Global existence of solutions to one-dimensional viscous quantum hydrodynamic equations

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The existence of global-in-time weak solutions to the one-dimensional viscous quantum hydrodynamic equations is proved. The model consists of the conservation laws for the particle density and particle current density, including quantum corrections from the Bohm potential and viscous stabilizations arising from quantum Fokker-Planck interaction terms in the Wigner equation. The model equations are coupled self-consistently to the Poisson equation for the electric potential and are supplemented with periodic boundary and initial conditions. When a diffusion term linearly proportional to the velocity is introduced in the momentum equation, the positivity of the particle density is proved. This term, which introduces a strong regularizing effect, may be viewed as a classical conservative friction term due to particle interactions with the background temperature. Without this regularizing viscous term, only the nonnegativity of the density can be shown. The existence proof relies on the Faedo-Galerkin method together with a priori estimates from the energy functional.

1. Introduction

Diffusive corrections in quantum models are of great importance in open quantum systems modeling, for instance, an electron ensemble interacting with a background heat bath. Applications of such systems include quantum semiconductor structures in which non-classical diffusive effects may be relevant in some regimes. Caldeira and Leggett [3] and Diósi [8] have derived closed equations for a dissipative quantum-mechanical system related to quantum Brownian motion. Their approach was later improved by Castella et al. [4] and leads to a Wigner equation with Fokker-Planck-type operator modeling

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interactions that may take into account basic quantum and classical mechanisms. Thus, one may interpret the Wigner-Fokker-Plank equation as a quantum Liouville equation equated to an interaction operator of quantum Fokker-Planck type.

Motivated by multi-scale modeling and by the fact that computational approaches for the Wigner-Fokker-Plank equation are expensive, due mainly to its high dimensionality (see for example [11] and references therein), associated macroscopic models were derived in an effort to produce asymptotically correct macroscopic reductions. For instance, employing a moment method and a suitable closure condition to the classical Wigner equation in the absence of interactions, quantum hydrodynamic equations are obtained [7, 15]. Another derivation comes from the mixed-state Schrödinger system via the Madelung transform [16].

When particle interactions are taken into account, a non-classical (quantum) Fokker-Planck interaction operator balances the Wigner equation, with classical and non-classical second-order derivative terms (that may be interpreted as viscous terms) appearing in the macroscopic model. We refer to [17,21,23] for a derivation of a non-classical viscous perturbation due to quantum interactions and to [13,14] for an analysis of stationary quantum-regularized models with a classical mass-conservative viscous effect. This leads to a broad class of *viscous quantum hydrodynamic equations*, which are the subject of this paper.

The (scaled) viscous quantum hydrodynamic equations in one space dimension for the particle density ρ, the velocity u, and the electric potential V read as follows:

$$\rho_t + (\rho u)_x = \nu \rho_{xx}, \tag{1}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x - \rho V_x - \frac{\delta^2}{2}\rho\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x = \nu(\rho u)_{xx} + \varepsilon u_{xx} - \frac{\rho u}{\tau}, \tag{2}$$

$$\lambda^2 V_{xx} = \rho - C(x), \tag{3}$$

The pressure $p(\rho)$ is assumed to depend on the particle density; typical examples are $p(\rho) = p_0\rho^\alpha$ for some $p_0 > 0$ and $\alpha \geq 1$. The function $C(x)$ is the doping profile modeling charged background ions in, for instance, semiconductor crystals. The viscosity $\nu > 0$ is related to effects depending also on the scaled Planck constant $\delta > 0$ through the well-known Lindblad condition. This condition guarantees the quantum mechanically correct evolution of the system and the convergence to the classical Fokker-Planck dynamics from stochastic calculus as $\delta \to 0$ (see [3,8,27]). The parameter $\tau > 0$ models momentum relaxation time due to classical friction mechanisms, and the parameter ε accounts for classical mass-conservative viscous effects due to classical particle-particle and particle-lattice interactions. Finally, $\lambda > 0$ is the scaled Debye length of the device.

Equations (1)-(3) are considered on the one-dimensional torus $\mathbb{T}$ (with size one) and are complemented with the initial conditions

$$\rho(0,\cdot) = \rho_0, \quad (\rho u)(0,\cdot) = \rho_0 u_0 \quad \text{in } \mathbb{T}. \tag{4}$$

Equation (1) expresses a mass balance law that becomes mass conservative with respect to the effective current density $J_0 = \rho u - \nu\rho_x$. The second equation is the classical balance equation for the particle current density or momentum ρu including the electric force term ρV_x, the relaxation-time term $-\rho u/\tau$, and the quantum correction with the Bohm potential $(\sqrt{\rho})_{xx}/\sqrt{\rho}$. The electric potential V is self-consistently given by the Poisson equation (3). In the absence of viscous and quantum effects, i.e. $\nu = \varepsilon = \delta = 0$, the above equations represent the hydrodynamic semiconductor equations [2]. When no viscous effects are present, $\nu = \varepsilon = 0$, we obtain the quantum hydrodynamic equations, studied in, e.g., [13,14,20]. For more recent papers, we refer to [1,19,26,28,29].

The viscous quantum hydrodynamic model for $\varepsilon = 0$ can be derived from the Wigner-Fokker-Planck equation by a moment method [17,23]. The viscous regularizations arise from the quantum Fokker-Planck interaction operator. More precisely, the part of the scattering operator yielding the non-classical viscous terms is proportional to

$$Q_{\mathrm{QFP}}(w) = \nu w_{xx},$$

where $w(x, k, t)$ is the Wigner function on the position-wave vector space (x, k), and $\nu > 0$ depends on the quantum friction. Introducing the moments $\rho = \int_{\mathbb{R}^3} w dk$ and $\rho u = \int_{\mathbb{R}^3} wk dk$ gives

$$\int_{\mathbb{R}^3} Q_{\mathrm{QFP}}(w) dk = \nu \rho_{xx}, \quad \int_{\mathbb{R}^3} Q_{\mathrm{QFP}}(w) k dk = \nu(\rho u)_{xx},$$

which are the non-classical viscous terms in the quantum fluid system (1)-(2), respectively. In this view, they are *not* artificial regularizations, but coming from the choice of the quantum interaction operator of Fokker-Plank type in the Wigner equation.

The classical diffusive velocity term proportional to $\varepsilon > 0$ is a heuristic regularization which allows us to prove the existence of solutions with *positive* particle densities (see Theorem 1). In Theorem 2 we perform the limit $\varepsilon \to 0$, obtaining *nonnegative* particle densities for the system (1)-(3). It is possible to derive the velocity term from the Wigner equation by introducing the following *heuristic* quantum interaction operator:

$$Q_{\mathrm{rQFP}}(w) = \varepsilon \partial_x^2 \Big(\frac{w}{\int_{\mathbb{R}^3} w dk} \Big).$$

Indeed, we obtain for the first moments of the operator

$$\begin{aligned} \int_{\mathbb{R}^3} Q_{\mathrm{rQFP}}(w) dk' &= \varepsilon \partial_x^2 \int_{\mathbb{R}^3} \frac{w}{\int_{\mathbb{R}^3} w dk} dk' = 0, \\ \int_{\mathbb{R}^3} Q_{\mathrm{rQFP}}(w) k' dk' &= \varepsilon \partial_x^2 \int_{\mathbb{R}^3} \frac{wk'}{\int_{\mathbb{R}^3} w dk} dk' = \varepsilon \partial_x^2 \frac{\rho u}{\rho} = \varepsilon u_{xx}, \end{aligned}$$

i.e., the contribution to the momentum equation (2) equals εu_{xx}.

There are only few mathematical results for these viscous quantum hydrodynamic model due to difficulties coming from the third-order derivatives in the quantum correction. The existence of classical solutions to the one-dimensional stationary model with $\varepsilon = 0$ and with physical boundary conditions was shown in [23]. The transient equations are considered in [5,6,9], and the local-in-time existence and exponential stability of solutions were proved. Global-in-time solutions in one space dimension are obtained if the initial energy is assumed to be sufficiently small [5]. In [23,24], numerical solutions of the model and applications to resonant tunneling diodes were presented. We also mention that in the inviscid case ($\nu = \varepsilon = 0$) there is a recent proof of non-global-in-time existence for a quantum hydrodynamic equation (corresponding to a reduced model in the absence of the coupling with the Poisson equation) in bounded domains with prescribed data corresponding to high boundary and initial energy [12].

