound for $\log n$. Using this information, the test function θ/n in the temperature equation provides an H^1 bound for θ which helps to define the product $n\theta_x$ in (2). As a consequence, in the one-dimensional case, the heat equation can be written in the usual way (2). Hence, we obtain the following theorem.

Theorem 1.3. *Let the assumptions of Theorem 1.1 hold and let $d = 1$. Let $(n_\varepsilon, \theta_\varepsilon, V_\varepsilon)$ be a weak solution to (4), (8)-(10), guaranteed by Theorem 1.1. Then* *there exists a subsequence (not relabeled) such that* (12)-(16) *hold for any* $T > 0$ *and* $p < \infty$ *and, moreover,*

$$\theta_\varepsilon \rightharpoonup \theta \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T})) \quad \text{as } \varepsilon \to 0.$$

The limit (n, θ, V) *solves the energy-transport model* (4)-(7) *in the usual weak sense.*

The proof of this theorem shows that the same result holds when we replace the periodic boundary conditions by

$$n = 1, \quad n_x = 0 \quad \text{on } \partial\Omega, \ t > 0,$$

where $\Omega \subset \mathbb{R}$ is an interval (see [21]). These boundary conditions have been used in numerical simulations of tunneling diodes. Also non-homogeneous boundary conditions for n_x can be allowed in the analysis, see [16].

The paper is organized as follows. In Section 2, we prove Theorem 1.2. Compared to the results in [20], we need an additional estimate on n_ε, which can be only obtained through the approximation procedure proposed in [20]. Therefore, we sketch the approximate problem and derive the needed a priori estimates. Section 3 is devoted to the proof of Theorem 1.3. Here, the approximative problem simplifies which allows us to derive the gradient bound on θ_ε.

2. **Proof of Theorem 1.2.** For the semiclassical limit $\varepsilon \to 0$, we need a priori estimates which are independent of ε. These estimates are derived from the entropy inequality (11). In order to make the calculations rigorous, we have to use the approximation procedure proposed in [20].

Proposition 1. *Let the assumptions of Theorem 1.1 hold. Then there exists a weak solution* $(n_\varepsilon, \theta_\varepsilon, V_\varepsilon)$ *to* (8)-(10) *and* (4) *satisfying the following bounds for all* $T > 0$*:*

$$\|n_\varepsilon \log n_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} + \|\sqrt{n_\varepsilon}\|_{L^2(0,T;H^1(\mathbb{T}^d))} + \|n_\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le K, \tag{21}$$

$$\varepsilon\|\sqrt{n_\varepsilon}\|_{L^2(0,T;H^2(\mathbb{T}^d))} \le K, \tag{22}$$

$$\|\theta_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} + \|\sqrt{n_\varepsilon}\theta_\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}^d))} \le K, \tag{23}$$

where the constant $K > 0$ *is independent of* ε.

Compared to [20], the uniform bound for (n_ε) in $L^2(0,T;L^2(\mathbb{T}^d))$ is new here. It cannot be derived from the gradient bounds for (n_ε) which yield only an estimate in $L^2(0,T;L^{3/2}(\mathbb{T}^d))$ (from (32) below).

Proof. By Lemma 2 of [20], there exists a weak solution $(\rho_k, \theta_k, V_k) \in H^2(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d) \times H^2(\mathbb{T}^d)$, satisfying $\rho_k \ge \kappa_\tau > 0$ and $0 < m_L \le \theta_\tau \le M_L$ in $\mathbb{T}^d$ for some constant $\kappa_\tau > 0$ to the approximate problem in $\mathbb{T}^d$

$$\begin{aligned}
&\frac{1}{\tau}\big(\rho_\tau^2 - \sigma_\tau(\rho_\tau^2)\big) + \frac{\varepsilon^2}{12}\nabla^2 : (\rho_\tau^2 \nabla^2 \log \rho_\tau^2) + \delta(\Delta^2 \log \rho_\tau^2 + \log \rho_\tau^2) \\
&\quad = \operatorname{div}\big(\nabla(\rho_\tau^2\theta_\tau) - \rho_\tau^2 \nabla V_\tau\big),
\end{aligned} \tag{24}$$

$$-\operatorname{div}(\rho_\tau^2 \nabla\theta_\tau) = \frac{\rho_\tau^2}{\tau_e}(\theta_L(x) - \theta_\tau), \ \lambda^2 \Delta V_\tau = \sigma_\tau(\rho_\tau^2) - C(x), \ \int_{\mathbb{T}^d} V_\tau dx = 0, \tag{25}$$

