THE ZERO-ELECTRON-MASS LIMIT IN THE HYDRODYNAMIC MODEL FOR PLASMAS

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Abstract. The limit of vanishing ratio of the electron mass to the ion mass in the isentropic transient Euler-Poisson equations with periodic boundary conditions is proved. The equations consist of the balance laws for the electron density and current density for given ion density, coupled to the Poisson equation for the electrostatic potential. The limit is related to the low-Mach-number limit of Klainerman and Majda. In particular, the limit velocity satisfies the incompressible Euler equations with damping. The difference to the zero-Mach-number limit comes from the electrostatic potential which needs to be controlled. This is done by a reformulation of the equations in terms of the enthalpy, higher-order energy estimates and a careful use of the Poisson equation.

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1. Introduction

A couple of years ago, the last two authors started a program to derive rigorously some asymptotic limits, namely the relaxation limit, the quasineutral limit and the zero-electron-mass limit, in quasi-hydrodynamic models for plasmas. Whereas most of these limits could be rigorously proved, the zero-electron-mass limit in the hydrodynamic equations remained unsolved. In this paper we fill this gap for the hydrodynamic model with given ion density.

First, consider the (scaled) hydrodynamic equations for the electron density n_e with charge $q_e = -1$, the density n_i of the positively charged ions with charge $q_i = +1$, the respective velocities v_e, v_i and the electrostatic potential ϕ,

$$\begin{aligned} \partial_t n_\alpha + \nabla\cdot(n_\alpha v_\alpha) &= 0, \quad \alpha = e, i, \\ m_\alpha \partial_t(n_\alpha v_\alpha) + m_\alpha \nabla\cdot(n_\alpha v_\alpha \otimes v_\alpha) + \nabla p_\alpha(n_\alpha) &= -q_\alpha n_\alpha \nabla\phi - m_\alpha \frac{n_\alpha v_\alpha}{\tau_\alpha}, \\ -\lambda^2\Delta\phi &= n_i - n_e - C(x) \quad \text{for } x \in \mathbb{T}^d,\ t > 0, \end{aligned}$$

where $d \geq 1$ and $\mathbb{T}^d$ denotes the d-dimensional torus. The initial conditions are given by

$$n_\alpha(\cdot, 0) = n_{I,\alpha}, \quad v_\alpha(\cdot, 0) = v_{I,\alpha} \quad \text{in } \mathbb{T}^d,\ \alpha = e, i.$$

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In the above equations, p_α are the pressure functions, usually given by $p_\alpha(n) = c_\alpha n^{\gamma_\alpha}$, $n \geq 0$, where $c_\alpha > 0$ and $\gamma_\alpha \geq 1$ are constants. In this work, we only assume that p_α is smooth and strictly increasing. The function $C(x)$ models fixed charged background ions (doping profile). The (scaled) physical parameters are the particle mass m_α, the relaxation time τ_α and the Debye length λ. We assume that the value of the integral $\int_{\mathbb{T}^d} \phi dx$ is fixed; for instance, $\int_{\mathbb{T}^d} \phi dx = 0$.

In this paper we restrict ourselves to a situation in which the ion density is given, i.e., we wish to perform rigorously the limit $m_e \to 0$ in the system

$$\partial_t n_e + \nabla \cdot (n_e v_e) = 0, \tag{1}$$

$$m_e \partial_t (n_e v_e) + m_e \nabla \cdot (n_e v_e \otimes v_e) + \nabla p_e(n_e) = n_e \nabla \phi - m_e \frac{n_e v_e}{\tau_e}, \tag{2}$$

$$\lambda^2 \Delta \phi = n_e - N \quad \text{for } x \in \mathbb{T}^d,\ t > 0, \tag{3}$$

where $N = n_i - C$ is given. (In fact, we need that N is a constant.) The parameter m_e is essentially the ratio of the electron mass to the ion mass (see, e.g., [14] for details on the scaling). We assume that the ion is much heavier than the electron such that the limit $m_e \to 0$ makes sense. The limit has the goal to achieve simpler models containing the essential physical phenomena. We notice that in plasma physics, zero-electron-mass assumptions are widely used (see, e.g., [9, 17]).

Concerning the existence of solutions to the hydrodynamic model in the one-dimensional or multi-dimensional situation, either in the torus or in the whole space, we refer to [2] for precise references. Whereas in the one-dimensional case, the existence of general global-in-time weak entropy solutions has been shown, there are only results for smooth solutions for small times or initial data close to an equilibrium state in the multi-dimensional case, which we consider here.

The relaxation limit in the hydrodynamic (Euler-Poisson) equations to the drift-diffusion model, which has been first studied by Marcati and Natalini [21], has been solved in [10, 12, 13] for weak entropy solutions (also see [1]). The quasineutral limit in the hydrodynamic model has been analyzed for transient smooth solutions by Cordier and Grenier [4] in the one-dimensional case and independently in [25, 29] in the multi-dimensional case. We refer also to [8] for an analysis of the limit in Vlasov-Poisson equations and [6] for a combined quasineutral-relaxation limit. The limit for steady states has been considered in [23, 24, 27]. The zero-electron-mass limit in the transient equations has been achieved only under restrictive assumptions; see [7]. For steady states, we refer to [23].

These asymptotic limits have been also studied in the drift-diffusion equations which are obtained in the relaxation limit. The quasineutral limit has been proved in [5, 15, 30]. In [14] the zero-electron-mass limit in these equations could be shown (which is easier than in the hydrodynamic model). We mention that such limits have been recently analyzed in quasi-hydrodynamic *quantum* models; see [11, 16].

Before we present the main ideas of this paper, it is convenient to write the main part of the system (1)-(2) in symmetric hyperbolic form. Setting $n = n_e$, $v = v_e$, $p(n) = p_e(n_e)$, and $\varepsilon^2 = m_e$ and introducing the *enthalpy* $h = h(n_e)$, defined by $h'(n) = p'(n)/n$ and

$h(1) = 0$, as a new variable, the system (1)-(3) can be rewritten as

$$\begin{aligned} (\partial_t + v\cdot\nabla)h + p'(n)\nabla\cdot v &= 0, \\ \varepsilon^2(\partial_t + v\cdot\nabla)v + \nabla h &= \nabla\phi - \varepsilon^2 v, \\ \Delta\phi &= n(h) - N, \quad x\in\mathbb{T}^d,\ t>0, \end{aligned} \tag{4}$$

where we assume that $\int_{\mathbb{T}^d}\phi dx = 0$, with initial conditions

$$h(\cdot,0) = h_I^\varepsilon, \quad v(\cdot,0) = v_I^\varepsilon \quad \text{in } \mathbb{T}^d. \tag{5}$$

Here, we have set $\tau_e = \lambda = 1$ in order to simplify the notation. Clearly, for smooth solutions, this system is equivalent to (1)-(3). As we suppose that the pressure function is invertible, so does $h(n)$ and we denote its inverse by $n(h)$. The objective of this paper is to perform the limit $\varepsilon\to 0$ in (4).

1.1. **Formal asymptotic analysis.** Assume that $N>0$ is a constant. In order to derive the limiting system when $\varepsilon\to 0$, we substitute the formal expansions

$$h = h^0 + \varepsilon h^1 + \varepsilon^2 h^2 + \cdots, \quad v = v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots, \quad \phi = \phi^0 + \varepsilon\phi^1 + \varepsilon^2\phi^2 + \cdots$$

in the system (4) and equate equal powers of ε. The lowest-order terms satisfy the equations

$$(\partial_t + v^0\cdot\nabla)h^0 + p'(n(h^0))\nabla\cdot v^0 = 0, \quad \nabla(h^0-\phi^0) = 0, \quad \Delta\phi^0 = n(h^0) - N. \tag{6}$$

The second equation implies that $h^0-\phi^0$ is a function of time only. Combining this fact with the third equation, we find that h^0 solves $\Delta h^0 = n(h^0) - N$. Employing the assumption $\int\phi^0 dx = 0$, it is not difficult to see that $\phi^0 = 0$ and $h^0 = h(N)\in\mathbb{R}$ are the unique solutions of the corresponding equations, such that $\int h^0 dx = \text{meas}(\mathbb{T}^d)h(N)$. In particular, the first equation in (6) becomes $\nabla\cdot v^0 = 0$. The first-order terms satisfy

$$\nabla(h^1-\phi^1) = 0, \quad \Delta\phi^1 = n'(h^0)h^1.$$

The solutions $h^1 = \phi^1 = 0$ are consistent with these equations. At second order, we find

$$(\partial_t + v^0\cdot\nabla)v^0 + v^0 = \nabla(\phi^2 - h^2), \quad \Delta\phi^2 = n'(h^0)h^2. \tag{7}$$

From $\nabla\cdot v^0 = 0$ and the first equation, v^0 and $\phi^2 - h^2$ can be found. Then, h^2 is the solution of the third equation, written in the form $\Delta h^2 = n'(h^0)h^2 - \Delta(\phi^2-h^2)$ and finally, ϕ^2 is given by $\phi^2 = h^2 + (\phi^2 - h^2)$. These considerations motivate to choose the initial data as

$$h_I^\varepsilon = h_I^0 + \varepsilon^2 h_I^2, \quad v_I^\varepsilon = v_I^0 + \varepsilon v_I^1. \tag{8}$$

The formal analysis shows that the zero-electron-mass limit has some similarities with the low-Mach-number limit in the compressible Euler system [20]. It is possible to use ideas from Klainerman and Majda [18, 19] to deal with the term $\varepsilon^{-1}\nabla h$ in (4) (after division by ε). However, we have another singularity from $\varepsilon^{-1}\nabla\phi$ which cannot be fixed by their method. Our idea is to control this term by a careful use of the mass conservation and the Poisson equation. To describe the idea more precisely, introduce the new variables

$$\tilde{h} = \frac{h-h^0}{\varepsilon}, \quad \tilde{\phi} = \frac{\phi-\phi^0}{\varepsilon}$$

as in [20, Ch. 2.4], where ϕ^0 is any constant fixed by $\int \phi dx$ (if $\int \phi dx = 0$ then $\phi^0 = 0$). The system (4) can be written as

$$A(\varepsilon\tilde{h})(\partial_t + v\cdot\nabla)\tilde{h} + \frac{1}{\varepsilon}\nabla\cdot v = 0, \tag{9}$$

$$(\partial_t + v\cdot\nabla)v + \frac{1}{\varepsilon}\nabla\tilde{h} = \frac{1}{\varepsilon}\nabla\tilde{\phi} - v, \tag{10}$$

$$\Delta\tilde{\phi} = \frac{1}{\varepsilon}(n(\varepsilon\tilde{h} + h^0) - n(h^0)), \quad x \in \mathbb{T}^d,\ t > 0, \tag{11}$$

where $A(\varepsilon\tilde{h}) = 1/p'(\varepsilon\tilde{h} + h^0)$.