However, no *global-in-time* existence result *without* smallness conditions seems to be available up to now for the transient system (1)-(3). In this paper, we prove such a result, first for the full system with $\varepsilon > 0$ and then, by passing to the limit $\varepsilon \to 0$, we obtain a global existence result for the non-classical viscous quantum hydrodynamic model (1)-(3) with $\varepsilon = 0$.

The main problem of the existence analysis lies in the strongly nonlinear third-order differential operator and the dispersive structure of the momentum equation. There are several attempts in the literature to deal with the quantum term. Integrating the stationary momentum equation leads to a second-order differential equation to which maximum principle arguments can be applied [13]. A fourth-order wave equation is obtained after differentiating the equation with respect to the spatial variable. This approach was employed in [25] to prove the existence of global solutions to the quantum hydrodynamic equations with $\nu = 0$ (and $\varepsilon = 0$), but only for initial data close to thermal equilibrium. The main idea of [5] was to introduce a bi-Laplacian regularization in the viscous model and to employ energy estimates to conclude local existence of solutions. Global existence of solutions to the inviscid model $\nu = 0$ (and $\varepsilon = 0$) with nonnegative particle density was achieved recently by a wave function polar decomposition method [1].

In this paper, we pursue a different strategy. We employ the Faedo-Galerkin method, introduced by Feireisl in [10] for the analysis of the classical compressible Navier-Stokes equations, applied to (1)-(3) for $\varepsilon > 0$ with the initial conditions (4).

The existence proof relies on the following ideas. First, for given u in a finite-dimensional Galerkin space, we solve (1). Since u is given and (1) is parabolic for $\varepsilon > 0$, a lower bound for the particle density can be concluded from the maximum principle. Classically, this bound depends on the L^∞ norm of u_x which is prohibitive to set up the fixed-point argument. We prove that the lower bound for ρ depends only on the L^2 norm of u_x (Lemma 3).

In the second step we solve the Poisson equation and then the momentum equation in the Galerkin space, for given ρ, which yields the existence of local-in-time solutions via Banach's fixed-point theorem.

A priori bounds (and thus global-in-time existence) are obtained from the energy inequality defined as follows. Let the enthalpy function h be defined by $h'(y) = p'(y)/y$ for $y > 0$ and $h(1) = 0$, and let H be a primitive of h. Furthermore, let the energy, consisting of the internal, kinetic, electric, and quantum energy, be given by

$$E(\rho, u) = \int_{\mathbb{T}} \Big(H(\rho) + \frac{1}{2}\rho u^2 + \frac{\lambda^2}{2} V_x^2 + \frac{\delta^2}{2} (\sqrt{\rho})_x^2 \Big) dx. \tag{5}$$

Then we show that

$$\frac{dE}{dt} + \nu \int_{\mathbb{T}} \big(\rho u_x^2 + \delta^2 (\sqrt{\rho})_{xx}^2 \big) dx + \varepsilon \int_{\mathbb{T}} u_x^2 dx \le K, \tag{6}$$

where the constant $K > 0$ depends only on $C(x)$, ν, and λ. This yields H^2 estimates for

$\sqrt{\rho}$ and, for fixed $\varepsilon > 0$, L^2 estimates for u_x, needed in the proof of the lower bound for ρ.

We consider the one-dimensional equations since we need several times in the proof the embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ which is valid in one space dimension only. We comment on the multi-dimensional situation in Remark 6.

Our first main result reads as follows.

Theorem 1. *Let $T > 0$, $\varepsilon > 0$, and $C \in L^2(\mathbb{T})$. Let the pressure function $p \in C^1([0,\infty))$ be monotone, and let the primitive H of the enthalpy satisfy $H(y) \geq -h_0$ for some $h_0 > 0$. Furthermore, let the initial datum $(\rho_0, u_0) \in H^1(\mathbb{T}) \times L^\infty(\mathbb{T})$ satisfy $\int_{\mathbb{T}} \rho_0(x)dx = \int_{\mathbb{T}} C(x)dx$, $\rho(x) \geq \eta_0 > 0$ for $x \in \mathbb{T}$ and for some $\eta_0 > 0$, and $E(\rho_0, u_0) < \infty$. Then there exists a constant $\eta > 0$ and a weak solution (ρ, u, V) to (1)-(3) satisfying*

$$\begin{aligned}
&\rho(t,x) \geq \eta > 0 \quad \text{for } t > 0,\ x \in \mathbb{T}, \quad V \in L^\infty(0,T;H^2(\mathbb{T})),\\
&\rho_t \in L^2(0,T;L^2(\mathbb{T})), \quad (\rho u)_t \in L^2(0,T;H^{-2}(\mathbb{T})),\\
&\rho \in L^\infty(0,T;H^1(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T})),\\
&u \in L^2(0,T;H^1(\mathbb{T})) \cap L^\infty(0,T;L^2(\mathbb{T})),
\end{aligned}$$

where the lower bound $\eta > 0$ depends on ε. The initial conditions (4) are satisfied in the sense of $H^{-2}(\mathbb{T})$.

The condition $H(y) \geq -h_0$ is satisfied, for instance, if the pressure is given by $p(\rho) = p_0\rho^\alpha$, where $p_0 > 0$ and $\alpha > 1$, since in this case $H(y) = (\alpha-1)^{-1}(y^\alpha - \alpha y) + \text{const.}$, and the minimum of H is achieved at $y = 1$. The regularity of ρ and u implies that $\rho u \in L^2(0,T;H^1(\mathbb{T}))$ and $\rho u^2 \in L^2(0,T;W^{1,1}(\mathbb{T}))$.

Let $(\rho_\varepsilon, u_\varepsilon)$ be a solution to (1)-(3) in the sense of the above theorem. In the limit $\varepsilon \to 0$ we loose the lower bound for ρ_ε since it depends on ε. Furthermore, it is not clear how to pass to the limit in $\rho_\varepsilon u_\varepsilon^2$, since we have only weak convergence of $\sqrt{\rho_\varepsilon}u_\varepsilon$ in L^2. Moreover, we loose the control on u_ε and obtain results for the current density $J = \lim_{\varepsilon\to 0} \rho_\varepsilon u_\varepsilon$ only. In order to overcome these difficulties, we multiply the momentum equation by $\rho_\varepsilon^{3/2}$ and pass to the limit $\varepsilon \to 0$ in the resulting equation. This allows us to control the convective part since

$$\rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon^2)_x = \left(\sqrt{\rho_\varepsilon}(\rho_\varepsilon u_\varepsilon)^2\right)_x - 3(\sqrt{\rho_\varepsilon})_x(\rho_\varepsilon u_\varepsilon)^2.$$

Our second main result is summarized in the following theorem.

Theorem 2. *Let $T > 0$ and let the assumptions of Theorem 1 hold. Then, for $\varepsilon = 0$ there exists a weak solution (ρ, J, V) to (1)-(3) with the regularity*

$$\begin{aligned}
&\rho(t,x) \geq 0 \quad \text{for } t > 0,\ x \in \mathbb{T}, \quad V \in L^\infty(0,T;H^2(\mathbb{T})),\\
&\rho_t \in L^2(0,T;L^2(\mathbb{T})), \quad (\rho^{3/2}J)_t \in L^2(0,T;H^{-1}(\mathbb{T})),\\
&\sqrt{\rho} \in L^\infty(0,T;H^1(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T})), \quad J \in L^2(0,T;H^1(\mathbb{T})),
\end{aligned}$$

satisfying $\rho_t + J_x = \nu\rho_{xx}$ and $\lambda^2 V_{xx} = \rho - C(x)$ almost everywhere in $(0,T) \times \mathbb{T}$ and, for *all* $\phi \in L^\infty(0,T;\, H^1(\mathbb{T}))$,

$$\int_0^T \langle(\rho^{3/2}J)_t, \phi\rangle_{H^{-1},H^1} dt - \frac{3}{2}\int_0^T\int_{\mathbb{T}} \sqrt{\rho}\rho_t J\phi dxdt \tag{7}$$
$$- \int_0^T\int_{\mathbb{T}} J^2\big(3(\sqrt{\rho})_x\phi + \sqrt{\rho}\phi_x\big)dxdt + \int_0^T\int_{\mathbb{T}} \big((p(\rho))_x - \rho V_x\big)\rho^{3/2}\phi dxdt$$
$$+ \frac{\delta^2}{2}\int_0^T\int_{\mathbb{T}} (\sqrt{\rho})_{xx}\big(5\rho^{3/2}(\sqrt{\rho})_x\phi + \rho^2\phi_x\big)dxdt$$
$$= -\nu\int_0^T\int_{\mathbb{T}} J_x\rho\big(3(\sqrt{\rho})_x\phi + \sqrt{\rho}\phi_x\big)dxdt - \frac{1}{\tau}\int_0^T\int_{\mathbb{T}} \rho^{3/2}J\phi dxdt.$$

The initial conditions are fulfilled in the following sense:

$\rho(0,\cdot) = \rho_0$ *in* $L^2(\mathbb{T})$, $\quad (\rho^{3/2}J)(0,\cdot) = \rho_0^{5/2}u_0$ *in* $H^{-1}(\mathbb{T})$.