where $\rho_\tau(x,t) = \rho_k(x)$, $\theta_\tau(x,t) = \theta_k(x)$, and $V_\tau(x,t) = V_k(x)$ for $x \in \mathbb{T}^d$ and $t \in ((k-1)\tau, k\tau]$, $k \in \mathbb{N}$, are piecewise constant functions in time, approximating $\sqrt{n(x,t)}$, $\theta(x,t)$, and $V(x,t)$ at $t = k\tau$, respectively. Furthermore, $(\sigma_\tau(\rho_\tau^2))(\cdot,t) = \rho_\tau^2(\cdot, t-\tau)$ for $\tau \le t \le T$ is a shift operator. This approximate problem is inspired from [19, 21]. The solution satisfies the discrete entropy estimate (see (26) and the preceding estimates in [20])

$$\frac{1}{\tau}\int_{\mathbb{T}^d}\big(\phi_1(\rho_\tau^2)-\phi_1(\sigma_\tau(\rho_\tau^2))\big)dx+\frac{\varepsilon^2}{12}\int_{\mathbb{T}^d}\rho_\tau^2|\nabla^2\log\rho_\tau^2|^2dx+\delta\int_{\mathbb{T}^d}(\Delta\log\rho_\tau^2)^2dx$$
$$+\frac{\delta}{2}\int_{\mathbb{T}^d}(\log\rho_\tau^2)^2dx+2m_L\int_{\mathbb{T}^d}|\nabla\rho_\tau|^2dx+\lambda^{-2}\int_{\mathbb{T}^d}\rho_\tau^2\sigma_\tau(\rho_\tau^2)dx\le K,$$

where $\phi_1(s)=s(\log s-1)+1$ and $K>0$ is here and in the following a generic constant independent of τ, δ, and ε. Lemma 2.2 in [19] shows that there exists a constant $K_0>0$ only depending on the space dimension d such that

$$K_0\int_{\mathbb{T}^d}(\Delta\rho_\tau)^2dx\le\int_{\mathbb{T}^d}\rho_\tau^2|\nabla^2\log\rho_\tau^2|^2dx,$$

which, together with the above entropy estimate, provides a uniform H^2-bound for $\varepsilon\rho_\tau$. Furthermore, the following bound holds [20, p. 1039]

$$\int_0^T\int_{\mathbb{T}^d}\rho_\tau^2|\nabla\theta_\tau|^2dx\,ds\le K(T). \tag{26}$$

In view of the uniform L^∞-bound for θ_τ, this and the above entropy estimate imply that

$$\int_0^T\int_{\mathbb{T}^d}|\nabla(\rho_\tau\theta_\tau)|^2dx\,ds\le 2\int_0^T\int_{\mathbb{T}^d}\big(\rho_\tau^2|\nabla\theta_\tau|^2+\theta_\tau^2|\nabla\rho_\tau|^2\big)dx\,ds\le K(T). \tag{27}$$

Hence, $(\rho_\tau\theta_\tau)$ is bounded in $L^2(0,T;H^1(\mathbb{T}^d))$. By the Gagliardo-Nirenberg inequality, with $\alpha=d/(4+d)$,

$$\|\rho_\tau\|_{L^{8/d+2}(0,T;L^{8/d+2}(\mathbb{T}^d))}^{8/d+2}\le K\int_0^T\|\rho_\tau\|_{H^2(\mathbb{T}^d)}^{(8+2d)\alpha/d}\|\rho_\tau\|_{L^2(\mathbb{T}^d)}^{(8+2d)(1-\alpha)/d}ds$$
$$\le K\|\rho_\tau\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}^{(8+2d)(1-\alpha)/d}\int_0^T\|\rho_\tau\|_{H^2(\mathbb{T}^d)}^2dx\,ds\le K\varepsilon^{-2}.$$

Thus, (ρ_τ^2) is bounded in $L^{4/d+1}(0,T;L^{4/d+1}(\mathbb{T}^d))$ uniformly in τ and δ (but not in ε), and the same holds for the time-shifted sequence $(\sigma_\tau(\rho_\tau^2))$.

It is proved in [20] that, as $(\tau,\delta)\to0$, a subsequence of $(\rho_\tau^2,\theta_\tau,V_\tau)$, which is not relabeled, converges to a weak solution (n,θ,V), which still depends on ε, to (8)-(10) in the following sense:

$$\rho_\tau\to\sqrt{n}\quad\text{strongly in } L^2(0,T;W^{1,4}(\mathbb{T}^d)), \tag{28}$$
$$\rho_\tau^2-\sigma_\tau(\rho_\tau^2)\to0\quad\text{strongly in } L^{11/10}(0,T;H^{-2}(\mathbb{T}^d)), \tag{29}$$
$$\nabla^2\rho_\tau\rightharpoonup\nabla^2\sqrt{n}\quad\text{weakly in } L^2(0,T;L^2(\mathbb{T}^d)),$$
$$\theta_\tau\rightharpoonup^*\theta\quad\text{weakly}^*\text{ in } L^\infty(0,T;L^\infty(\mathbb{T}^d)). \tag{30}$$

The above bounds on (ρ_τ^2) and $(\sigma_\tau(\rho_\tau^2))$ show that, up to subsequences,

$$\rho_\tau^2\rightharpoonup n,\quad \sigma_\tau(\rho_\tau^2)\rightharpoonup z\quad\text{weakly in } L^{4/d+1}(0,T;L^{4/d+1}(\mathbb{T}^d)).$$