1.2. **Main ideas.** For the proof of the limit $\varepsilon \to 0$ we need to derive uniform estimates up to sth-order derivatives with $s > d/2 + 1$. Here, we will describe only how to derive the lowest-order estimates, which is sufficient to illustrate the idea. We assume that there are $L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))$ estimates for $\tilde{h}$ and v. Friedrich's energy estimate for symmetric hyperbolic systems and integration by parts yield

$$\frac{d}{dt}\int_{\mathbb{T}^d}(A(\varepsilon\tilde{h})|\tilde{h}|^2 + |v|^2)dx + \int_{\mathbb{T}^d}|v|^2dx \le c\int_{\mathbb{T}^d}(|\tilde{h}|^2 + |v|^2)dx - \frac{1}{\varepsilon}\int_{\mathbb{T}^d}\tilde{\phi}\nabla\cdot v dx,$$

where the constant $c > 0$ depends on the $L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))$ bounds for $\tilde{h}$ and v. Replacing the term $\varepsilon^{-1}\nabla\cdot v$ by the mass conservation equation, we are left to control the integrals

$$\int_{\mathbb{T}^d}\tilde{\phi}A(\varepsilon\tilde{h})\tilde{h}_t dx + \int_{\mathbb{T}^d}\tilde{\phi}A(\varepsilon\tilde{h})v\cdot\nabla\tilde{h}dx.$$

The second integral can be easily controlled (after integration by parts) by the $L^\infty(0,T;$ $W^{1,\infty}(\mathbb{T}^d))$ bounds for $\tilde{h}$ and v. In order to deal with the first integral we employ the Poisson equation,

$$\Delta\tilde{\phi}_t = n'(\varepsilon\tilde{h} + h^0)\tilde{h}_t.$$

Then we arrive at

$$\int_{\mathbb{T}^d}\tilde{\phi}A(\varepsilon\tilde{h})\tilde{h}_t dx = \int_{\mathbb{T}^d}\frac{A(\varepsilon\tilde{h})}{n'(\varepsilon\tilde{h} + h^0)}\Delta\tilde{\phi}_t\tilde{\phi}dx.$$

Again after integration by parts, we obtain an integral with a "good" sign, $-\partial_t\|\nabla\tilde{\phi}\|_{L^2}^2$ and other integrals which can be estimated by $\|\varepsilon\nabla\tilde{\phi}_t\|_{L^2}$ and $\|\partial_t(n'(\varepsilon\tilde{h} + h^0))\|_{L^2}$. Using the Poisson equation to bound the first expression, it can be seen that both terms contain the derivative $\varepsilon\tilde{h}_t$ as above but now including the factor ε. Indeed, by (9), we are now able to control this expression in some norm in terms of the $L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))$ estimates for $\tilde{h}$ and v.

For higher-order derivatives, we need to take care of the nonlinear terms arising from the partial derivatives, but finally, we end up with estimates for $\tilde{h}$ and v which are appropriate to employ the standard continuation argument (see below for details).

We remark that a more general strategy to perform the low-Mach-number limit, and possibly also the zero-electron-mass limit, has been suggested by Métivier and Schochet in [22]. However, the technique of Klainerman and Majda is quite fundamental and sufficient for our purpose. For other results on the incompressible limit of the compressible Euler equations, we refer to [3, 28].

1.3. **Main results.** In order to formulate our main theorems, we introduce as in [20] the following notations:

$$\|\cdot\|_s = \|\cdot\|_{H^s(\mathbb{T}^d)}, \quad |||\cdot|||_{s,T} = \sup_{0<t<T} \|\cdot\|_s \quad \text{for } s \in \mathbb{R}, \quad \|\cdot\|_\infty = \|\cdot\|_{L^\infty(\mathbb{T}^d)}.$$

Theorem 1. *Let n be a smooth strictly increasing function and let $N > 0$. Furthermore, let $s > d/2 + 1$ and let the initial data $(h_I^\varepsilon, v_I^\varepsilon)$ satisfy $v_I^\varepsilon = v_I^0$ and*

$$\left\| \frac{h_I^\varepsilon - h^0}{\varepsilon} \right\|_s + \|v_I^\varepsilon\|_s \le M_0,$$

where $h^0 = h(N)$ and $M_0 > 0$ is a constant independent of ε. Then there exist constants $T_0 > 0$ and $M_0' > 0$, independent of ε and $\varepsilon_0(M_0) > 0$ such that for all $0 < \varepsilon < \varepsilon_0(M_0)$, the problem (4)-(5) *has a classical solution $(h^\varepsilon, v^\varepsilon, \phi^\varepsilon)$ in $[0, T_0]$ satisfying*

$$\left|\left|\left| \frac{h^\varepsilon - h^0}{\varepsilon} \right|\right|\right|_{s,T_0} + |||v^\varepsilon|||_{s,T_0} + \left|\left|\left| \frac{\nabla \phi^\varepsilon}{\varepsilon} \right|\right|\right|_{s,T_0} \le M_0'.$$

Theorem 2. *Let the assumptions of Theorem 1 hold with $\nabla \cdot v_I^0 = 0$ and*

$$\left\| \frac{h_I^\varepsilon - h^0}{\varepsilon^2} \right\|_s \le M_1. \tag{12}$$

Let $(h^\varepsilon, v^\varepsilon, \phi^\varepsilon)$ be a classical solution to (4)-(5) *in $[0, T_0]$ with $T_0 > 0$ independent of ε. Then, as $\varepsilon \to 0$,*

$$\begin{aligned} h^\varepsilon \to h^0, \quad \nabla\phi^\varepsilon \to 0 \qquad & \textit{strongly in } L^\infty(0, T_0; H^\alpha(\mathbb{T}^d)) \cap C^{0,1}([0, T_0]; L^2(\mathbb{T}^d)), \\ v^\varepsilon \to v^0 \qquad & \textit{strongly in } C^0([0, T_0]; H^\alpha(\mathbb{T}^d)) \quad \textit{for all } \alpha < s, \end{aligned}$$

where v^0 is the (unique) classical solution of the following incompressible Euler equations with damping,

$$\begin{gathered} \nabla \cdot v^0 = 0, \quad (\partial_t + v^0 \cdot \nabla) v^0 + v^0 = \nabla \pi, \quad x \in \mathbb{T}^d, \ t > 0, \\ v^0(\cdot, 0) = v_I^0 \quad \textit{in } \mathbb{T}^d, \end{gathered} \tag{13}$$

and π is the limit of

$$\nabla \left(\frac{\phi^\varepsilon - h^\varepsilon}{\varepsilon} \right) \rightharpoonup^* \nabla \pi \qquad \textit{weakly* in } L^\infty(0, T_0; L^2(\mathbb{T}^d)).$$

The paper is organized as follows. In section 2, uniform estimates are shown and Theorem 1 is proved by means of a continuation argument. Section 3 is devoted to the proof of Theorem 2.

2. Uniform local existence

First we recall for convenience some Moser-type inequalities which we will use in the subsequent analysis. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi-index. The differential operator D^α is defined by $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$; D^s for $s \in \mathbb{N}$ denotes the sth derivative.

- Let $s \geq 0$, f, $g \in H^s(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$, and α a multi-index with $|\alpha| \leq s$. Then, for some constant $c_s > 0$,

$$\|D^\alpha(fg)\|_0 \leq c_s(\|f\|_\infty \|D^s g\|_0 + \|g\|_\infty \|D^s f\|_0). \tag{14}$$

- Let $s \geq 1$, $f \in H^s(\mathbb{T}^d)$ with $Df \in L^\infty(\mathbb{T}^d)$, $g \in H^{s-1}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$ and $|\alpha| \leq s$. Then, for some constant $c_s > 0$,

$$\|D^\alpha(fg) - fD^\alpha g\|_0 \leq c_s(\|Df\|_\infty \|D^{s-1}g\|_0 + \|g\|_\infty \|D^s f\|_0). \tag{15}$$

For the proof of Theorem 1 we employ the basic theory of smooth solutions to hyperbolic systems of Majda [20]. The key result is contained in the following lemma.