Equation (7) is the weak formulation of

$$(\rho^{3/2}J)_t - (\rho^{3/2})_t J + (\sqrt{\rho}J^2)_x - 3J^2(\sqrt{\rho})_x - \frac{\delta^2}{2}\big(\rho^2(\sqrt{\rho})_{xx}\big)_x + \frac{5\delta^2}{8}(\rho^2)_x(\sqrt{\rho})_{xx}$$
$$- \nu(\rho^{3/2}J_x)_x + \nu J_x(\rho^{3/2})_x + \rho^{3/2}\Big((p(\rho))_x - \rho V_x + \frac{J}{\tau}\Big) = 0,$$

which is obtained from (2) after multiplication of $\rho^{3/2}$ and setting $J = \rho u$. If the limit density ρ is positive and smooth, we can divide the above equation by $\rho^{3/2}$ and recover the original formulation (2). We remark that Chen and Dreher [5] have shown the existence of global solutions to (1)-(3) which possess more regularity (essentially $\rho(\cdot,t) \in H^3$ and $J(\cdot,t) \in H^2$), thus allowing for the original formulation. However, their proof only works if the doping profile is constant and if the initial energy is sufficiently small. Theorem 2 is valid for any doping profile in $L^2(\mathbb{T})$ and for any value of the initial energy, but we obtain less regular solutions than [5].

The paper is organized as follows. In the next section, we solve, for $\varepsilon > 0$ and given velocity u, equation (1) for ρ, prove a lower bound for ρ only depending on the L^2 norm of u_x, and solve (2) locally in time. In section 3 we show the energy estimates for (5) and infer a global existence result for the nonlinear Faedo-Galerkin problem. Theorem 1 is proved in section 4, whereas section 5 is concerned with the proof of Theorem 2. We remark that the a priori estimates derived from the energy functional (5) and its corresponding energy production were already employed in [5,12,18].

2. Linear Faedo-Galerkin approximation

In this section, we prove the existence of solutions to the linearized viscous quantum hydrodynamic equations with $\varepsilon > 0$. Let $T > 0$ and let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{T})$ which is also an orthogonal basis of $H^1(\mathbb{T})$. For instance, one may take the eigenfunctions of $-\partial_x^2$ with eigenvalues $\mu_n > 0$, given by

$$e_{2n}(x) = \sqrt{2}\cos(2n\pi x), \quad \mu_{2n} = 8n^2\pi^2,$$
$$e_{2n+1}(x) = \sqrt{2}\sin(2n\pi x), \quad \mu_{2n+1} = 8n^2\pi^2, \quad n \in \mathbb{N}_0.$$

Introduce the finite-dimensional space $X_n = \text{span}(e_0, \dots, e_n)$. We denote by $C^k(0, T; Z)$ the space of C^k functions on $[0, T]$ with values in the Banach space Z. Furthermore, let $(\rho_0, u_0) \in C^\infty(\mathbb{T})^2$ be some initial data satisfying $\rho_0(x) \geq \eta_0 > 0$ for $x \in \mathbb{T}$ and $\int_{\mathbb{T}} \rho_0 dx = \int_{\mathbb{T}} C(x)dx$. Finally, let $v \in C^0(0, T; X_n)$ be given. We notice that v can be written as

$$v(t, x) = \sum_{i=1}^{n} \lambda_i(t) e_i(x), \quad t \in [0, T],\ x \in \mathbb{T},$$

for some $\lambda_i(t)$, and we have

$$\|v\|_{C^0(0,T;X_n)} = \max_{t \in [0,T]} \sum_{i=1}^{n} |\lambda_i(t)|.$$

As a consequence, v can be bounded in $C^0(0, T; C^k(\mathbb{T}))$ for any $k \in \mathbb{N}$, and there exists a constant $K_k > 0$ depending on k such that

$$\|v\|_{C^0(0,T;C^k(\mathbb{T}))} \leq K_k \|v\|_{C^0(0,T;L^2(\mathbb{T}))}.$$

Now, we define the approximate system. Let ρ be the classical solution to

$$\rho_t + (\rho v)_x = \nu \rho_{xx}, \quad x \in \mathbb{T},\ t > 0, \tag{8}$$
$$\rho(0, x) = \rho_0(x), \quad x \in \mathbb{T}. \tag{9}$$

The solution satisfies $\rho \in C^0(0, T; C^k(\mathbb{T}))$ for any $k \in \mathbb{N}$. Furthermore, it holds $\int_{\mathbb{T}} \rho dx = \int_{\mathbb{T}} \rho_0 dx = \int_{\mathbb{T}} C(x)dx$. We introduce the operator $S : C^0(0, T; X_n) \to C^0(0, T; C^3(\mathbb{T}))$ by $S(v) = \rho$. Since v is smooth, the maximum principle shows that $\rho = S(v)$ is bounded from above and below, i.e., for $\|v\|_{C^0(0,T;L^2(\mathbb{T}))} \leq c$, there exist positive constants $K_0(c)$ and $K_1(c)$ depending on c such that

$$0 < K_0(c) \leq (S(v))(t, x) \leq K_1(c), \quad t \in [0, T],\ x \in \mathbb{T}. \tag{10}$$

Furthermore, since the equation for ρ is linear, there exists $K_2 > 0$ depending on k and n such that for all v_1, $v_2 \in C^0(0, T; X_n)$,

$$\|S(v_1) - S(v_2)\|_{C^0(0,T;C^k(\mathbb{T}))} \leq K_2 \|v_1 - v_2\|_{C^0(0,T;L^2(\mathbb{T}))}. \tag{11}$$

We claim that the lower bound for $S(v)$ only depends on the $L^2(0, T; L^2(\mathbb{T}))$ norm of v_x,

$$\rho = S(v) \geq \eta = \eta(\|v_x\|_{L^2(0,T;L^2(\mathbb{T}))}) > 0 \quad \text{in } [0, T] \times \mathbb{T}. \tag{12}$$

This result is a consequence of the following lemma whose proof is presented at the end of this section.