In view of (29), we can identify $z=n$, and because of (28), $\rho_\tau^2\to n$ a.e. in $\mathbb{T}^d$, $t>0$. This and the above weak convergence of ρ_τ^2 to n imply that, since $4/d+1>2$ for $d\le3$,

$$\rho_\tau^2\to n\quad\text{strongly in } L^2(0,T;L^2(\mathbb{T}^d)).$$

Hence,

$$\rho_\tau^2\sigma_\tau(\rho_\tau^2)\rightharpoonup n^2\quad\text{weakly in } L^1(0,T;L^1(\mathbb{T}^d)).$$

By (30), we infer that $\rho_\tau\theta_\tau \rightharpoonup \sqrt{n}\theta$ weakly in $L^2(0,T;L^2(\mathbb{T}^d))$. Then (27) implies that, up to a subsequence,

$$\rho_\tau\theta_\tau \rightharpoonup \sqrt{n}\theta \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T}^d)).$$

By the weakly lower semicontinuity of the convex function $\phi_1(s) = s(\log s - 1) + 1$ and of the Sobolev norms, we find that

$$\begin{aligned}
&\int_{\mathbb{T}^d} \phi_1(n(\cdot,t))dx + K_0\frac{\varepsilon^2}{12}\int_0^T\int_{\mathbb{T}^d}(\Delta\sqrt{n})^2 dx\,ds + 2m_L\int_0^T\int_{\mathbb{T}^d}|\nabla\sqrt{n}|^2 dx\,ds \\
&\qquad + \lambda^{-2}\int_0^T\int_{\mathbb{T}^d} n^2 dx\,ds \\
&\quad \le \liminf_{(\tau,\delta)\to 0}\Big(\int_{\mathbb{T}^d}\phi_1(\rho_\tau^2)dx + K_0\frac{\varepsilon^2}{12}\int_0^T\int_{\mathbb{T}^d}(\Delta\rho_\tau)^2 dx\,ds \\
&\qquad + 2m_L\int_0^T\int_{\mathbb{T}^d}|\nabla\rho_\tau|^2 dx\,ds + \lambda^{-2}\int_0^T\int_{\mathbb{T}^d}\rho_\tau^2\sigma_\tau(\rho_\tau^2)dx\,ds\Big) \\
&\quad \le K.
\end{aligned}$$

Furthermore, the bounds (23) hold. This proves the proposition. □

From the estimates of Proposition 1 we derive more bounds.

Lemma 2.1. *The following uniform estimates hold for all* $T > 0$*:*

$$\|n_\varepsilon\theta_\varepsilon\|_{L^{8/7}(0,T;W^{1,4/3}(\mathbb{T}^d))} + \|V_\varepsilon\|_{L^2(0,T;H^2(\mathbb{T}^d))} \le K, \tag{31}$$

$$\|n_\varepsilon\|_{L^2(0,T;W^{1,1}(\mathbb{T}^d))} + \|\partial_t n_\varepsilon\|_{L^{8/7}(0,T;H^{-3}(\mathbb{T}^d))} \le K. \tag{32}$$

Proof. First, we prove some bounds on $\sqrt{n_\varepsilon}\theta_\varepsilon$ and n_ε in Lebesgue spaces. By the Gagliardo-Nirenberg inequality with $\alpha = d/4$, we find that

$$\begin{aligned}
\|\sqrt{n_\varepsilon}\theta_\varepsilon\|_{L^{8/d}(0,T;L^4(\mathbb{T}^d))}^{8/d} &\le K\int_0^T \|\sqrt{n_\varepsilon}\theta_\varepsilon\|_{H^1(\mathbb{T}^d)}^{8\alpha/d}\|\sqrt{n_\varepsilon}\theta_\varepsilon\|_{L^2(\mathbb{T}^d)}^{8(1-\alpha)/d} ds \\
&\le K\|\sqrt{n_\varepsilon}\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}^{8(1-\alpha)/d}\|\theta_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}^{8(1-\alpha)/d} \\
&\quad \times\int_0^T \|\sqrt{n_\varepsilon}\theta_\varepsilon\|_{H^1(\mathbb{T}^d)}^2 ds \\
&\le K,
\end{aligned}$$

using (21) and (23). Moreover, by (21) and (23) again, we obtain for $d \le 3$,

$$\begin{aligned}
\|\nabla(n_\varepsilon\theta_\varepsilon)\|_{L^{8/7}(0,T;L^{4/3}(\mathbb{T}^d))} &\le \|\sqrt{n_\varepsilon}\|_{L^{8/3}(0,T;L^4(\mathbb{T}^d))}\|\nabla(\sqrt{n_\varepsilon}\theta_\varepsilon)\|_{L^2(0,T;L^2(\mathbb{T}^d))} \\
&\quad + \|\sqrt{n_\varepsilon}\theta_\varepsilon\|_{L^{8/3}(0,T;L^4(\mathbb{T}^d))}\|\nabla\sqrt{n_\varepsilon}\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le K.
\end{aligned}$$

Because of the uniform L^2-bound on $n_\varepsilon\theta_\varepsilon$, this shows that $(n_\varepsilon\theta_\varepsilon)$ is bounded in $L^{8/7}(0,T;W^{1,4/3}(\mathbb{T}^d))$. We remark that the L^2-bound on n_ε and elliptic regularity imply that (V_ε) is bounded in $L^2(0,T;H^2(\mathbb{T}^d))$. This proves (31).