Lemma 3. *Suppose that it holds, for some $T^* > 0$ (maybe depending on ε) and $M > 0$ (independent of ε),*

$$\|\tilde{h}\|_{L^\infty(0,T^*;W^{1,\infty}(\mathbb{T}^d))} + \|v\|_{L^\infty(0,T^*;W^{1,\infty}(\mathbb{T}^d))} \leq M. \tag{16}$$

Then there exist $\varepsilon_0 = \varepsilon_0(M) > 0$ and $c(M) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, it holds

$$|||\tilde{h}|||_{s,T^*} + |||v|||_{s,T^*} + |||\nabla\tilde{\phi}|||_{s,T^*} \leq e^{c(M)T^*}(M_0 + c(M)T^*). \tag{17}$$

We assume Lemma 3 for the moment and proceed with the proof of Theorem 1.

Proof of Theorem 1. In view of the existence results of [20], the proof follows from a continuation argument. We follow the lines of [20, p. 59]. First, let $M_1 > M_0$ and fix $T_1 > 0$ such that

$$e^{c(M_1)T_1}(M_0 + c(M_1)T_1) \leq M_1.$$

Writing $\phi = \Delta^{-1}(n(h) - N)$ (with periodic boundary conditions) and using the properties of the linear operator Δ^{-1}, the first two equations in (4) form a symmetric hyperbolic system with the unknowns (h, v). Therefore, the local existence results of [20] show that there exists a local smooth solution (h, v). Let $T(\varepsilon)$ be the maximal time of existence of such a smooth solution. If $T(\varepsilon) = +\infty$ for all $\varepsilon > 0$, we are done. Otherwise, $T(\varepsilon) < +\infty$ for some $\varepsilon > 0$ and then

$$\limsup_{t \to T(\varepsilon)-}(|||\tilde{h}|||_{s,t} + |||v|||_{s,t} + |||\nabla\tilde{\phi}|||_{s,t}) = +\infty.$$

There exists $t_1(\varepsilon) < T(\varepsilon)$ such that

$$|||\tilde{h}|||_{s,t_1(\varepsilon)} + |||v|||_{s,t_1(\varepsilon)} + |||\nabla\tilde{\phi}|||_{s,t_1(\varepsilon)} \leq M_1 \tag{18}$$

and

$$|||\tilde{h}|||_{s,t} + |||v|||_{s,t} + |||\nabla\tilde{\phi}|||_{s,t} > M_1 \quad \text{for all } t > t_1(\varepsilon). \tag{19}$$

Lemma 3, applied with some $T^* > 0$ and $M = M_1$, provides the existence of $\varepsilon_0 > 0$ (independent of T^*) such that (17) holds for all $\varepsilon < \varepsilon_0$. If $t_1(\varepsilon) \geq T_1$ for all $0 < \varepsilon < \varepsilon_0$, the proof is finished. Otherwise, we argue by contradiction; there exists $0 < \varepsilon_1 < \varepsilon_0$ such that $t_1(\varepsilon_1) < T_1$. In particular,

$$|||\tilde{h}|||_{s,t_1(\varepsilon_1)} + |||v|||_{s,t_1(\varepsilon_1)} + |||\nabla\tilde{\phi}|||_{s,t_1(\varepsilon_1)} \leq M_1.$$

Now, we apply Lemma 3 with $T^* = t_1(\varepsilon_1)$ and $M = M_1$. This does not change the value of ε_0 since it only depends on M_1. Then, by (17),

$$\begin{aligned} |||\tilde{h}|||_{s,t_1(\varepsilon_1)} + |||v|||_{s,t_1(\varepsilon_1)} + |||\nabla\tilde{\phi}|||_{s,t_1(\varepsilon_1)} &\leq e^{c(M_1)t_1(\varepsilon_1)}(M_0 + c(M_1)t_1(\varepsilon_1)) \\ &< e^{c(M_1)T_1}(M_0 + c(M_1)T_1) \leq M_1. \end{aligned}$$

Thanks to the strict inequality sign, we may extend, again by local existence results, the time interval $[0, t_1(\varepsilon_1)]$ to $[0, t_2(\varepsilon_1)]$ for some $t_2(\varepsilon_1) > t_1(\varepsilon_1)$ such that

$$|||\tilde{h}|||_{s,t_2(\varepsilon_1)} + |||v|||_{s,t_2(\varepsilon_1)} + |||\nabla\tilde{\phi}|||_{s,t_2(\varepsilon_1)} \leq M_1.$$

But this contradicts the definition of $t_1(\varepsilon)$. Hence, $t_1(\varepsilon) \geq T_1 > 0$ for all $0 < \varepsilon < \varepsilon_0$, which shows that there exists a solution on $[0, T_1]$ and T_1 does not depend on ε. □

Proof of Lemma 3. Step 1: preparations. First we collect some useful inequalities which we will employ several times in this proof. Let f be a smooth function and let (16) hold. Then there exist constants $c(M)$, $c_0(M)$, $c_1(M) > 0$ such that

$$\begin{aligned} &0 < c_0(M) \leq f^{(m)}(\varepsilon\tilde{h}) \leq c_1(M) \quad \text{for all } m \in \mathbb{N} \text{ and sufficiently small } \varepsilon, \\ &\sup_{(0,T)} \|\nabla f(\varepsilon\tilde{h})\|_\infty \leq \varepsilon c(M), \quad \sup_{(0,T)} \|\partial_t f(\varepsilon\tilde{h})\|_\infty \leq c(M). \end{aligned} \tag{20}$$

In fact, the first estimates can be derived directly by elementary computations. The last one is a consequence of the first, using (16) and (9),

$$\begin{aligned} |||\partial_t f(\varepsilon\tilde{h})|||_{\infty,T} &\leq |||f'(\varepsilon\tilde{h})|||_{\infty,T}|||\varepsilon\partial_t\tilde{h}|||_{\infty,T} \\ &\leq c(M)(|||\varepsilon v \cdot \nabla\tilde{h}|||_{\infty,T} + |||A^{-1}(\varepsilon\tilde{h})\nabla \cdot v|||_{\infty,T}) \leq c(M). \end{aligned}$$

We will use the following notations. Let α with $|\alpha| \leq s$ be a multi-index. Then we define $|D^{|\alpha|}u| = \sup_\alpha |D^\alpha u|$. Furthermore, we abbreviate

$$h_\alpha = D^\alpha\tilde{h}, \quad v_\alpha = D^\alpha v, \quad \phi_\alpha = D^\alpha\tilde{\phi}.$$

Step 2: Friedrich's energy estimates. We divide (9) by $A(\varepsilon\tilde{h})$, apply the operator D^α and multiply by $A(\varepsilon\tilde{h})$. Differentiating also (10) and (11), the resulting equations are

$$A(\varepsilon\tilde{h})\partial_t h_\alpha + A(\varepsilon\tilde{h})v \cdot \nabla h_\alpha + \frac{1}{\varepsilon}\nabla \cdot v_\alpha = F_\alpha, \tag{21}$$

$$\partial_t v_\alpha + v \cdot \nabla v_\alpha + \frac{1}{\varepsilon}\nabla h_\alpha = \frac{1}{\varepsilon}\nabla\phi_\alpha - v_\alpha + G_\alpha, \tag{22}$$

$$\Delta\phi_\alpha = n'(\varepsilon\tilde{h} + h^0)h_\alpha + H_\alpha, \tag{23}$$

where

$$
\begin{aligned}
F_\alpha &= A(\varepsilon\tilde h)(v\cdot\nabla h_\alpha - D^\alpha(v\cdot\nabla\tilde h)) + \frac{1}{\varepsilon}\big(\nabla\cdot v_\alpha - A(\varepsilon\tilde h)D^\alpha(A^{-1}(\varepsilon\tilde h)\nabla\cdot v)\big),\\
G_\alpha &= v\cdot\nabla v_\alpha - D^\alpha(v\cdot\nabla v),\\
H_\alpha &= \frac{1}{\varepsilon}D^\alpha(n(\varepsilon\tilde h + h^0)) - n'(\varepsilon\tilde h + h^0)h_\alpha. \qquad (24)
\end{aligned}
$$

Friedrich's energy estimate for (21)-(22) (i.e., multiplying (21) by h_α and (22) by v_α, integrating over $\mathbb{T}^d$, taking the sum and integrating by parts) give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\big(A(\varepsilon\tilde h)|h_\alpha|^2 + |v_\alpha|^2\big)dx + \int_{\mathbb{T}^d}|v_\alpha|^2dx\\
&\quad\le B\int_{\mathbb{T}^d}(|h_\alpha|^2+|v_\alpha|^2)dx + \int_{\mathbb{T}^d}(|F_\alpha|^2+|G_\alpha|^2)dx + \frac{1}{\varepsilon}\int_{\mathbb{T}^d}\nabla\phi_\alpha\cdot v_\alpha dx \qquad (25)\\
&\quad= I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
B &= |||\partial_t(A(\varepsilon\tilde h))|||_{\infty,T} + |||\nabla A(\varepsilon\tilde h)|||_{\infty,T}|||v|||_{\infty,T} + |||A(\varepsilon\tilde h)|||_{\infty,T}|||\nabla\cdot v|||_{\infty,T}\\
&\quad + c|||\nabla v|||_{\infty,T} + 1.
\end{aligned}
$$