Lemma 3. *Let $T > 0$ and $v \in L^2(0, T; H^1(\mathbb{T}))$. Let ρ be the solution to (8)-(9) with initial datum $\rho_0 \in L^\infty(\mathbb{T})$ satisfying $\rho_0(x) \geq \eta_0 > 0$ for $x \in \mathbb{T}$. Then there exists a constant $\eta > 0$ only depending on ν, ρ_0, and the $L^2(0, T; L^2(\mathbb{T}))$ norm of v_x such that*

$$\rho(t, x) \geq \eta > 0, \quad t \in [0, T],\ x \in \mathbb{T}.$$

Next, for given $\rho = S(v)$, we wish to solve the following linear problem on X_n for u_n:

$$(\rho u_n)_t + (\rho v u_n + p(\rho))_x - \rho(V[\rho])_x - \frac{\delta^2}{2}\rho\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x \tag{13}$$
$$= \nu(\rho u_n)_{xx} + \varepsilon(u_n)_{xx} - \frac{\rho u_n}{\tau},$$

where $V[\rho] \in C^0(0,T;C^2(\mathbb{T}))$ is the unique solution to

$$\lambda^2(V[\rho])_{xx} = \rho - C(x) \quad \text{in } \mathbb{T} \tag{14}$$

satisfying $\int_{\mathbb{T}} V dx = 0$. More explicitly, we are looking for a function $u_n \in C^0(0,T;\, X_n)$ verifying, for all test functions $\phi \in C^1(0,T;X_n)$ with $\phi(T,\cdot) = 0$,

$$\int_{\mathbb{T}} \rho u_n \phi_t dx + \int_{\mathbb{T}} \big(\rho v u_n + p(\rho)\big)\phi_x dx + \int_{\mathbb{T}} \rho(V[\rho])_x \phi dx - \frac{\delta^2}{2}\int_{\mathbb{T}} \frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}(\rho\phi)_x dx$$
$$- \int_{\mathbb{T}} \big(\nu\rho u_n + \varepsilon u_n\big)_x \phi_x dx - \frac{1}{\tau}\int_{\mathbb{T}} \rho u_n \phi dx = \int_{\mathbb{T}} \rho_0 u_0 \phi(0,\cdot) dx.$$

For given $\rho \in N_\eta = \{\rho \in L^1(\mathbb{T}) : \inf_{x\in\mathbb{T}} \rho \geq \eta > 0\}$, we introduce the following family of operators, following [10]:

$$M[\rho] : X_n \to X_n^*, \quad \langle M[\rho]u, w\rangle = \int_{\mathbb{T}} \rho u w dx, \quad u, w \in X_n.$$

These operators are symmetric and positive definite with the smallest eigenvalue

$$\inf_{\|w\|_{L^2(\mathbb{T})}=1} \langle M[\rho]w, w\rangle = \inf_{\|w\|_{L^2(\mathbb{T})}=1} \int_{\mathbb{T}} \rho w^2 dx \geq \inf_{x\in\mathbb{T}} \rho(x) > \eta,$$

employing the bound (12). Hence, as we are working in finite dimensions, the operators are invertible with

$$\|M^{-1}[\rho]\|_{\mathcal{L}(X_n^*,X_n)} \leq \eta^{-1},$$

where $\mathcal{L}(X_n^*, X_n)$ is the set of bounded linear mappings from X_n^* to X_n. Moreover, similar as in [10], it holds:

$$\|M^{-1}[\rho_1] - M^{-1}[\rho_2]\|_{\mathcal{L}(X_n^*,X_n)} \leq K(n,\eta)\|\rho_1 - \rho_2\|_{L^1(\mathbb{T})} \tag{15}$$

for all ρ_1, $\rho_2 \in N_\eta$. With these notations, we can rephrase problem (13) as an ordinary differential equation on the finite-dimensional space X_n:

$$\frac{d}{dt}\big(M[\rho(t)]u_n(t)\big) = N[v, u_n(t)], \quad t > 0, \quad M[\rho_0]u_n(0) = M[\rho_0]u_0, \tag{16}$$

where

$$\langle N[v,u_n], \phi\rangle = \int_{\mathbb{T}} \Big(- \big(\rho v u_n + p(\rho)\big)_x + \frac{\delta^2}{2}\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x + \rho(V[\rho])_x$$
$$+ \nu(\rho u_n)_{xx} + \varepsilon(u_n)_{xx} - \frac{1}{\tau}\rho u_n\Big)\phi dx, \quad \phi \in X_n.$$

Recall that $\rho = S(v) \in C^0(0,T;C^3(\mathbb{T}))$ is bounded from below, so the above integral is well defined. The operator $N[v,\cdot]$, defined for every $t \in [0,T]$ as an operator from X_n to X_n^*, is continuous in time. Then, standard theory of finite-dimensional systems of differential equations provides the existence of a unique C^1 solution of (16). In other words, there exists a unique solution $u_n \in C^1(0,T;X_n)$ to (13).

Proof of Lemma 3. We introduce the function

$$L(t,x) = \ln \frac{1}{\rho}\left(t, x + \int_0^t \int_{\mathbb{T}} v(s,y)dyds\right),$$

which is a solution to

$$L_t - \nu L_{xx} = v_x - \tilde{v}L_x - \nu(L_x)^2,$$

where

$$\tilde{v} = v - \int_{\mathbb{T}} vdx.$$

Since

$$|\tilde{v}L_x| = \left|\sqrt{\frac{1}{2\nu}}\tilde{v}\sqrt{2\nu}L_x\right| \leq \frac{\tilde{v}^2}{4\nu} + \nu(L_x)^2,$$

we obtain

$$L_t - \nu L_{xx} \leq v_x + \frac{\tilde{v}^2}{4\nu}.$$

The idea is to show an upper bound for L which only depend on η_0, ν, and the L^2-norm of v_x and from which the lower bound for ρ follows. This is achieved by estimating the solution ψ to a certain parabolic problem and using the comparison principle to obtain $L \leq \psi$. We introduce the functions ψ_1, which is a solution to

$$(\psi_1)_t - \nu(\psi_1)_{xx} = v_x, \quad x \in \mathbb{T},\ t > 0,$$
$$\psi_1(0,x) = 0, \quad x \in \mathbb{T},$$

and ψ_2, which solves

$$(\psi_2)_t - \nu(\psi_2)_{xx} = \frac{\tilde{v}^2}{4\nu}, \quad x \in \mathbb{T},\ t > 0,$$
$$\psi_1(0,x) = L(0,x) = \ln \frac{1}{\rho_0(x)}, \quad x \in \mathbb{T}.$$

First, notice that $\tilde{v}_x = v_x$ and $\int_{\mathbb{T}} \tilde{v}\, dx = 0$. Hence, by the Poincaré inequality in one space dimension,

$$\|\tilde{v}^2\|_{L^1(0,T;L^\infty(\mathbb{T}))} = \int_0^T \|\tilde{v}\|^2_{L^\infty(\mathbb{T})}dt \leq \int_0^T \|\tilde{v}_x\|^2_{L^2(\mathbb{T})}dt = \|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))}.$$

This shows that

$$\psi_2(t,x) \le \frac{1}{4\nu}\|v_x\|^2_{L^2(L^2)} + \ln\frac{1}{\eta_0}, \quad t>0, x\in\mathbb{T}. \tag{17}$$

Multiplying the equation for ψ_1 by $-(\psi_1)_{xx}$ and integrating over $\mathbb{T}$, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}(\psi_1)_x^2dx + \nu\int_{\mathbb{T}}(\psi_1)_{xx}^2dx &= -\int_{\mathbb{T}} v_x(\psi_1)_{xx}\,dx\\ &\le \frac{1}{4\nu}\int_{\mathbb{T}} v_x^2dx + \nu\int_{\mathbb{T}}(\psi_1)_{xx}^2dx,\end{aligned}$$

from which we conclude that

$$\|(\psi_1)_x\|^2_{L^\infty(0,T;L^2(\mathbb{T}))} \le \frac{1}{2\nu}\|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))}.$$

Finally, for any $t>0$, the integral of $\psi_1(t,\cdot)$ over $\mathbb{T}$ vanishes, and an application of the Poincaré inequality then gives

$$\|\psi_1\|_{L^\infty(0,T;L^\infty(\mathbb{T}))} \le \frac{1}{2\nu}\|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))}. \tag{18}$$

Consider now the sum $\psi=\psi_1+\psi_2$ which is a solution to

$$\begin{aligned}&\psi_t - \nu\psi_{xx} = v_x + \frac{\tilde{v}^2}{4\nu}, \quad x\in\mathbb{T},\ t>0,\\ &\psi(0,x) = L(0,x) = \ln\frac{1}{\rho_0(x)}, \quad x\in\mathbb{T}.\end{aligned}$$

Then, by the comparison principle, $L\le\psi$ in $[0,T]\times\mathbb{T}$, and, together with (18) and (17), we obtain for any $t>0$ and $x\in\mathbb{T}$:

$$L(t,x) \le \frac{1}{\nu}\|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))} + \ln\frac{1}{\eta_0}.$$

This leads to

$$\rho(t,x) \ge \eta_0\exp\Big(-\frac{1}{\nu}\|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))}\Big),$$

for every $x\in\mathbb{T}$, $t>0$. $\square$

3. Solution of the nonlinear approximate problem

In this section, we show that there exists a solution to the system (8)-(9) and (13) on $\mathbb{T}$. More precisely, we prove the following result.