Next, since $(\sqrt{n_\varepsilon})$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$ and $(\nabla\sqrt{n_\varepsilon})$ is bounded in $L^2(0,T;L^2(\mathbb{T}^d))$, the sequence $\nabla n_\varepsilon = 2\sqrt{n_\varepsilon}\nabla\sqrt{n_\varepsilon}$ is uniformly bounded in $L^2(0,T;L^1(\mathbb{T}^d))$. As a consequence, we conclude the bound on (n_ε) in $L^2(0,T;W^{1,1}(\mathbb{T}^d))$.

Furthermore, by the Gagliardo-Nirenberg inequality with $\alpha = d/12$,

$$\begin{aligned}\|\sqrt{n_\varepsilon}\|^8_{L^8(0,T;L^{12/5}(\mathbb{T}^d))} &\le K\int_0^T \|\sqrt{n_\varepsilon}\|^{8\alpha}_{H^1(\mathbb{T}^d)}\|\sqrt{n_\varepsilon}\|^{8(1-\alpha)}_{L^2(\mathbb{T}^d)}ds \\ &\le \|\sqrt{n_\varepsilon}\|^{8(1-\alpha)}_{L^\infty(0,T;L^2(\mathbb{T}^d))}\int_0^T \|\sqrt{n_\varepsilon}\|^{2d/3}_{H^1(\mathbb{T}^d)}ds \le K,\end{aligned}$$

by taking into account the bound (21) and $2d/3 \le 2$. Then, by the Hölder inequality,

$$\|n_\varepsilon\|_{L^{8/3}(0,T;L^{3/2}(\mathbb{T}^d))} \le \|\sqrt{n_\varepsilon}\|_{L^4(0,T;L^4(\mathbb{T}^d))}\|\sqrt{n_\varepsilon}\|_{L^8(0,T;L^{12/5}(\mathbb{T}^d))} \le K. \tag{33}$$

It remains to estimate $\partial_t n_\varepsilon$. We observe that, by the Gagliardo-Nirenberg inequality with $\alpha = (d+12)/24$ and (22),

$$\begin{aligned}\|\nabla\sqrt{n_\varepsilon}\|^{16/5}_{L^{16/5}(0,T;L^{12/5}(\mathbb{T}^d))} &\le K\int_0^T \|\sqrt{n_\varepsilon}\|^{16\alpha/5}_{H^2(\mathbb{T}^d)}\|\sqrt{n_\varepsilon}\|^{16(1-\alpha)/5}_{L^2(\mathbb{T}^d)}ds \\ &\le K\|\sqrt{n_\varepsilon}\|^{16(1-\alpha)/5}_{L^\infty(0,T;L^2(\mathbb{T}^d))}\int_0^T \|\sqrt{n_\varepsilon}\|^{2(d+12)/15}_{H^2(\mathbb{T}^d)}ds \\ &\le K\|n_\varepsilon\|^{8(1-\alpha)/5}_{L^\infty(0,T;L^1(\mathbb{T}^d))}\|\sqrt{n_\varepsilon}\|^{2(d+12)/15}_{L^2(0,T;H^2(\mathbb{T}^d))} \\ &\le K\varepsilon^{-2(d+12)/15},\end{aligned}$$

and by the Gagliardo-Nirenberg inequality with $\alpha = d/12$ and (22),

$$\begin{aligned}\|\sqrt{n_\varepsilon}\|^8_{L^8(0,T;L^3(\mathbb{T}^d))} &\le K\int_0^T \|\sqrt{n_\varepsilon}\|^{8\alpha}_{H^2(\mathbb{T}^d)}\|\sqrt{n_\varepsilon}\|^{8(1-\alpha)}_{L^2(\mathbb{T}^d)}ds \\ &\le K\|\sqrt{n_\varepsilon}\|^{8(1-\alpha)}_{L^\infty(0,T;L^2(\mathbb{T}^d))}\|\sqrt{n_\varepsilon}\|^{2d/3}_{L^2(0,T;H^2(\mathbb{T}^d))} \\ &\le K\varepsilon^{-2d/3}.\end{aligned}$$