Step 3: control of I_1, I_2 and I_3. The inequalities (20) show that

$$I_1 \le c(M)\int_{\mathbb{T}^d}(|h_\alpha|^2+|v_\alpha|^2)dx.$$

The integral I_2 can be estimated in a similar way as done by Klainerman and Majda [18, 19]. For convenience, we present the details. We employ Moser-type estimates (14)-(15) to obtain for $|\alpha|\ge 1$ (noting that $F_0 = G_0 = 0$)

$$
\begin{aligned}
\|F_\alpha\|_0 &\le \|A(\varepsilon\tilde h)\|_\infty\|v\cdot\nabla h_\alpha - D^\alpha(v\cdot\nabla\tilde h)\|_0 + \frac{1}{\varepsilon}\|\nabla\cdot v_\alpha - A(\varepsilon\tilde h)D^\alpha(A^{-1}(\varepsilon\tilde h)\nabla\cdot v)\|_0\\
&\le c\|A(\varepsilon\tilde h)\|_\infty\Big(\|\nabla v\|_\infty\|D^{|\alpha|-1}\nabla\tilde h\|_0 + \|D^{|\alpha|}v\|_0\|\nabla\tilde h\|_\infty\Big)\\
&\quad + \frac{c}{\varepsilon}\Big(\|\nabla A(\varepsilon\tilde h)\|_\infty\|D^{|\alpha|-1}(A^{-1}(\varepsilon\tilde h)\nabla\cdot v)\|_0 + \|D^{|\alpha|}A(\varepsilon\tilde h)\|_0\|A^{-1}(\varepsilon\tilde h)\nabla\cdot v\|_\infty\Big)\\
&\le c(M)\Big(\|D^{|\alpha|}\tilde h\|_0 + \|D^{|\alpha|}v\|_0\Big) + c(M)\Big(\|A^{-1}(\varepsilon\tilde h)\|_\infty\|D^{|\alpha|}v\|_0\\
&\quad + \|\nabla\cdot v\|_\infty\|D^{|\alpha|-1}A^{-1}(\varepsilon\tilde h)\|_0 + \frac{1}{\varepsilon}\|D^{|\alpha|}A(\varepsilon\tilde h)\|_0\Big)\\
&\le c(M)\Big(\|D^{|\alpha|}\tilde h\|_0 + \|D^{|\alpha|}v\|_0 + 1\Big) \qquad (26)
\end{aligned}
$$

and

$$
\begin{aligned}
\|G_\alpha\|_0 &\le \|v\cdot\nabla v_\alpha - D^\alpha(v\cdot\nabla v)\|_0\\
&\le c\left(\|\nabla v\|_\infty\|D^{|\alpha|-1}\nabla v\|_0 + \|D^{|\alpha|}v\|_0\|\nabla v\|_\infty\right) \le c(M)\left(\|D^{|\alpha|}v\|_0 + 1\right).
\end{aligned}
$$

This yields

$$I_2 \le c(M)\left(\|D^{|\alpha|}\tilde{h}\|_0^2 + \|D^{|\alpha|}v\|_0^2 + 1\right).$$

Now we turn to the delicate integral I_3. This term cannot be controlled directly from the Riesz transformation (derived from the coupling with the Poisson equation), as done in [2], for instance. Our idea is to replace the singular term $\varepsilon^{-1}\nabla\phi_\alpha$, after integration by parts in I_3, by equation (21),

$$\begin{aligned} I_3 &= -\frac{1}{\varepsilon}\int_{\mathbb{T}^d}\phi_\alpha\nabla\cdot v_\alpha dx \\ &= -\int_{\mathbb{T}^d}\phi_\alpha F_\alpha dx + \int_{\mathbb{T}^d}\phi_\alpha A(\varepsilon\tilde{h})v\cdot\nabla h_\alpha dx + \int_{\mathbb{T}^d}\phi_\alpha A(\varepsilon\tilde{h})\partial_t h_\alpha dx \\ &= K_1 + K_2 + K_3. \end{aligned}$$

Step 4: control of K_1, K_2 and K_3. With the help of the estimate (26), it is not difficult to see that the first integral K_1 can be controlled by

$$K_1 \le \frac{1}{2}\int_{\mathbb{T}^d}(|\phi_\alpha|^2 + |F_\alpha|^2)dx \le c(M)\int_{\mathbb{T}^d}(|\phi_\alpha|^2 + |D^{|\alpha|}\tilde{h}|^2 + |D^{|\alpha|}v|^2)dx + c(M).$$

We integrate by parts in the second integral since ∇h_α cannot be estimated by $D^{|\alpha|}\tilde{h}$,

$$\begin{aligned} K_2 &= -\int_{\mathbb{T}^d}\nabla\phi_\alpha\cdot vA(\varepsilon\tilde{h})h_\alpha dx - \int_{\mathbb{T}^d}\phi_\alpha\nabla A(\varepsilon\tilde{h})\cdot vh_\alpha dx - \int_{\mathbb{T}^d}\phi_\alpha A(\varepsilon\tilde{h})\nabla\cdot vh_\alpha dx \\ &\le c(M)\int_{\mathbb{T}^d}(|\nabla\phi_\alpha|^2 + |\phi_\alpha|^2 + |h_\alpha|^2)dx. \end{aligned}$$

The difficult term is now K_3. In order to control K_3, we cannot use (21) since this would (again) give a term containing $\varepsilon^{-1}\nabla\cdot v_\alpha$. Our strategy is to employ the Poisson equation in the following way. Taking the time derivative of (23), we have

$$\Delta\partial_t\phi_\alpha = n'(\varepsilon\tilde{h}+h^0)\partial_t h_\alpha + \partial_t(n'(\varepsilon\tilde{h}+h^0))h_\alpha + \partial_t H_\alpha,$$

recalling the definition (24) of H_α. Then, multiplying this equation by $\phi_\alpha A(\varepsilon\tilde{h})/n'(\varepsilon\tilde{h}+h^0)$ and substituting the expression into K_3, we arrive at

$$\begin{aligned} K_3 &= \int_{\mathbb{T}^d}\frac{A(\varepsilon\tilde{h})}{n'(\varepsilon\tilde{h}+h^0)}\phi_\alpha\Delta\partial_t\phi_\alpha dx - \int_{\mathbb{T}^d}\frac{A(\varepsilon\tilde{h})}{n'(\varepsilon\tilde{h}+h^0)}\phi_\alpha\partial_t(n'(\varepsilon\tilde{h}+h^0))h_\alpha dx \\ &\quad - \int_{\mathbb{T}^d}\frac{A(\varepsilon\tilde{h})}{n'(\varepsilon\tilde{h}+h^0)}\phi_\alpha\partial_t H_\alpha dx = L_1 + L_2 + L_3. \end{aligned}$$

Step 5: control of L_1 and L_2. After integration by parts we obtain

$$\begin{aligned} L_1 &= -\frac{1}{2}\int_{\mathbb{T}^d} \frac{A(\varepsilon\tilde h)}{n'(\varepsilon\tilde h + h^0)}\partial_t|\nabla\phi_\alpha|^2 dx - \int_{\mathbb{T}^d} \partial_t\nabla\phi_\alpha\cdot\nabla\left(\frac{A(\varepsilon\tilde h)}{n'(\varepsilon\tilde h + h^0)}\right)\phi_\alpha dx \\ &= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} \frac{A(\varepsilon\tilde h)}{n'(\varepsilon\tilde h + h^0)}|\nabla\phi_\alpha|^2 dx + \frac{1}{2}\int_{\mathbb{T}^d}\partial_t\left(\frac{A(\varepsilon\tilde h)}{n'(\varepsilon\tilde h + h^0)}\right)|\nabla\phi_\alpha|^2 dx \\ &\quad - \int_{\mathbb{T}^d} \partial_t\nabla\phi_\alpha\cdot\nabla\left(\frac{A(\varepsilon\tilde h)}{n'(\varepsilon\tilde h + h^0)}\right)\phi_\alpha dx. \end{aligned}$$

The bound (20) allows to estimate the second and third integral:

$$L_1 \le -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} \frac{A(\varepsilon\tilde h)}{n'(\varepsilon\tilde h + h^0)}|\nabla\phi_\alpha|^2 dx + c(M)\int_{\mathbb{T}^d}|\nabla\phi_\alpha|^2 dx + c(M)\int_{\mathbb{T}^d}(|\varepsilon\partial_t\nabla\phi_\alpha|^2 + |\phi_\alpha|^2)dx.$$

In order to estimate the integral over $|\varepsilon\partial_t\nabla\phi_\alpha|^2$, we employ again the Poisson equation (23), now written in the form

$$\Delta\partial_t\phi_\alpha = \frac{1}{\varepsilon}D^\alpha\partial_t(n(\varepsilon\tilde h + h^0)).$$

For $|\alpha| = 0$, we have, observing (20),

$$\|\varepsilon\partial_t\nabla\tilde\phi\|_0 \le c\|\partial_t(n(\varepsilon\tilde h + h^0))\|_0 \le c(M).$$

For $1 \le |\alpha| \le s$, we proceed by induction. Elliptic estimates give

$$\begin{aligned} \|\varepsilon\partial_t\nabla\phi_\alpha\|_0 &\le c\left(\|D^\alpha\partial_t(n(\varepsilon\tilde h + h^0))\|_{-1} + \|\varepsilon\partial_t\phi_\alpha\|_0\right) \\ &\le c\left(\|D^{|\alpha|-1}(n'(\varepsilon\tilde h + h^0)\varepsilon\partial_t\tilde h)\|_0 + \|\varepsilon\partial_t\phi_\alpha\|_0\right). \end{aligned}$$

The last term is bounded by the induction hypothesis. In order to bound the first term, we employ (21). This gives controllable terms since the time derivative provides the factor ε in front of $\tilde h_t$ and $\varepsilon\tilde h_t$ can be estimated. Therefore, by Moser-type calculus,

$$\|\varepsilon\partial_t\nabla\phi_\alpha\|_0 \le c(M)(\|D^{|\alpha|}\tilde h\|_0 + \|D^{|\alpha|}v\|_0 + 1).$$