Proposition 4. *Let the assumptions of Theorem 1 hold and let the initial data be smooth with positive particle density. Then there exists a solution* $(\rho,u_n)\in C^0(0,T;C^3(\mathbb{T}))\times$

$C^1(0,T;X_n)$ to (8)-(9) and (13), with $v = u_n$ and $\rho = \rho_n = S(u_n)$, satisfying the following estimates:

$$\rho_n(t,x) \geq \eta(\varepsilon) > 0, \quad t \in [0,T],\ x \in \mathbb{T}, \tag{19}$$
$$\|\sqrt{\rho_n}\|_{L^\infty(0,T;H^1(\mathbb{T}))} + \|\sqrt{\rho_n}\|_{L^2(0,T;H^2(\mathbb{T}))} \leq K, \tag{20}$$
$$\|\sqrt{\rho_n}u_n\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|\sqrt{\rho_n}(u_n)_x\|_{L^2(0,T;L^2(\mathbb{T}))} \leq K, \tag{21}$$
$$\varepsilon\|(u_n)_x\|_{L^2(0,T;L^2(\mathbb{T}))} \leq K, \tag{22}$$
$$\|V[\rho_n]\|_{L^\infty(0,T;H^1(\mathbb{T}))} \leq K, \tag{23}$$

where $\eta(\varepsilon) > 0$ depends on ε, the initial data and the $L^2(\mathbb{T})$ norm of $C(x)$, and $K > 0$ only depends on ν, λ, the initial data, and $C(x)$. The potential $V[\rho_n]$ is defined by (14) with $\rho = \rho_n$.

Proof. Integrating (16) over $(0,t)$, we can write the problem as the following nonlinear equation:

$$u_n(t) = M^{-1}[(S(u_n))(t)]\Big(M[\rho_0](u_0) + \int_0^t N[u_n, u_n(s)]ds\Big) \quad \text{in } X_n.$$

Taking into account (11) and (15), this equation can be solved with the fixed-point theorem of Banach, at least on a short time interval $[0,T']$, where $T' \leq T$, in the space $C^0(0,T';X_n)$. In fact, we obtain even $u_n \in C^1(0,T';X_n)$. We have to show that we can choose $T' = T$. It is sufficient to prove that u_n is bounded in X_n on the whole interval $[0,T']$. This is achieved by employing the energy estimate. We multiply (8) by $\phi = h(\rho_n) - V[\rho_n] - u_n^2/2 - (\delta^2/2)(\sqrt{\rho_n})_{xx}/\sqrt{\rho_n}$, use the test function u_n in (13), with $v = u_n$ and $\rho = \rho_n$, and add both equations. This leads to

$$\begin{aligned}
0 &= \int_{\mathbb{T}} \Big((\rho_n)_t h(\rho_n) - (\rho_n)_t \frac{u_n^2}{2} + (\rho_n u_n)_t u_n\Big)dx \\
&\quad + \int_{\mathbb{T}} \big(- (\rho_n)_t V[\rho_n] - (\rho_n u_n)_x V[\rho_n] + \nu(\rho_n)_{xx} V[\rho_n] - \rho_n (V[\rho_n])_x u_n\big)dx \\
&\quad + \int_{\mathbb{T}} \big((\rho_n u_n)_x h(\rho_n) + (p(\rho_n))_x u_n\big)dx \\
&\quad - \frac{\delta^2}{2}\int_{\mathbb{T}} \Big((\rho_n u_n)_x \frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}} + \rho_n u_n \Big(\frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}}\Big)_x + (\rho_n)_t \frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}}\Big)dx \\
&\quad + \int_{\mathbb{T}} \Big(- \frac{1}{2}(\rho_n u_n)_x u_n^2 + (\rho_n u_n^2)_x u_n\Big)dx \\
&\quad + \nu \int_{\mathbb{T}} \Big(- (\rho_n)_{xx} h(\rho_n) + \frac{1}{2}(\rho_n)_{xx} u_n^2 + \frac{\delta^2}{2}(\rho_n)_{xx}\frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}} - (\rho_n u_n)_{xx} u_n\Big)dx \\
&\quad - \varepsilon \int_{\mathbb{T}} u_n (u_n)_{xx} dx + \frac{1}{\tau}\int_{\mathbb{T}} \rho_n u_n^2 dx \\
&= I_1 + \cdots + I_8.
\end{aligned}$$

Notice that at this point, we need a pointwise solution to (8) such that this equation can be multiplied by ϕ. If (8) was solved in a Galerkin space only, we could not use ϕ as a test function since it is not admissible. On the other hand, u_n is an admissible test function for the Galerkin equation (13).

We estimate the above expression integral by integral. The first integral can be reformulated as

$$I_1 = \partial_t \int_{\mathbb{T}} \Big(H(\rho) + \frac{1}{2}\rho_n u_n^2 \Big) dx,$$

where we recall that H is a primitive of h. The second integral becomes, after integrating by parts and employing the Poisson equation,

$$\begin{aligned} I_2 &= \int_{\mathbb{T}} \big(-\lambda^2 (V[\rho_n])_{xxt} V[\rho_n] + \nu \rho_n (V[\rho_n])_{xx} \big) dx \\ &= \int_{\mathbb{T}} \Big(\frac{\lambda^2}{2} \partial_t (V[\rho_n])_x^2 + \nu \lambda^{-2} \rho_n (\rho_n - C(x)) \Big) dx. \end{aligned}$$

Integrating by parts in the first member of the third integral, we see that

$$I_3 = \int_{\mathbb{T}} \big(\rho_n u_n h'(\rho_n)(\rho_n)_x + p'(\rho_n)(\rho_n)_x u_n \big) dx = 0,$$

since, by definition, $p'(\rho_n) = \rho_n h'(\rho_n)$. Again by integrating by parts, the fourth integral simplifies to

$$I_4 = -\delta^2 \int_{\mathbb{T}} (\sqrt{\rho_n})_t (\sqrt{\rho_n})_{xx} dx = \frac{\delta^2}{2} \partial_t \int_{\mathbb{T}} (\sqrt{\rho_n})_x^2 dx.$$

The fifth integral vanishes since, in view of the periodic boundary conditions,

$$I_5 = \frac{1}{2} \int_{\mathbb{T}} (\rho_n u_n^3)_x dx = 0.$$

Integrating by parts in the sixth integral gives

$$\begin{aligned} I_6 = \nu \int_{\mathbb{T}} \Big(& h'(\rho_n)(\rho_n)_x^2 - (\rho_n)_x u_n (u_n)_x + \delta^2 (\sqrt{\rho_n})_{xx}^2 + \frac{\delta^2}{3} \frac{((\sqrt{\rho_n})_x^3)_x}{\sqrt{\rho_n}} \\ & + (\rho_n u_n)_x (u_n)_x \Big) dx \\ = \nu \int_{\mathbb{T}} \Big(& (G(\rho_n))_x^2 + \rho_n (u_n)_x^2 + \delta^2 (\sqrt{\rho_n})_{xx}^2 + \frac{16}{3} \delta^2 (\sqrt[4]{\rho_n})_x^4 \Big) dx, \end{aligned}$$

where $G'(y) = \sqrt{h'(y)}$, $y \geq 0$. Summarizing, we obtain

$$\begin{aligned} \partial_t \int_{\mathbb{T}} & \Big(H(\rho_n) + \frac{1}{2}\rho_n u_n^2 + \frac{\lambda^2}{2} (V[\rho_n])_x^2 + \frac{\delta^2}{2} (\sqrt{\rho_n})_x^2 \Big) dx \\ & + \nu \int_{\mathbb{T}} \Big((G(\rho_n))_x^2 + \rho_n (u_n)_x^2 + \delta^2 (\sqrt{\rho_n})_{xx}^2 + \frac{16}{3} \delta^2 (\sqrt[4]{\rho_n})_x^4 \Big) dx \\ & + \varepsilon \int_{\mathbb{T}} (u_n)_x^2 dx + \frac{1}{\tau} \int_{\mathbb{T}} \rho_n u_n^2 dx \\ = & -\nu \lambda^{-2} \int_{\mathbb{T}} \rho_n (\rho_n - C(x)) dx \qquad (24) \\ \leq & \frac{\nu}{2\lambda^2} \Big(- \int_{\mathbb{T}} \rho_n^2 dx + \int_{\mathbb{T}} C(x)^2 dx \Big). \qquad (25) \end{aligned}$$

From this estimate the uniform bounds (20)-(23) follow. Using (12) and (22), we infer the lower bound (19). Then, the estimate (21) shows that (u_n) is bounded in $L^\infty(0,T;L^2(\mathbb{T}))$ with a bound which depends on ε. Together with the Lipschitz estimates (11) and (15), this allows us to apply the fixed-point theorem recursively until $T' = T$. □

We end this section by proving some estimates uniform in n and ε.