Both estimates yield

$$\varepsilon^{(d+12)/24}\|\nabla\sqrt{n_\varepsilon}\|_{L^{16/5}(0,T;L^{12/5}(\mathbb{T}^d))} + \varepsilon^{d/12}\|\sqrt{n_\varepsilon}\|_{L^8(0,T;L^3(\mathbb{T}^d))} \le K.$$

Therefore, taking into account (21) and (22),

$$\begin{aligned}&\varepsilon^2\|\sqrt{n_\varepsilon}\nabla^2\sqrt{n_\varepsilon} - \nabla\sqrt{n_\varepsilon}\otimes\nabla\sqrt{n_\varepsilon}\|_{L^{8/5}(0,T;L^{6/5}(\mathbb{T}^d))} \\ &\quad\le \varepsilon^{(12-d)/12}\|\varepsilon^{d/12}\sqrt{n_\varepsilon}\|_{L^8(0,T;L^3(\mathbb{T}^d))}\|\varepsilon\sqrt{n_\varepsilon}\|_{L^2(0,T;H^2(\mathbb{T}^d))} \\ &\qquad+ \varepsilon^{(12-d)/12}\|\varepsilon^{(d+12)/24}\nabla\sqrt{n_\varepsilon}\|^2_{L^{16/5}(0,T;L^{12/5}(\mathbb{T}^d))} \\ &\quad\le K\varepsilon^{(12-d)/12}.\end{aligned} \tag{34}$$

Furthermore, using (31), (33), and the continuous embedding $H^2(\mathbb{T}^d) \hookrightarrow W^{1,6}(\mathbb{T}^d)$ for $d \le 3$,

$$\begin{aligned}\|\nabla(n_\varepsilon\theta_\varepsilon) - n_\varepsilon\nabla V_\varepsilon\|_{L^{8/7}(0,T;L^{6/5}(\mathbb{T}^d))} &\le \|\nabla(n_\varepsilon\theta_\varepsilon)\|_{L^{8/7}(0,T;L^{6/5}(\mathbb{T}^d))} \\ &\quad+ \|n_\varepsilon\|_{L^{8/3}(0,T;L^{3/2}(\mathbb{T}^d))}\|\nabla V_\varepsilon\|_{L^2(0,T;L^6(\mathbb{T}^d))} \le K.\end{aligned}$$

Since the continuous embedding $H^1(\mathbb{T}^d) \hookrightarrow L^6(\mathbb{T}^d)$ (for $d \le 3$) implies that the embedding $L^{6/5}(\mathbb{T}^d) = (L^6(\mathbb{T}^d))' \hookrightarrow H^{-1}(\mathbb{T}^d)$ is also continuous, we estimate $\partial_t n_\varepsilon$

as follows:

$$\begin{aligned}
\|\partial_t n_\varepsilon\|_{L^{8/7}(0,T;H^{-3}(\mathbb{T}^d))} &\le \frac{\varepsilon^2}{6}\|\sqrt{n_\varepsilon}\nabla^2\sqrt{n_\varepsilon} - \nabla\sqrt{n_\varepsilon}\otimes\nabla\sqrt{n_\varepsilon}\|_{L^{8/7}(0,T;H^{-1}(\mathbb{T}^d))} \\
&\quad + \|\nabla(n_\varepsilon\theta_\varepsilon) - n_\varepsilon\nabla V_\varepsilon\|_{L^{8/7}(0,T;H^{-2}(\mathbb{T}^d))} \\
&\le \varepsilon^2 K\|\sqrt{n_\varepsilon}\nabla^2\sqrt{n_\varepsilon} - \nabla\sqrt{n_\varepsilon}\otimes\nabla\sqrt{n_\varepsilon}\|_{L^{8/7}(0,T;L^{6/5}(\mathbb{T}^d))} \\
&\quad + \|\nabla(n_\varepsilon\theta_\varepsilon) - n_\varepsilon\nabla V_\varepsilon\|_{L^{8/7}(0,T;L^{6/5}(\mathbb{T}^d))} \\
&\le K.
\end{aligned}$$

Hence, $(\partial_t n_\varepsilon)$ is bounded in $L^{8/7}(0,T;H^{-3}(\mathbb{T}^d))$. □

Now, we are able to prove Theorem 1.2, i.e. to pass to the limit $\varepsilon\to 0$ in (8)-(10). Estimate (32) allows us to apply the Aubin lemma [24, Corollary 4] (also see [13]) to conclude the existence of a subsequence of (n_ε), which is not relabeled, such that, as $\varepsilon\to 0$,

$$n_\varepsilon \to n \quad \text{strongly in } L^2(0,T;L^p(\mathbb{T}^d)) \text{ for all } p<\tfrac{3}{2}. \tag{35}$$

Here, we have used the compact embedding $W^{1,1}(\mathbb{T}^d)\hookrightarrow L^p(\mathbb{T}^d)$ for $p<3/2$. Furthermore, by (21) and (32), for the same subsequence,

$$\begin{aligned}
\sqrt{n_\varepsilon} &\rightharpoonup \sqrt{n} \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T}^d)), \\
\partial_t n_\varepsilon &\rightharpoonup \partial_t n \quad \text{weakly in } L^{8/7}(0,T;H^{-3}(\mathbb{T}^d)).
\end{aligned}$$