Hence, the integral L_1 is bounded by

$$L_1 \le -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\frac{A(\varepsilon\tilde h)|\nabla\phi_\alpha|^2}{n'(\varepsilon\tilde h + h^0)}dx + c(M)\int_{\mathbb{T}^d}(|D^{|\alpha|}\tilde h|^2 + |D^{|\alpha|}v|^2 + |\nabla\phi_\alpha|^2 + |\phi_\alpha|^2)dx + c(M).$$

The estimate of L_2 uses again the fact that the term $\|\varepsilon\partial_t\tilde h\|_\infty$ can be controlled, employing (21). More precisely, we have

$$L_2 \le c(M)\|\partial_t(n'(\varepsilon\tilde h + h^0))\|_\infty\int_{\mathbb{T}^d}(|h_\alpha|^2 + |\phi_\alpha|^2)dx \le c(M)\int_{\mathbb{T}^d}(|h_\alpha|^2 + |\phi_\alpha|^2)dx.$$

Step 6: control of L_3. First we write, recalling the definition (24) of H_α,

$$L_3 = -\int_{\mathbb{T}^d} \frac{A(\varepsilon\tilde{h})}{n'(\varepsilon\tilde{h}+h^0)}\phi_\alpha\partial_t\left(\frac{1}{\varepsilon}D^\alpha(n(\varepsilon\tilde{h}+h^0)) - n'(\varepsilon\tilde{h}+h^0)h_\alpha\right)dx.$$

Before we describe our idea how to deal with this terms, we notice that $H_\alpha = 0$ for all α with $|\alpha| = 1$ and thus $L_3 = 0$. Therefore, we may assume that $2 \le |\alpha| \le s$. The natural idea is to take the time derivative in the above integral and then to employ Moser-type inequalities. However, in this case we would obtain terms like $\|\partial_t\nabla\tilde{h}\|_\infty$ which cannot be controlled by M. Our idea is to reformulate the integral in such a way that only terms like $\varepsilon\tilde{h}_t$ and not $\tilde{h}_t$ appear. For convenience, we set $g(\varepsilon\tilde{h}) = A(\varepsilon\tilde{h})/n'(\varepsilon\tilde{h}+h^0)$. Then, by a slight abuse of notation for D^α, a reformulation gives

$$\begin{aligned} L_3 &= -\int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha\partial_t\left(\frac{1}{\varepsilon}D^{\alpha-1}(n'(\varepsilon\tilde{h}+h^0)\varepsilon D\tilde{h}) - D^\alpha(n'(\varepsilon\tilde{h}+h^0)\tilde{h})\right)dx \\ &\quad - \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha\partial_t\left(D^\alpha(n'(\varepsilon\tilde{h}+h^0)\tilde{h}) - n'(\varepsilon\tilde{h}+h^0)h_\alpha\right)dx \\ &= \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha\partial_t D^{\alpha-1}(D(n'(\varepsilon\tilde{h}+h^0))\tilde{h})dx - \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha\Big(D^\alpha\partial_t(n'(\varepsilon\tilde{h}+h^0)\tilde{h}) \\ &\quad - n'(\varepsilon\tilde{h}+h^0)D^\alpha\partial_t\tilde{h} - \partial_t(n'(\varepsilon\tilde{h}+h^0))h_\alpha\Big)dx. \end{aligned}$$

We write these two integrals in the following way, using integration by parts:

$$\begin{aligned} &\int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha\partial_t D^{\alpha-1}(D(n'(\varepsilon\tilde{h}+h^0))\tilde{h})dx \\ &\quad = \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha D^{\alpha-1}\left(\partial_t(Dn'(\varepsilon\tilde{h}+h^0))\tilde{h} + Dn'(\varepsilon\tilde{h}+h^0)\partial_t\tilde{h}\right)dx \\ &\quad = -\int_{\mathbb{T}^d} D(g(\varepsilon\tilde{h})\phi_\alpha)D^{\alpha-2}(\partial_t(Dn'(\varepsilon\tilde{h}+h^0))\tilde{h})dx \\ &\qquad + \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha D^{\alpha-1}\big(Dn'(\varepsilon\tilde{h}+h^0)\partial_t\tilde{h}\big)dx = N_1 + N_2 \end{aligned}$$

and

$$\begin{aligned} &-\int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha\left(D^\alpha\partial_t(n'(\varepsilon\tilde{h}+h^0)\tilde{h}) - n'(\varepsilon\tilde{h}+h^0)D^\alpha\partial_t\tilde{h} - \partial_t(n'(\varepsilon\tilde{h}+h^0))h_\alpha\right)dx \\ &= \int_{\mathbb{T}^d} D(g(\varepsilon\tilde{h})\phi_\alpha)\left(D^{\alpha-1}\partial_t(n'(\varepsilon\tilde{h}+h^0)\tilde{h}) - n'(\varepsilon\tilde{h}+h^0)D^{\alpha-1}\partial_t\tilde{h}\right)dx \\ &\quad - \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha Dn'(\varepsilon\tilde{h}+h^0)D^{\alpha-1}\partial_t\tilde{h}dx + \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha\partial_t(n'(\varepsilon\tilde{h}+h^0))h_\alpha dx \\ &= \int_{\mathbb{T}^d} D(g(\varepsilon\tilde{h})\phi_\alpha)\left(D^{\alpha-1}(n'(\varepsilon\tilde{h}+h^0)\partial_t\tilde{h}) - n'(\varepsilon\tilde{h}+h^0)D^{\alpha-1}\partial_t\tilde{h}\right)dx \\ &\quad + \int_{\mathbb{T}^d} D(g(\varepsilon\tilde{h})\phi_\alpha)D^{\alpha-1}(\partial_t(n'(\varepsilon\tilde{h}+h^0))\tilde{h})dx - \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha Dn'(\varepsilon\tilde{h}+h^0)D^{\alpha-1}\partial_t\tilde{h}dx \end{aligned}$$

$$+\int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\phi_\alpha\partial_t(n'(\varepsilon\tilde{h}+h^0))h_\alpha dx = N_3+N_4+N_5+N_6.$$

Before we estimate $N_1,\ldots,N_6$, we notice the following useful inequalities. Let f be a smooth function and $2\le|\alpha|\le s$. Then, by Moser-type calculus (14) and employing (21),

$$\begin{aligned}\|D^{\alpha-1}\partial_t f(\varepsilon\tilde{h})\|_0 &= \|D^{\alpha-1}(f'(\varepsilon\tilde{h})\varepsilon\partial_t\tilde{h})\|_0\\ &\le c\left(\|f'(\varepsilon\tilde{h})\|_\infty\|\varepsilon\partial_t h_{\alpha-1}\|_0+\|\varepsilon\partial_t\tilde{h}\|_\infty\|D^{\alpha-1}f'(\varepsilon\tilde{h})\|_0\right)\\ &\le c(M)\left(\|\varepsilon\nabla h_{\alpha-1}\|_0+\|\nabla\cdot v_{\alpha-1}\|_0+\|\varepsilon F_{\alpha-1}\|_0+\|h_{\alpha-1}\|_0\right)\\ &\le c(M)(\|D^{|\alpha|}\tilde{h}\|_0+\|D^{|\alpha|}v\|_0+1). \qquad (27)\end{aligned}$$

Furthermore, by applying Gagliardo-Nirenberg's inequality, it is not difficult to verify that

$$\|D^\alpha f(\varepsilon\tilde{h})\|_0\le\varepsilon c(\|\tilde{h}\|_\infty)\|D^\alpha\tilde{h}\|_0. \qquad (28)$$

For the term N_1 we use first integration by parts and then (14) and (20):

$$\begin{aligned}N_1 &= -\int_{\mathbb{T}^d} D(g(\varepsilon\tilde{h})\phi_\alpha)\left(D^{\alpha-1}(\partial_t(n'(\varepsilon\tilde{h}+h^0))\tilde{h})-D^{\alpha-2}(\partial_t(n'(\varepsilon\tilde{h}+h^0))D\tilde{h})\right)dx\\ &= -N_4+\int_{\mathbb{T}^d} D(g(\varepsilon\tilde{h})\phi_\alpha)D^{\alpha-2}(\partial_t(n'(\varepsilon\tilde{h}+h^0))D\tilde{h})dx\\ &\le -N_4+\|\nabla(g(\varepsilon\tilde{h})\phi_\alpha)\|_0\|D^{\alpha-2}(\partial_t(n'(\varepsilon\tilde{h}+h^0))D\tilde{h})\|_0\\ &\le -N_4+c(M)(\|\varepsilon\phi_\alpha\|_0+\|\nabla\phi_\alpha\|_0)(\|\partial_t n'(\varepsilon\tilde{h}+h^0)\|_\infty\|h_{\alpha-1}\|_0\\ &\quad+\|\nabla\tilde{h}\|_\infty\|D^{\alpha-2}\partial_t n'(\varepsilon\tilde{h}+h^0)\|_0)\\ &\le -N_4+c(M)\int_{\mathbb{T}^d}(|\phi_\alpha|^2+|\nabla\phi_\alpha|^2+|D^{|\alpha|}\tilde{h}|^2+|D^{|\alpha|}v|^2)dx+c(M),\end{aligned}$$

where in the last inequality we have employed (27). In a similar way, using (28), we obtain