Lemma 5. *The following estimates holds:*

$$\|\partial_t\rho_n\|_{L^2(0,T;L^2(\mathbb{T}))} + \|\sqrt{\rho_n}\|_{L^6(0,T;W^{1,6}(\mathbb{T}))} \le K, \tag{26}$$
$$\|\rho_n\|_{L^\infty(0,T;H^1(\mathbb{T}))} + \|\rho_n\|_{L^2(0,T;H^2(\mathbb{T}))} \le K, \tag{27}$$
$$\|\partial_t(\rho_n u_n)\|_{L^2(0,T;H^{-2}(\mathbb{T}))} \le K, \tag{28}$$
$$\|\rho_n^\alpha\partial_t(\rho_n u_n)\|_{L^2(0,T;H^{-1}(\mathbb{T}))} \le K, \tag{29}$$

for all $\alpha \ge 1/2$, where $K > 0$ is independent of n and ε.

Proof. By the Galiardo-Nirenberg inequality with $\theta = 1/3$, we have

$$\begin{aligned}\|(\sqrt{\rho_n})_x\|^6_{L^6(0,T;L^6(\mathbb{T}))} &\le K\int_0^T \|(\sqrt{\rho_n})_x\|^{6\theta}_{H^1(\mathbb{T})}\|(\sqrt{\rho_n})_x\|^{6(1-\theta)}_{L^2(\mathbb{T})}dt \\ &\le K\|\sqrt{\rho_n}\|^4_{L^\infty(0,T;H^1(\mathbb{T}))}\int_0^T \|\sqrt{\rho_n}\|^2_{H^2(\mathbb{T})}dt \le K,\end{aligned} \tag{30}$$

taking into account the bound (20). This shows that $\sqrt{\rho_n}$ is bounded in $L^6(0,T;W^{1,6}(\mathbb{T}))$. The function ρ_n solves (8)-(9), with $v = u_n$, written as

$$\partial_t\rho_n = -\sqrt{\rho_n}\sqrt{\rho_n}(u_n)_x - 2\sqrt{\rho_n}u_n(\sqrt{\rho_n})_x + 2\nu\sqrt{\rho_n}(\sqrt{\rho_n})_{xx} + 2\nu(\sqrt{\rho_n})^2_x.$$

In view of (20), (21), and (30), we infer that $\partial_t\rho_n \in L^2(0,T;L^2(\mathbb{T}))$. Furthermore, by (30), $(\rho_n)_{xx} = 2\sqrt{\rho_n}(\sqrt{\rho_n})_{xx} + 2(\sqrt{\rho_n})^2_x$ is bounded in $L^2(0,T;L^2(\mathbb{T}))$.

We claim that $\partial_t(\rho_n u_n)$ is bounded in $L^2(0,T;H^{-2}(\mathbb{T}))$. We have to verify that all terms in (13), with $\rho = \rho_n$ and $v = u_n$, except $\partial_t(\rho_n u_n)$ lie in this space. This is clear for the terms $(p(\rho_n))_x$, $\rho_n(V[\rho_n])_x$, $\varepsilon(u_n)_{xx}$, and $\rho_n u_n/\tau$. Furthermore, $\rho_n u_n^2 = (\sqrt{\rho_n}u_n)^2$ is bounded in $L^\infty(0,T;L^1(\mathbb{T}))$, by (21), such that $(\rho_n u_n^2)_x$ is bounded in $L^\infty(0,T;W^{-1,1}(\mathbb{T})) \hookrightarrow L^\infty(0,T;H^{-2}(\mathbb{T}))$; $\nu(\rho_n u_n)_{xx} = \nu(2\sqrt{\rho_n}u_n(\sqrt{\rho_n})_x + \rho_n(u_n)_x)_x$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$; and

$$\rho_n\Big(\frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}}\Big)_x = \big(\sqrt{\rho_n}(\sqrt{\rho_n})_{xx}\big)_x - 2(\sqrt{\rho_n})_x(\sqrt{\rho_n})_{xx}$$

is bounded in $L^2(0,T;H^{-1})$. This shows the claim.

Next, let $\alpha \ge 1/2$. We want to show that $\rho_n^\alpha\partial_t(\rho_n u_n)$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$. The term

$$\rho_n^\alpha(\rho_n u_n^2)_x = 2\rho_n^{\alpha-1/2}(\sqrt{\rho_n})_x\rho_n u_n^2 + 2\rho_n^\alpha\sqrt{\rho_n}u_n\sqrt{\rho_n}(u_n)_x$$

is bounded in $L^2(0,T;L^1(\mathbb{T}))$ and hence also in $L^2(0,T;H^{-1}(\mathbb{T}))$. Notice that we have used here that $\alpha \ge 1/2$. If $\alpha \ge 0$ only, the bound depends on ε through the lower bound of

ρ_n. Furthermore, $\rho_n^\alpha(p(\rho_n))_x$, $\rho^{\alpha+1}(V[\rho_n])_x$, and $\rho^{\alpha+1}u_n/\tau$ are bounded in $L^2(0,T;L^2(\mathbb{T}))$. The first term of

$$\rho_n^{\alpha+1}\Big(\frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}}\Big)_x = \big(\rho_n^{\alpha+1/2}(\sqrt{\rho_n})_{xx}\big)_x - 2(\alpha+1)\rho_n^\alpha(\sqrt{\rho_n})_x(\sqrt{\rho_n})_{xx}$$

is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$, the second term in $L^2(0,T;L^1(\mathbb{T}))$, so the sum is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$. Similarly, the sequences

$$\begin{aligned}\varepsilon\rho_n^\alpha(u_n)_{xx} &= \big(\rho_n^\alpha\varepsilon(u_n)_x\big)_x - 2\alpha\rho_n^{\alpha-1/2}(\sqrt{\rho_n})_x\varepsilon(u_n)_x,\\ \rho_n^\alpha(\rho_n u_n)_{xx} &= \big(\rho_n^{\alpha+1/2}\sqrt{\rho_n}(u_n)_x\big)_x - 2(\alpha-1)\rho_n^\alpha(\sqrt{\rho_n})_x\sqrt{\rho_n}(u_n)_x\\ &\quad + \rho_n^{\alpha-1/2}(\rho_n)_{xx}\sqrt{\rho_n}u_n\end{aligned}$$

are bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$. □

4. Proof of Theorem 1

In this section, we perform the limit $n\to\infty$, for fixed $\varepsilon>0$, in the system (8)-(9), (13), and (14), with $\rho=\rho_n$ and $v=u_n$.