The uniform bounds in (23) and (31) lead to (up to subsequences)

$$\begin{aligned}
\theta_\varepsilon &\rightharpoonup^* \theta \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^\infty(\mathbb{T}^d)), \qquad (36)\\
\nabla V_\varepsilon &\rightharpoonup \nabla V \quad \text{weakly in } L^2(0,T;L^6(\mathbb{T}^d)),
\end{aligned}$$

since $H^2(\mathbb{T}^d)$ embeddes continuously into $W^{1,6}(\mathbb{T}^d)$ for $d\le 3$. We infer that

$$\begin{aligned}
n_\varepsilon\nabla V_\varepsilon &\rightharpoonup n\nabla V \quad \text{weakly in } L^1(0,T;L^{6p/(6+p)}(\mathbb{T}^d)), \\
n_\varepsilon\theta_\varepsilon &\rightharpoonup n\theta \quad \text{weakly in } L^2(0,T;L^p(\mathbb{T}^d)).
\end{aligned}$$

We remark that $6p/(6+p)>1$ if $p>6/5$ which is possible since we can choose $p\in[1,3/2)$. Estimate (31) implies that, up to a subsequence,

$$\nabla(n_\varepsilon\theta_\varepsilon) \rightharpoonup \nabla(n\theta) \quad \text{weakly in } L^{8/7}(0,T;L^{4/3}(\mathbb{T}^d)). \tag{37}$$

Furthermore, for test functions $\psi\in L^{8/3}(0,T;W^{2,6}(\mathbb{T}^d))$, by (34),

$$\begin{aligned}
&\varepsilon^2\int_0^T\int_{\mathbb{T}^d}\left(\sqrt{n_\varepsilon}\nabla^2\sqrt{n_\varepsilon} - \nabla\sqrt{n_\varepsilon}\otimes\nabla\sqrt{n_\varepsilon}\right):\nabla^2\psi dx\,ds \\
&\quad \le \varepsilon^2\left\|\sqrt{n_\varepsilon}\nabla^2\sqrt{n_\varepsilon} - \nabla\sqrt{n_\varepsilon}\otimes\nabla\sqrt{n_\varepsilon}\right\|_{L^{8/5}(0,T;L^{6/5}(\mathbb{T}^d))}\|\psi\|_{L^{8/3}(0,T;W^{2,6}(\mathbb{T}^d))} \\
&\quad \le \varepsilon^{(12-d)/12}K\|\psi\|_{L^{8/3}(0,T;W^{2,6}(\mathbb{T}^d))} \to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned}$$

The above convergence results are sufficient to pass to the limit in the mass balance equation (8) and in the linear Poisson equation (10).

Next, the convergences (35) and (36) imply that

$$\sqrt{n_\varepsilon}\theta_\varepsilon \rightharpoonup \sqrt{n}\theta \quad \text{weakly in } L^4(0,T;L^{2p}(\mathbb{T}^d)), \quad p<\frac{3}{2}.$$

This, together with the second bound in (23), yields

$$\nabla(\sqrt{n_\varepsilon}\theta_\varepsilon) \rightharpoonup \nabla(\sqrt{n}\theta) \quad \text{weakly in } L^2(0,T;L^2(\mathbb{T}^d)).$$

Since, by (35), $\sqrt{n_\varepsilon} \to \sqrt{n}$ strongly in $L^4(0,T;L^{2p}(\mathbb{T}^d))$ for $p < 3/2$, we infer that

$$\sqrt{n_\varepsilon}\nabla(\sqrt{n_\varepsilon}\theta_\varepsilon) \rightharpoonup \sqrt{n}\nabla(\sqrt{n}\theta) \quad \text{weakly in } L^{4/3}(0,T;L^1(\mathbb{T}^d)). \tag{38}$$

Convergences (37) and (38) allow us to perform the limit $\varepsilon \to 0$ in the temperature equation (9) ending the proof.

3. **Proof of Theorem 1.3.** First, we prove some a priori estimates derived from the entropy expression (20).

Proposition 2. *Let the assumptions of Theorem 1.1 hold and let $d = 1$. Then there exists a weak solution $(n_\varepsilon, \theta_\varepsilon, V_\varepsilon)$ to (8)-(10) and (4) satisfying the following bounds for all $T > 0$:*

$$\|n_\varepsilon - \log n_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}))} + \varepsilon\|\log n_\varepsilon\|_{L^2(0,T;H^2(\mathbb{T}))} + \|\log n_\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}))} \le K, \tag{39}$$

$$\|\theta_\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}))} \le K, \tag{40}$$

where the constant $K > 0$ is independent of ε.