$$\begin{aligned}N_2 &\le c(M)\|\phi_\alpha\|_0\|D^{\alpha-1}(Dn'(\varepsilon\tilde{h}+h^0)\partial_t\tilde{h})\|_0\\ &\le c(M)\|\phi_\alpha\|_0\left(\|Dn'(\varepsilon\tilde{h}+h^0)\|_\infty\|D^{|\alpha|-1}\partial_t\tilde{h}\|_0+\|\partial_t\tilde{h}\|_\infty\|D^{|\alpha|}n'(\varepsilon\tilde{h}+h^0)\|_0\right)\\ &\le c(M)\|\phi_\alpha\|_0\left(\|\varepsilon D^{|\alpha|-1}\partial_t\tilde{h}\|_0+\|\varepsilon\partial_t\tilde{h}\|_\infty\|D^{|\alpha|}\tilde{h}\|_0\right)\\ &\le c(M)\int_{\mathbb{T}^d}(|\phi_\alpha|^2+|D^{|\alpha|}\tilde{h}|^2+|D^{|\alpha|}v|^2)dx+c(M),\end{aligned}$$

since the integral over $\|\varepsilon\partial_t h_{\alpha-1}\|_\infty$ can be bounded. The third integral N_3 is estimated by means of (15):

$$\begin{aligned}N_3 &\le \|D(g(\varepsilon\tilde{h})\phi_\alpha)\|_0\left(\|Dn'(\varepsilon\tilde{h}+h^0)\|_\infty\|D^{|\alpha|-2}\partial_t\tilde{h}\|_0+\|\partial_t\tilde{h}\|_\infty\|D^{|\alpha|-1}n'(\varepsilon\tilde{h}+h^0)\|_0\right)\\ &\le c(M)(\|\varepsilon\phi_\alpha\|_0+\|\nabla\phi_\alpha\|_0)\left(\|\varepsilon D^{|\alpha|-2}\partial_t\tilde{h}\|_0+\|\partial_t\tilde{h}\|_\infty\|\varepsilon D^{|\alpha|-1}\tilde{h}\|_0\right)\\ &\le c(M)\int_{\mathbb{T}^d}(|\phi_\alpha|^2+|\nabla\phi_\alpha|^2+|D^{|\alpha|}\tilde{h}|^2+|D^{|\alpha|}v|^2)dx+c(M).\end{aligned}$$

Here, the condition $|\alpha| \geq 2$ is essential to obtain, after differentiation, terms like $\varepsilon\partial_t\tilde{h}$ instead of $\partial_t\tilde{h}$ only. The remaining integrals N_5 and N_6 are estimated in a similar way:

$$\begin{aligned} N_5 &\leq c(M)\|\phi_\alpha\|_0\|Dn'(\varepsilon\tilde{h}+h^0)\|_\infty\|D^{|\alpha|-1}\partial_t\tilde{h}\|_0 \\ &\leq c(M)\int_{\mathbb{T}^d}(|\phi_\alpha|^2+|D^{|\alpha|}\tilde{h}|^2+|D^{|\alpha|}v|^2)dx+c(M), \\ N_6 &\leq c(M)\int_{\mathbb{T}^d}(|\phi_\alpha|^2+|h_\alpha|^2)dx. \end{aligned}$$

Summarizing, we have found that

$$L_3 \leq c(M)\int_{\mathbb{T}^d}(|\phi_\alpha|^2+|\nabla\phi_\alpha|^2+|D^{|\alpha|}\tilde{h}|^2+|D^{|\alpha|}v|^2)dx+c(M).$$

Step 7: end of the proof. The bounds for L_1, L_2 and L_3 in Steps 5 and 6 show that the integral K_3 from Step 4 can also be controlled. The control of K_1, K_2 and K_3 then controls I_3 from Step 3. Finally, the bounds for I_1, I_2 and I_3 give the inequality (see (25))

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\left(A(\varepsilon\tilde{h})|h_\alpha|^2+|v_\alpha|^2+\frac{A(\varepsilon\tilde{h})}{2n'(\varepsilon\tilde{h}+h^0)}|\nabla\phi_\alpha|^2\right)dx+\int_{\mathbb{T}^d}|v_\alpha|^2dx \\ &\quad\leq c(M)\int_{\mathbb{T}^d}(|D^{|\alpha|}\tilde{h}|^2+|D^{|\alpha|}v|^2+|\nabla\phi_\alpha|^2+|\phi_\alpha|^2)dx+c(M). \end{aligned}$$

Summing up over all multi-indices α with the same norm gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\left(A(\varepsilon\tilde{h})|D^{|\alpha|}\tilde{h}|^2+|D^{|\alpha|}v|^2+\frac{A(\varepsilon\tilde{h})}{2n'(\varepsilon\tilde{h}+h^0)}|D^{|\alpha|}\nabla\tilde{\phi}|^2\right)dx+\int_{\mathbb{T}^d}|D^{|\alpha|}v|^2dx \\ &\quad\leq c(M)\left(\int_{\mathbb{T}^d}(|D^{|\alpha|}\tilde{h}|^2+|D^{|\alpha|}v|^2+|D^{|\alpha|}\nabla\tilde{\phi}|^2+|\tilde{\phi}|^2)dx+1\right). \end{aligned}$$

By Gronwall's inequality, we obtain

$$\begin{aligned} \sup_{0<t<T^*}\|(\tilde{h},v,\nabla\tilde{\phi})(\cdot,t)\|_s+\|v\|_{L^2(0,T^*;H^s(\mathbb{T}^d))} &\leq (M_0+1)e^{c(M)T^*}-1 \\ &\leq e^{c(M)T^*}(M_0+c(M)T^*). \end{aligned}$$

This gives the assertion of the Lemma. □

We also need estimates for the time derivative of $\tilde{h}$, v and $\nabla\tilde{\phi}$.

Lemma 4. *Let the assumptions in lemma 3 and (12) hold,* $\nabla\cdot v_I^0=0$, *then there exists* $\varepsilon_1\in(0,\varepsilon_0)$ *such that for all* $0<\varepsilon<\varepsilon_1$, *it holds*

$$|||\tilde{h}_t|||_{0,T^*}+|||v_t|||_{0,T^*}+|||\nabla\tilde{\phi}_t|||_{0,T^*}\leq c(M,M_0,M_1,T^*).$$

Proof. Taking the time derivative of (9)-(11), we obtain

$$A(\varepsilon\tilde{h})\partial_t\tilde{h}_t + A(\varepsilon\tilde{h})v\cdot\nabla\tilde{h}_t + \frac{1}{\varepsilon}\nabla\cdot v_t = F_t, \tag{29}$$

$$\partial_t v_t + v\cdot\nabla v_t + \frac{1}{\varepsilon}\nabla\tilde{h}_t = \frac{1}{\varepsilon}\nabla\tilde{\phi}_t - v_t + G_t, \tag{30}$$

$$\Delta\tilde{\phi}_t = n'(\varepsilon\tilde{h}+h^0)\tilde{h}_t, \tag{31}$$

where

$$\begin{aligned} F_t &= A(\varepsilon\tilde{h})(v\cdot\nabla\tilde{h}_t - \partial_t(v\cdot\nabla\tilde{h})) + \frac{1}{\varepsilon}\big(\nabla\cdot v_t - A(\varepsilon\tilde{h})\partial_t(A^{-1}(\varepsilon\tilde{h})\nabla\cdot v)\big), \\ G_t &= v\cdot\nabla v_t - \partial_t(v\cdot\nabla v) = -v_t\cdot\nabla v. \end{aligned}$$

Friedrich's estimates for (29)-(30) give

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}(A(\varepsilon\tilde{h})|\tilde{h}_t|^2 + |v_t|^2)dx + \int_{\mathbb{T}^d}|v_t|^2dx \\ &\quad\le \int_{\mathbb{T}^d}(|F_t|^2+|G_t|^2)dx + c(M)\int_{T^d}(|\tilde{h}_t|^2+|v_t|^2)dx + \frac{1}{\varepsilon}\int_{\mathbb{T}^d}\nabla\tilde{\phi}_t\cdot v_t dx. \end{aligned}$$

The terms F_t and G_t can be easily controlled by

$$\begin{aligned} \|F_t\|_0 &\le \|A(\varepsilon\tilde{h})\|_\infty(\|\nabla\tilde{h}\|_\infty\|v_t\|_0 + c(M)\|\nabla\cdot v\|_\infty\|\tilde{h}_t\|_0) \le c(M)(\|\tilde{h}_t\|_0 + \|v_t\|_0), \\ \|G_t\|_0 &\le \|\nabla v\|_\infty\|v_t\|_0 \le c(M)\|v_t\|_0. \end{aligned}$$

The only delicate integral in the Friedrich's estimate is the term involving $1/\varepsilon$. In order to control it we use (29):

$$\begin{aligned} \frac{1}{\varepsilon}\int_{\mathbb{T}^d}\nabla\tilde{\phi}_t\cdot v_t dx &= -\frac{1}{\varepsilon}\int_{\mathbb{T}^d}\tilde{\phi}_t\nabla\cdot v_t dx \\ &= -\int_{\mathbb{T}^d}\tilde{\phi}_t F_t dx + \int_{\mathbb{T}^d}\tilde{\phi}_t A(\varepsilon\tilde{h})v\cdot\nabla\tilde{h}_t dx + \int_{\mathbb{T}^d}\tilde{\phi}_t A(\varepsilon\tilde{h})\tilde{h}_{tt}dx \\ &= P_1 + P_2 + P_3. \end{aligned}$$

The above estimate for F_t gives

$$P_1 \le c(M)\int_{\mathbb{T}^d}(|\tilde{\phi}_t|^2 + |\tilde{h}_t|^2 + |v_t|^2)dx.$$