In view of (26), (27), and the compactness of the embeddings $H^1(\mathbb{T})\hookrightarrow L^\infty(\mathbb{T})$ and $H^2(\mathbb{T})\hookrightarrow H^1(\mathbb{T})$, the Aubin lemma provides the existence of a subsequence of (ρ_n) (not relabeled) such that, as $n\to\infty$,

$$\begin{aligned}\rho_n &\to \rho \quad \text{strongly in } L^2(0,T;H^1(\mathbb{T})) \text{ and } L^\infty(0,T;L^\infty(\mathbb{T})),\\ \rho_n &\rightharpoonup \rho \quad \text{weakly in } L^2(0,T;H^2(\mathbb{T})),\\ \partial_t\rho_n &\rightharpoonup \partial_t\rho \quad \text{weakly in } L^2(0,T;L^2(\mathbb{T})).\end{aligned}$$

Since (ρ_n) is bounded from below, $(\sqrt{\rho_n})_x$ converges weakly (up to a subsequence) to $(\sqrt{\rho})_x$ in $L^2(0,T;L^2(\mathbb{T}))$. Moreover, since $\varepsilon>0$ is fixed, by (21) and (22), u_n converges weakly to a function u in $L^2(0,T;H^1(\mathbb{T}))$. These results show that

$$\partial_t\rho_n + (\rho_n u_n)_x - \nu(\rho_n)_{xx} \rightharpoonup \partial_t\rho + (\rho u)_x - \nu\rho_{xx} \quad \text{weakly in } L^1(0,T;L^2(\mathbb{T}))$$

and that

$$(p(\rho_n))_x - \frac{\delta^2}{2}\rho_n\Big(\frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}}\Big)_x - \nu(\rho_n u_n)_{xx} - \varepsilon(u_n)_{xx} + \frac{1}{\tau}\rho_n u_n$$

converges weakly in $L^1(0,T;H^{-1}(\mathbb{T}))$ to

$$(p(\rho))_x - \frac{\delta^2}{2}\rho\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x - \nu(\rho u)_{xx} - \varepsilon u_{xx} + \frac{1}{\tau}\rho u.$$

We also have $V[\rho_n]\to V[\rho]$ in $L^\infty(0,T;H^2(\mathbb{T}))$. In order to pass to the limit in the convection term, we observe first that $\rho_n u_n \rightharpoonup \rho u$ weakly* in $L^2(0,T;L^\infty(\mathbb{T}))$ since (ρ_n) converges strongly in $L^\infty(0,T;L^\infty(\mathbb{T}))$ and (u_n) converges weakly* in $L^2(0,T;L^\infty(\mathbb{T}))$.

On the other hand, taking into account (28) and the bound for $(\rho_n u_n)$ in $L^2(0,T;H^1(\mathbb{T}))$, Aubin's lemma implies that $\rho_n u_n \to \rho u$ strongly in $L^2(0,T;L^\infty(\mathbb{T}))$. Thus,

$$\rho_n u_n^2 \rightharpoonup \rho u^2 \quad \text{weakly* in } L^1(0,T;L^\infty(\mathbb{T})).$$

Thus, passing to the limit $n \to \infty$ in (13), with $\rho = \rho_n$ and $v = u_n$, shows that $(\rho, u, V[\rho])$ is a solution to (1)-(3) for $\varepsilon > 0$. This finishes the proof of Theorem 1.

Remark 6. The construction of approximate solutions of section 2 can be generalized to the multi-dimensional quantum hydrodynamic equations

$$\rho_t + \operatorname{div}(\rho u) = \nu \Delta \rho, \tag{31}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \rho \nabla V - \frac{\delta^2}{2} \rho \nabla \Big(\frac{\Delta \sqrt{n}}{\sqrt{n}} \Big) = \nu \Delta(\rho u) + \varepsilon \Delta u - \frac{\rho u}{\tau}, \tag{32}$$

$$\lambda^2 \Delta V = \rho - C(x), \quad x \in \mathbb{T}^d. \tag{33}$$

Indeed, let X_n be a finite-dimensional space defined, for instance, by the span of the first $n+1$ eigenfunctions of $-\Delta$ on $L^2(\mathbb{T}^d)$. Further, let ρ_n be the classical solution to (31) with u replaced by some given function $v \in C^0(0,T;X_n)$, and let $V[\rho_n]$ be the unique solution to (33). By the maximum principle, ρ_n is strictly positive with a bound depending on the L^∞ norm of $\operatorname{div} v$. Finally, we can define u_n to be the solution to (32), projected on X_n, in the sense of section 2. The nonlinear finite-dimensional problem then is solved by employing Banach's fixed-point theorem, giving a local-in-time solution $u_n \in C^0(0,T';X_n)$ on the time interval $[0,T']$.

The energy estimate (25) can also be generalized to the multi-dimensional problem (see, e.g., [5]):

$$\begin{aligned}
&\partial_t \int_{\mathbb{T}^d} \Big(H(\rho_n) + \frac{1}{2}\rho_n |u_n|^2 + \frac{\lambda^2}{2} |\nabla V[\rho_n]|^2 + \frac{\delta^2}{2} |\nabla \sqrt{\rho_n}|^2 \Big) dx \\
&\quad + \nu \int_{\mathbb{T}^d} \big(|\nabla G(\rho_n)|^2 + \rho_n \|\nabla u_n\|^2 + \delta^2 \rho_n \|\nabla^2 \log \rho_n\|^2 \big) dx \\
&\quad + \varepsilon \int_{\mathbb{T}^d} \|\nabla u_n\|^2 dx + \frac{1}{\tau} \int_{\mathbb{T}^d} \rho_n |u_n|^2 dx \le \frac{\nu}{4\lambda^2} \int_{\mathbb{T}^d} C(x)^2 dx,
\end{aligned}$$

where $\|\cdot\|$ denotes the ℓ^2 norm of a matrix and $\nabla^2 \log \rho_n$ the Hessian of $\log \rho_n$. By the estimate (1.3) of [22] (also see Proposition A.1 in [5]),

$$\int_{\mathbb{T}^d} \rho_n \|\nabla^2 \log \rho_n\|^2 dx \ge c \int_{\mathbb{T}^d} \|\nabla^2 \rho_n\|^2 dx,$$

for some constant $c > 0$, which provides a uniform $L^2(0,T';H^2(\mathbb{T}^d))$ estimate for $\sqrt{\rho_n}$. Moreover, $\sqrt{\rho_n}$ is uniformly bounded in $L^\infty(0,T';H^1(\mathbb{T}^d))$.

Provided that the lower bound for ρ_n is independent of n and only depends on the $L^2(0,T';L^2(\mathbb{T}^d))$ norm of ∇u_n, it is possible to perform the limit $n \to \infty$. The most difficult parts are the limits in the third-order expression and the convective term. Similarly as in the one-dimensional case, we can prove uniform bounds for ρ_n and $\rho_n u_n$ which enable us to apply Aubin's lemma, thus providing the strong convergence of these sequences and allowing us to pass to the limit $n \to \infty$. The problem, however, is to prove the lower bound for ρ_n only depending on the $L^2(0,T';L^2(\mathbb{T}^d))$ norm of ∇u_n. Indeed, the proof of Lemma 3 relies on certain L^∞ estimates for ψ_i and Sobolev embeddings which are only valid in one space dimension, and we are not able to extend them to the multi-dimensional situation.

5. Proof of Theorem 2

Let $(\rho_\varepsilon, u_\varepsilon, V[\rho_\varepsilon])$ be the solution to (1)-(3), for $\varepsilon > 0$, constructed in the previous section. In this section, we will perform the limit $\varepsilon \to 0$.

By the ε-independent estimates (26) and (27), the Aubin lemma gives the existence of a subsequence (again not relabeled) such that

$$\begin{aligned} \rho_\varepsilon \to \rho \quad & \text{strongly in } L^2(0,T;H^1(\mathbb{T})) \text{ and in } L^\infty(0,T;L^\infty(\mathbb{T})), \\ \rho_\varepsilon \rightharpoonup \rho \quad & \text{weakly in } L^2(0,T;H^2(\mathbb{T})), \\ \partial_t\rho_\varepsilon \rightharpoonup \partial_t\rho \quad & \text{weakly in } L^2(0,T;L^2(\mathbb{T})). \end{aligned}$$

Furthermore, by (20), up to a subsequence,

$$\sqrt{\rho_\varepsilon} \rightharpoonup \sqrt{\rho} \quad \text{weakly* in } L^\infty(0,T;H^1(\mathbb{T})) \text{ and weakly in } L^2(0,T;H^2(\mathbb{T})).$$

By (21),

$$(\rho_\varepsilon u_\varepsilon)_x = 2(\sqrt{\rho_\varepsilon})_x\sqrt{\rho_\varepsilon}u_\varepsilon + \sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon}(u_\varepsilon)_x$$

is bounded in $L^2(0,T;L^2(\mathbb{T}))$, and hence, $(\rho_\varepsilon u_\varepsilon)$ is bounded in $L^2(0,T;H^1(\mathbb{T}))$. Therefore,

$$\rho_\varepsilon u_\varepsilon \rightharpoonup J \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T})).$$