Proof. The idea of the proof is to semi-discretize equation (8) in time as in the proof of Proposition 1. In one space dimension, we do not need the regularizing δ-terms (see (24)). Instead, we solve the problem in $\mathbb{T}$

$$\frac{1}{\tau}(\rho_\tau^2 - \sigma_\tau(\rho_\tau^2)) + \frac{\varepsilon^2}{12}\big(\rho_\tau^2(\log\rho_\tau^2)_{xx}\big)_{xx} = \big((\rho_\tau^2\theta_\tau)_x - \rho_\tau^2(V_\tau)_x\big)_x, \tag{41}$$

$$-\big(\rho_\tau^2(\theta_\tau)_x\big)_x = \frac{\rho_\tau^2}{\tau_e}(\theta_L(x) - \theta_\tau),\ \lambda^2(V_\tau)_{xx} = \sigma_\tau(\rho_\tau^2) - C(x),\ \int_{\mathbb{T}} V_\tau dx = 0. \tag{42}$$

The proof of the existence of a weak solution $(\rho_\tau, \theta_\tau, V_\tau)$ to this problem is performed similarly as in Step 2 of the proof of Lemma 2 in [20] by applying the Leray-Schauder fixed-point theorem. For this, we need a uniform estimate for $\log\rho_\tau^2$ in $H^1(\mathbb{T})$. This is achieved by employing $1 - \rho_\tau^{-2}$ as a test function in (41):

$$\begin{aligned}\frac{1}{\tau}\int_{\mathbb{T}}\big(\rho_\tau^2 - \sigma_\tau(\rho_\tau^2)\big)(1-\rho_\tau^{-2})dx &+ \frac{\varepsilon^2}{12}\int_{\mathbb{T}}\rho_\tau^2(\log\rho_\tau^2)_{xx}(1-\rho_\tau^{-2})_{xx}dx \\ &= -\int_{\mathbb{T}}\big((\rho_\tau^2\theta_\tau)_x - \rho_\tau^2(V_\tau)_x\big)(1-\rho_\tau^{-2})_x dx.\end{aligned} \tag{43}$$

The convexity of the function $\phi_0(s) = s - \log s$ implies that $\phi_0(s) - \phi_0(t) \le \phi_0'(s)(s-t) = (1-s^{-1})(s-t)$ for all $s, t > 0$, and the first integral is estimated as follows

$$\begin{aligned}&\frac{1}{\tau}\int_{\mathbb{T}}\big(\rho_\tau^2 - \sigma_\tau(\rho_\tau^2)\big)(1-\rho_\tau^{-2})dx \\ &\quad\ge \frac{1}{\tau}\int_{\mathbb{T}}\big(\phi_0(\rho_\tau^2)dx - \phi_0(\sigma_\tau(\rho_\tau^2))\big)dx = \frac{1}{\tau}\big(E_0(\rho_\tau^2) - E_0(\sigma_\tau(\rho_\tau^2))\big)\end{aligned}$$

The second integral in (43) can be written as

$$\begin{aligned}&\frac{\varepsilon^2}{12}\int_{\mathbb{T}}(\log\rho_\tau^2)_{xx}\big((\log\rho_\tau^2)_{xx} - (\log\rho_\tau^2)_x^2\big)dx \\ &\quad= \frac{\varepsilon^2}{12}\int_{\mathbb{T}}\Big((\log\rho_\tau^2)_{xx}^2 - \frac{1}{3}\big((\log\rho_\tau^2)_x^3\big)_x\Big)dx = \frac{\varepsilon^2}{12}\int_{\mathbb{T}}(\log\rho_\tau^2)_{xx}^2 dx,\end{aligned}$$

using the periodic boundary conditions (this is also true when assuming homogeneous Neumann boundary conditions). The right-hand side of (43) is formulated as

$$\int_{\mathbb{T}}\big(- \theta_\tau(\log\rho_\tau^2)_x^2 - (\theta_\tau)_x(\log\rho_\tau^2)_x + (V_\tau)_x(\log\rho_\tau^2)_x\big)dx.$$

In order to estimate these terms, we employ $-\rho_\tau^{-2}$ as a test function in the first equation of (42):

$$-\int_{\mathbb{T}}(\theta_\tau)_x(\log\rho_\tau^2)_x dx = -\frac{1}{\tau_e}\int_{\mathbb{T}}(\theta_L(x)-\theta_\tau)dx \le K,$$

since θ_τ is uniformly bounded in $L^\infty(0,T;L^\infty(\mathbb{T}))$, by the maximum principle. Furthermore, the test function $\log\rho_\tau^2$ in the Poisson equation in (42) leads to

$$\begin{aligned}\int_{\mathbb{T}}(V_\tau)_x(\log\rho_\tau^2)_x dx &= -\frac{1}{\lambda^2}\int_{\mathbb{T}}\big(\sigma_\tau(\rho_\tau^2) - C(x)\big)\log\rho_\tau^2 dx \\ &= -\frac{1}{\lambda^2}\int_{\mathbb{T}}\sigma_\tau(\rho_\tau^2)\log\rho_\tau^2 dx + \frac{1}{\lambda^2}\|C\|_{L^\infty(\mathbb{T})}\int_{\mathbb{T}}|\log\rho_\tau^2|dx.\end{aligned}$$