In order to avoid the term $\nabla\tilde{h}_t$ in P_2, we integrate by parts:

$$\begin{aligned} P_2 &= -\int_{\mathbb{T}^d}\Big(\nabla\tilde{\phi}_t A(\varepsilon\tilde{h})v + \tilde{\phi}_t\nabla A(\varepsilon\tilde{h})v + \tilde{\phi}_t A(\varepsilon\tilde{h})\nabla\cdot v\Big)\tilde{h}_t dx \\ &\le c(M)\int_{\mathbb{T}^d}(|\nabla\tilde{\phi}_t|^2 + |\tilde{\phi}_t|^2 + |\tilde{h}_t|^2)dx. \end{aligned}$$

For the estimate of the remaining integral P_3, we take the time derivative of (31) and multiply the resulting equation by $\tilde{\phi}_t A(\varepsilon\tilde{h})/n'(\varepsilon\tilde{h}+h^0)$. This yields

$$A(\varepsilon\tilde{h})\tilde{\phi}_t\tilde{h}_{tt} = \frac{A(\varepsilon\tilde{h})}{n'(\varepsilon\tilde{h}+h^0)}\tilde{\phi}_t\Delta\tilde{\phi}_{tt} - \frac{A(\varepsilon\tilde{h})}{n'(\varepsilon\tilde{h}+h^0)}\partial_t(n'(\varepsilon\tilde{h}+h^0))\tilde{\phi}_t\tilde{h}_t.$$

Substituting the product $A(\varepsilon\tilde{h})\tilde{\phi}_t\tilde{h}_{tt}$ in P_3 by the above expression, we obtain, setting $g(\varepsilon\tilde{h}) = A(\varepsilon\tilde{h})/n'(\varepsilon\tilde{h}+h^0)$,

$$\begin{aligned}
P_3 &= \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\tilde{\phi}_t\Delta\tilde{\phi}_{tt}dx - \int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\partial_t(n'(\varepsilon\tilde{h}+h^0))\tilde{\phi}_t\tilde{h}_t dx \\
&\le -\frac{1}{2}\int_{\mathbb{T}^d} g(\varepsilon\tilde{h})\partial_t|\nabla\tilde{\phi}_t|^2 dx - \int_{\mathbb{T}^d} \nabla g(\varepsilon\tilde{h})\cdot\nabla\tilde{\phi}_{tt}\tilde{\phi}_t dx \\
&\quad + c(M)\|\partial_t n'(\varepsilon\tilde{h}+h^0)\|_\infty \int_{\mathbb{T}^d} |\tilde{\phi}_t\tilde{h}_t| dx \\
&\le -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} g(\varepsilon\tilde{h})|\nabla\tilde{\phi}_t|^2 dx + \frac{1}{2}\int_{\mathbb{T}^d} \partial_t g(\varepsilon\tilde{h})|\nabla\tilde{\phi}_t|^2 dx \\
&\quad + c(M)\int_{\mathbb{T}^d} (|\tilde{\phi}_t|^2 + |\tilde{h}_t|^2 + |\varepsilon\nabla\tilde{\phi}_{tt}|^2) dx \\
&\le -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} g(\varepsilon\tilde{h})|\nabla\tilde{\phi}_t|^2 dx + c(M)\int_{\mathbb{T}^d} (|\nabla\tilde{\phi}_t|^2 + |\tilde{\phi}_t|^2 + |\tilde{h}_t|^2 + |\varepsilon\nabla\tilde{\phi}_{tt}|^2) dx.
\end{aligned}$$

We employ again the Poisson equation to deal with the term $\varepsilon\nabla\tilde{\phi}_{tt}$. From (29) we find that

$$\begin{aligned}
\Delta\tilde{\phi}_{tt} &= \partial_t(n'(\varepsilon\tilde{h}+h^0))\tilde{h}_t + n'(\varepsilon\tilde{h}+h^0)\tilde{h}_{tt} \\
&= \partial_t(n'(\varepsilon\tilde{h}+h^0))\tilde{h}_t - n'(\varepsilon\tilde{h}+h^0)v\cdot\nabla\tilde{h}_t - \frac{1}{\varepsilon}\frac{n'(\varepsilon\tilde{h}+h^0)}{A(\varepsilon\tilde{h})}\nabla\cdot v_t + \frac{n'(\varepsilon\tilde{h}+h^0)}{A(\varepsilon\tilde{h})}F_t.
\end{aligned}$$

Multiplying this equation by $\varepsilon^2\tilde{\phi}_{tt}$, integrating over $\mathbb{T}^d$ and integrating by parts yields

$$\begin{aligned}
\int_{\mathbb{T}^d} |\varepsilon\nabla\tilde{\phi}_{tt}|^2 dx &= -\varepsilon^2\int_{\mathbb{T}^d} \partial_t(n'(\varepsilon\tilde{h}+h^0))\tilde{h}_t\tilde{\phi}_{tt} dx + \varepsilon^2\int_{\mathbb{T}^d} n'(\varepsilon\tilde{h}+h^0)v\cdot\nabla\tilde{h}_t\tilde{\phi}_{tt} dx \\
&\quad + \varepsilon\int_{\mathbb{T}^d} \frac{n'(\varepsilon\tilde{h}+h^0)}{A(\varepsilon\tilde{h})}\nabla\cdot v_t\tilde{\phi}_{tt} dx - \varepsilon^2\int_{\mathbb{T}^d} \frac{n'(\varepsilon\tilde{h}+h^0)}{A(\varepsilon\tilde{h})}F_t\tilde{\phi}_{tt} dx \\
&= -\varepsilon^2\int_{\mathbb{T}^d} \partial_t(n'(\varepsilon\tilde{h}+h^0))\tilde{h}_t\tilde{\phi}_{tt} dx - \varepsilon^2\int_{\mathbb{T}^d} \nabla\cdot(n'(\varepsilon\tilde{h}+h^0)v)\tilde{h}_t\tilde{\phi}_{tt} dx \\
&\quad - \varepsilon^2\int_{\mathbb{T}^d} n'(\varepsilon\tilde{h}+h^0)\tilde{h}_t v\cdot\nabla\tilde{\phi}_{tt} dx + \varepsilon\int_{\mathbb{T}^d} \nabla\left(\frac{n'(\varepsilon\tilde{h}+h^0)}{A(\varepsilon\tilde{h})}\right)\cdot v_t\tilde{\phi}_{tt} dx \\
&\quad + \varepsilon\int_{\mathbb{T}^d} \frac{n'(\varepsilon\tilde{h}+h^0)}{A(\varepsilon\tilde{h})}v_t\cdot\nabla\tilde{\phi}_{tt} dx - \varepsilon^2\int_{\mathbb{T}^d} \frac{n'(\varepsilon\tilde{h}+h^0)}{A(\varepsilon\tilde{h})}F_t\tilde{\phi}_{tt} dx
\end{aligned}$$

$$\leq \frac{1}{2}\int_{\mathbb{T}^d}|\varepsilon\nabla\tilde\phi_{tt}|^2dx + c(M)\int_{\mathbb{T}^d}(|\tilde h_t|^2+|v_t|^2)dx,$$

where we have used Poincaré's inequality

$$\int_{\mathbb{T}^d}|\tilde\phi_{tt}|^2dx \leq c\int_{\mathbb{T}^d}|\nabla\tilde\phi_{tt}|^2dx,$$

which is allowed since the integral over $\tilde\phi$ and hence also over $\tilde\phi_{tt}$ vanishes. Thus, the estimate of P_3 becomes

$$P_3 \leq -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}g(\varepsilon\tilde h)|\nabla\tilde\phi_t|^2dx + c(M)\int_{\mathbb{T}^d}(|\nabla\tilde\phi_t|^2+|\tilde h_t|^2+|v_t|^2)dx.$$

We end up with an estimate of $\varepsilon^{-1}\int\nabla\tilde\phi_t\cdot v_t dx$, and thus, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\left(A(\varepsilon\tilde h)|\tilde h_t|^2+|v_t|^2+\frac{A(\varepsilon\tilde h)}{2n'(\varepsilon\tilde h+h^0)}|\nabla\tilde\phi_t|^2\right)dx+\int_{\mathbb{T}^d}|v_t|^2dx$$
$$\leq c(M)\int_{\mathbb{T}^d}(|\nabla\tilde\phi_t|^2+|\tilde h_t|^2+|v_t|^2)dx.$$

The proof of the lemma is completed by application of Gronwall's lemma if a bound for the initial data is available, i.e. $\|(\tilde h_t, v_t, \nabla\tilde\phi_t)(\cdot,0)\|_0 \leq c(M_0)$. In fact, from (9) and using assumption $v_I^\varepsilon = v_I^0$ with $\nabla\cdot v_I^0 = 0$, we have

$$\|\tilde h_t(\cdot,0)\|_0 \leq c(M_0)(\|\nabla(h_I^\varepsilon - h^0)\|_0 + \|v_I^1\|_0) \leq c(M_0),$$

and similarly for $v_t(\cdot,0)$ with the help of assumption (12) and $\|\nabla\tilde\phi_t(\cdot,0)\|_0 \leq C\|\tilde h_t(\cdot,0)\|_0$. □

3. Proof of Theorem 2

Let $(h^\varepsilon, v^\varepsilon, \phi^\varepsilon)$ be a (classical) solution to (4)-(5) in the interval $[0,T_0]$ with T_0 independent of ε. The estimates of Lemmas 3 and 4 show that the following uniform bounds hold:

$$\begin{aligned} |||\varepsilon^{-1}(h^\varepsilon-h^0)|||_{s,T_0} + |||v^\varepsilon|||_{s,T_0} + |||\varepsilon^{-1}\nabla\phi^\varepsilon|||_{s,T_0} &\leq M, \\ |||\varepsilon^{-1}h_t^\varepsilon|||_{0,T_0} + |||v_t^\varepsilon|||_{0,T_0} + |||\varepsilon^{-1}\nabla\phi_t^\varepsilon|||_{0,T_0} &\leq c(M). \end{aligned}$$

The inequalities imply that, as $\varepsilon \to 0$,

$$h^\varepsilon \to h^0, \quad \nabla\phi^\varepsilon \to 0 \qquad \text{strongly in } L^\infty(0,T_0;H^s(\mathbb{T}^d)) \cap C^{0,1}([0,T_0];L^2(\mathbb{T}^d)).$$

Furthermore, by Aubin's lemma (see Theorem 5 in [26]), there exists a subsequence of v^ε, which is not relabeled, such that

$$v^\varepsilon \to v^0 \qquad \text{strongly in } L^\infty(0,T_0;H^\alpha(\mathbb{T}^d)) \text{ for all } \alpha < s,$$

where $v^0 \in C^0([0,T_0];C^1(\mathbb{T}^d)) \cap C^{0,1}([0,T_0];L^2(\mathbb{T}^d))$.