Thus, by (26) and (27), letting $\varepsilon \to 0$ in the mass conservation equation (8) with $\rho = \rho_\varepsilon$ and $v = u_\varepsilon$ yields

$$\rho_t + J_x = \nu\rho_{xx} \quad \text{in } L^2(0,T;L^2(\mathbb{T})).$$

In order to let $\varepsilon \to 0$ in the momentum equation, we need to multiply (13) by $\rho_\varepsilon^{3/2}$. The reason is that we cannot control (u_ε) but only $(\rho_\varepsilon u_\varepsilon)$ which makes it difficult to pass to the limit in $(\rho_\varepsilon u_\varepsilon^2)_x$. The $L^2(0,T;H^1(\mathbb{T}))$ bound for $(\rho_\varepsilon u_\varepsilon)$ together with (28) implies that, by Aubin's lemma,

$$\rho_\varepsilon u_\varepsilon \to J \quad \text{strongly in } L^2(0,T;L^\infty(\mathbb{T})).$$

Thus, for any test function $\phi \in L^\infty(0,T;H^1(\mathbb{T}))$, as $\varepsilon \to 0$,

$$\begin{aligned} \int_{\mathbb{T}} \rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon^2)_x\phi dx &= -\int_{\mathbb{T}} \big(3(\sqrt{\rho_\varepsilon})_x\phi + \sqrt{\rho_\varepsilon}\phi_x\big)(\rho_\varepsilon u_\varepsilon)^2 dx \\ &\to \int_{\mathbb{T}} \big(3(\sqrt{\rho})_x\phi + \sqrt{\rho}\phi_x\big)J^2 dx. \end{aligned}$$

By (29), $\rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon)_t$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$. Hence, also

$$(\rho_\varepsilon^{5/2}u_\varepsilon)_t = \rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon)_t + \frac{3}{2}\rho_\varepsilon(\rho_\varepsilon)_t(\sqrt{\rho_\varepsilon}u_\varepsilon)$$

is bounded in this space and we infer that

$$(\rho_\varepsilon^{5/2}u_\varepsilon)_t \rightharpoonup (\rho^{3/2}J)_t \quad \text{weakly in } L^2(0,T;H^{-1}(\mathbb{T})). \tag{34}$$

Using $\rho_\varepsilon^{3/2}\phi$ with $\phi \in L^\infty(0,T;H^1(\mathbb{T}))$ as a test function in the weak formulation of (2), it holds

$$\begin{aligned}
0 &= \int_0^T\int_{\mathbb{T}} \rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon)_t\phi dxdt - \int_0^T\int_{\mathbb{T}}\big(\rho_\varepsilon u_\varepsilon^2 + p(\rho_\varepsilon)\big)(\rho_\varepsilon^{3/2}\phi)_x dxdt \\
&\quad - \int_0^T\int_{\mathbb{T}}\rho_\varepsilon^{5/2}(V[\rho_\varepsilon])_x\phi dxdt + \frac{\delta^2}{2}\int_0^T\int_{\mathbb{T}}\frac{(\sqrt{\rho_\varepsilon})_{xx}}{\sqrt{\rho_\varepsilon}}(\rho_\varepsilon^{5/2}\phi)_x dxdt \\
&\quad + \nu\int_0^T\int_{\mathbb{T}}(\rho_\varepsilon u_\varepsilon)_x(\rho_\varepsilon^{3/2}\phi)_x + \varepsilon\int_0^T\int_{\mathbb{T}}(u_\varepsilon)_x\phi_x dxdt \\
&\quad + \frac{1}{\tau}\int_0^T\int_{\mathbb{T}}\rho_\varepsilon^{5/2}u_\varepsilon\phi dxdt \\
&= K_1 + \cdots + K_7.
\end{aligned}$$

Employing (34), we have

$$\begin{aligned}
K_1 &= \int_0^T\langle(\rho_\varepsilon^{5/2}u_\varepsilon)_t,\phi\rangle_{H^{-1},H^1}dt - \frac{3}{2}\int_0^T\int_{\mathbb{T}}\rho_\varepsilon^{3/2}(\rho_\varepsilon)_t u_\varepsilon\phi dxdt \\
&\to \int_0^T\langle(\rho^{3/2}J)_t,\phi\rangle_{H^{-1},H^1}dt - \frac{3}{2}\int_0^T\int_{\mathbb{T}}\sqrt{\rho}\rho_t J\phi dxdt.
\end{aligned}$$

For the second integral, we obtain

$$\begin{aligned}
K_2 &= \int_0^T\int_{\mathbb{T}}\big((\rho_\varepsilon u_\varepsilon)^2 + \rho_\varepsilon p(\rho_\varepsilon)\big)\big(3(\sqrt{\rho_\varepsilon})_x\phi + \sqrt{\rho_\varepsilon}\phi_x\big)dxdt \\
&\to \int_0^T\int_{\mathbb{T}}\big(J^2 + \rho p(\rho)\big)\big(3(\sqrt{\rho})_x\phi + \sqrt{\rho}\phi_x\big)dxdt,
\end{aligned}$$

since $(\rho_\varepsilon u_\varepsilon)^2$ converges strongly in $L^1(0,T;L^\infty(\mathbb{T}))$ and $(\sqrt{\rho_\varepsilon})_x$ converges weakly* in $L^\infty(0,T;L^2(\mathbb{T}))$. Furthermore, $(V[\rho_\varepsilon])_x$ converges weakly* in $L^\infty(0,T;\,H^1(\mathbb{T}))$ to $V[\rho]$:

$$K_3 \to \int_0^T\int_{\mathbb{T}}\rho^{5/2}(V[\rho])_x\phi dxdt.$$

The fourth integral can be written as

$$K_4 = \frac{\delta^2}{2}\int_0^T\int_{\mathbb{T}}(\sqrt{\rho_\varepsilon})_{xx}\Big(\frac{5}{2}\rho_\varepsilon(\rho_\varepsilon)_x\phi + \rho_\varepsilon^2\phi_x\Big)dxdt.$$

Since ρ_ε converges strongly in $L^\infty(0,T;L^\infty(\mathbb{T}))$ and in $L^2(0,T;H^1(\mathbb{T}))$, it follows that

$$K_4 \to \frac{\delta^2}{2}\int_0^T\int_{\mathbb{T}}(\sqrt{\rho})_{xx}\big(5\rho^{3/2}(\sqrt{\rho})_x\phi + \rho^2\phi_x\big)dxdt.$$

The weak convergence of $\rho_\varepsilon u_\varepsilon$ in $L^2(0,T;H^1(\mathbb{T}))$ and the strong convergences of $\sqrt{\rho_\varepsilon}$ in $L^\infty(0,T;L^\infty(\mathbb{T}))$ and of $(\rho_\varepsilon)_x$ in $L^2(0,T;L^2(\mathbb{T}))$ imply that

$$\begin{aligned} K_5 &= \nu\int_0^T\int_{\mathbb{T}}(\rho_\varepsilon u_\varepsilon)_x\Big(\frac{3}{2}\sqrt{\rho_\varepsilon}(\rho_\varepsilon)_x\phi + \rho_\varepsilon^{3/2}\phi_x\Big)dxdt \\ &\to \nu\int_0^T\int_{\mathbb{T}}J_x\Big(\frac{3}{2}\sqrt{\rho}\rho_x\phi + \rho^{3/2}\phi_x\Big)dxdt = \nu\int_0^T\int_{\mathbb{T}}J_x(\rho^{3/2}\phi)_x dxdt. \end{aligned}$$

Finally, the estimate (22) shows that $K_6 \to 0$, and

$$K_7 \to \frac{1}{\tau}\int_0^T \rho^{3/2}J\phi dxdt.$$

This proves that (ρ, J, V) solves the system (1)-(3) for $\varepsilon = 0$ and for smooth initial data. A standard approximation procedure gives the result for initial data $(\rho_0, u_0) \in H^1(\mathbb{T}) \times L^\infty(\mathbb{T})$ with positive particle density and finite energy. Theorem 2 is now proven.

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