We need to estimate the integral over $\sigma_\tau(\rho_\tau^2)\log\rho_\tau^2$. To this end, we employ $\log\rho_\tau^2$ as a test function in (41) and use the first equation in (42):

$$\begin{aligned}\frac{1}{\tau}&\int_{\mathbb{T}}\big(\rho_\tau^2 - \sigma_\tau(\rho_\tau^2)\big)\log\rho_\tau^2 dx + \int_{\mathbb{T}}\rho_\tau^2(\log\rho_\tau^2)_{xx}^2 dx \\ &= -\int_{\mathbb{T}}\rho_\tau^2(\theta_\tau)_x(\log\rho_\tau^2)_x dx - \frac{1}{\lambda^2}\int_{\mathbb{T}}(\rho_\tau^2 - C(x))\log\rho_\tau^2 dx \\ &= -\frac{1}{\tau_e}\int_{\mathbb{T}}\rho_\tau^2(\theta_L(x)-\theta)\log\rho_\tau^2 dx - \frac{1}{\lambda^2}\int_{\mathbb{T}}(\rho_\tau^2 - C(x))\log\rho_\tau^2 dx \\ &\le K\int_{\mathbb{T}}|\rho_\tau^2\log\rho_\tau^2|dx + K\int_{\mathbb{T}}|\log\rho_\tau^2|dx.\end{aligned}$$

Hence, using $|x\log x| \le x^2+1$ and $|\log x| \le x - \log x$ for $x>0$,

$$-\frac{1}{\tau}\int_{\mathbb{T}}\sigma_\tau(\rho_\tau^2)\log\rho_\tau^2 dx \le K + K\int_{\mathbb{T}}\rho_\tau^4 dx + KE_0(\rho_\tau^2) \le K(1+E_0(\rho_\tau^2)),$$

by (21). Putting the above estimates together, we arrive at

$$\frac{1}{\tau}\big(E_0(\rho_\tau^2) - E_0(\sigma_\tau(\rho_\tau^2))\big) + \frac{\varepsilon^2}{12}\int_{\mathbb{T}}(\log\rho_\tau^2)_{xx}^2 dx \le K(1+E_0(\rho_\tau^2)),$$

and the discrete Gronwall lemma implies the desired bound for $\log\rho_\tau^2$ in $H^2(\mathbb{T})$. We infer the existence of a weak solution $(\rho_\tau,\theta_\tau,V_\tau)$ to (41)-(42).

Employing the test function θ_τ/ρ_τ^2 in the first equation of (42), we find that

$$\int_{\mathbb{T}}(\theta_\tau)_x\big((\theta_\tau)_x - \theta_\tau(\log\rho_\tau^2)_x\big)dx = \frac{1}{\tau_e}\int_{\mathbb{T}}(\theta_L(x)-\theta_\tau)\theta_\tau dx.$$

By Young's inequality, it follows that

$$\frac{1}{2}\int_{\mathbb{T}}(\theta_\tau)_x^2 \le \frac{1}{2}\int_{\mathbb{T}}\theta_\tau^2(\log\rho_\tau^2)_x^2 dx + \frac{1}{2\tau_e}\int_{\mathbb{T}}\theta_L^2 dx + K.$$

In view of the uniform L^∞ bound for θ_τ and the uniform H^1 bound for $\log\rho_\tau^2$, the right-hand side is uniformly bounded. Thus, (θ_τ) is bounded in $L^2(0,T;H^1(\mathbb{T}))$.

We can pass to the limit $\tau\to 0$ in (41)-(42) to conclude the existence of a solution $(n_\varepsilon,\theta_\varepsilon,V_\varepsilon)$ to (8)-(10) and (4) satisfying the estimates (39)-(40). □

Now, we can prove Theorem 1.3. The limit $\varepsilon \to 0$ can be performed as in the proof of Theorem 1.2. The only difference is the treatment of the term involving $n_\varepsilon(\theta_\varepsilon)_x$. The estimate (32) and Aubin's lemma provide the existence of a subsequence (not relabeled) such that, as $\varepsilon \to 0$,

$$n_\varepsilon \to n \quad \text{strongly in } L^2(0,T;L^4(\mathbb{T})).$$

Furthermore, the uniform bound (40) on θ_ε leads, up to a subsequence, to

$$(\theta_\varepsilon)_x \rightharpoonup \theta_x \quad \text{weakly in } L^2(0,T;L^2(\mathbb{T})).$$

Hence, we have

$$n_\varepsilon(\theta_\varepsilon)_x \rightharpoonup n\theta_x \quad \text{weakly in } L^1(0,T;L^1(\mathbb{T})).$$

This ends the proof.

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