It remains to show that v^0 is a solution of the incompressible Euler equations with damping. It holds for all $\chi \in C^\infty([0,T_0])$, $\psi \in C_0^\infty(\mathbb{T}^d;\mathbb{R}^d)$ such that $\nabla\cdot\psi = 0$,

$$\begin{aligned}\int_0^{T_0}\int_{\mathbb{T}^d}(v_t^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon + v^\varepsilon)\cdot\psi\chi dxdt &= \frac{1}{\varepsilon^2}\int_0^{T_0}\int_{\mathbb{T}^d}\nabla(\phi^\varepsilon - h^\varepsilon)\cdot\psi\chi dxdt \\ &= \frac{1}{\varepsilon^2}\int_0^{T_0}\int_{\mathbb{T}^d}(\phi^\varepsilon - h^\varepsilon)\nabla\cdot\psi\chi dxdt = 0.\end{aligned}$$

Letting $\varepsilon \to 0$ in the above equation gives

$$\int_{Q_{T_0}}(v^0\psi\chi_t + v^0\cdot\nabla v^0\psi\chi + v^0\psi\chi)dxdt = 0.$$

By the definition of weak derivatives, we conclude that

$$v_t^0 = -P(v^0\cdot\nabla v^0 + v^0),$$

where P is the standard projection on the set of divergence-free vector fields. Since we already have $v^0 \in C^0([0,T_0];C^1(\mathbb{T}^d))\cap L^\infty(0,T_0;H^s(\mathbb{T}^d))$, which implies that $v^0\cdot\nabla v^0 + v^0 \in C^0([0,T_0];C^0(\mathbb{T}^d))\cap L^\infty(0,T_0;H^{s-1}(\mathbb{T}^d))$, we infer

$$v_t^0 \in C^0(\mathbb{T}^d\times[0,T_0])\cap L^\infty(0,T_0;H^{s-1}(\mathbb{T}^d)).$$

Thus, $v^0 \in C^1(\mathbb{T}^d\times[0,T_0])$ is a classical solution to

$$\nabla\cdot v^0 = 0, \quad P(v_t^0 + v^0\cdot\nabla v^0 + v^0) = 0, \quad v^0(x,0) = v_I^0(x), \quad x\in\mathbb{T}^d,\ t>0.$$

The second equation and the regularity of $v_t^0 + v^0\cdot\nabla v^0 + v^0$ show that there exists a function $\pi \in L^\infty(0,T_0;H^s(\mathbb{T}^d))$ such that

$$\nabla\cdot v^0 = 0, \quad v_t^0 + v^0\cdot\nabla v^0 + v^0 = \nabla\pi.$$

Taking into account the equation satisfied by v^ε and

$$v_t^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon + v^\varepsilon \rightharpoonup^* v_t^0 + v^0\cdot\nabla v^0 + v^0 = \nabla\pi \qquad \text{weakly* in } L^\infty(0,T_0;L^2(\mathbb{T}^d)),$$

we infer that

$$\frac{1}{\varepsilon}\nabla(\phi^\varepsilon - h^\varepsilon) = v_t^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon + v^\varepsilon \rightharpoonup^* \nabla\pi \qquad \text{weakly* in } L^\infty(0,T_0;L^2(\mathbb{T}^d)).$$

Finally, the uniqueness of smooth solutions to the incompressible Euler equations with damping implies the convergence of the whole sequences. This completes the proof.

References

[1] G. Alì, D. Bini and S. Rionero, Global existence and relaxation limit for smooth solutions to the Euler-Poisson model for semiconductors, *SIAM J. Appl. Math.* 32 (2000), 572-587.
[2] G. Alì and A. Jüngel, Global smooth solutions to the multi-dimensional hydrodynamic model for two-carrier plasmas, *J. Diff. Eqns.* 190 (2003), 663-685.
[3] K. Asano, On the incompressible limit of the compressible Euler equations, *Japan J. Appl. Math.* 4 (1987), 455-488.
[4] S. Cordier and E. Grenier, Quasineutral limit of an Euler-Poisson system arising from plasma physics, *Comm. Part. Diff. Eqns.* 25 (2000), 1099-1113.

[5] I. Gasser, C.D. Levermore, P. Markowich and C. Schmeiser, The initial time layer problem and the quasineutral limit in the semiconductor drift-diffusion model, *Eur. J. Appl. Math.* 12 (2001), 497-512.

[6] I. Gasser and P. Marcati, A quasi-neutral limit in a hydrodynamic model for charged fluids, *Monatsh. Math.* 138 (2003), 189-208.

[7] T. Goudon, A. Jüngel and Y.J. Peng, Zero-electron-mass limits in hydrodynamic models for plasmas, *Appl. Math. Lett.* 12 (1999), 75-79.

[8] E. Grenier, Oscillations in quasineutral plasmas, *Comm. Part. Diff. Eqns.* 21 (1996), 363-394.

[9] D. Hewett, Low-frequency electro-magnetic (Darwin) applications in plasma simulation, *Comput. Phys. Commum.* 84 (1994), 243-277.

[10] S. Junca and M. Rascle, Relaxation of the isothermal Euler-Poisson system to the drift-diffusion equations, *Quart. Appl. Math.* 58 (2000), 511-521.

[11] A. Jüngel, H.L. Li and A. Matsumura, The relaxation-time limit in the quantum hydrodynamic equations for semiconductors, *J. Diff. Eqns.* 225 (2006), 440-464.

[12] A. Jüngel and Y.J. Peng, A hierarchy of hydrodynamic models for plasmas : zero-relaxation-time limits, *Comm. Part. Diff. Eqns.* 24 (1999), 1007-1033.

[13] A. Jüngel and Y.J. Peng, Zero-relaxation-time limits in hydrodynamic models for plasmas revisited, *Z. Angew. Math. Phys.* 51 (2000), 385-396.

[14] A. Jüngel and Y.J. Peng, A hierarchy of hydrodynamic models for plasmas : zero-electron-mass limits in the drift-diffusion equations, *Ann. Inst. H. Poincaré, Anal. non lin.* 17 (2000), 83-118.

[15] A. Jüngel and Y.J. Peng, A hierarchy of hydrodynamic models for plasmas : quasi-neutral limits in the drift-diffusion equations, *Asympt. Anal.* 28 (2001), 49-73.

[16] A. Jüngel and I. Violet, The quasineutral limit in the quantum drift-diffusion equations, *Asympt. Anal.* 53 (2007), 139-157.

[17] F. Kazeminezhad, J. Dawson, J. Lebœuf, R. Sydora and D. Holland, A Vlasov ion zero mass electron model for plasma simulations, *J. Comput. Phys.* 102 (1992), 277-296.

[18] S. Klainerman and A. Majda, Singular perturbations of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.* 34 (1981), 481-524.

[19] S. Klainerman and A. Majda, Compressible and incompressible fluids, *Comm. Pure Appl. Math.* 35 (1982), 629-651.

[20] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer, New York, 1984.

[21] P. Marcati and R. Natalini, Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift diffusion equations, *Arch. Rat. Mech. Anal.* 129 (1995), 129-145.

[22] G. Métivier and S. Schochet, The incompressible limit of the non-isentropic Euler equations, *Arch. Rat. Mech. Anal.* 158 (2001), 61-90.

[23] Y.J. Peng, Some asymptotic analysis in steady-state Euler-Poisson equations for potential flow, *Asympt. Anal.* 36 (2003), 75-92.

[24] Y.J. Peng and Y.G. Wang, Boundary layers and quasi-neutral limit in steady state Euler-Poisson equations for potential flows, *Nonlinearity*, 17 (2004), 835-849.

[25] Y.J. Peng and Y.G. Wang, Convergence of compressible Euler-Poisson equations to incompressible type Euler equations, *Asympt. Anal.* 41 (2005), 141-160.

[26] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Math. Pura Appl.* 146 (1987), 65-96.

[27] M. Slemrod and N. Sternberg, Quasi-neutral limit for Euler-Poisson system, *J. Nonlinear Sci.* 11 (2001), 193-209.

[28] S. Ukai, The incompressible limit and the initial layer of the compressible Euler equation, *J. Math. Kyoto Univ.* 26 (1986), 323-331.

[29] S. Wang, Quasineutral limit of the Euler-Poisson system with and without viscosity, *Comm. Part. Diff. Eqns.* 29 (2004), 419-456.

[30] S. Wang, Z.P. Xin and P. Markowich, Quasineutral limit of the drift-diffusion model for semiconductors : the case of general sign-changing doping profile, *SIAM J. Math. Anal.* 37 (2006), 1854-1889.

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