EFFECTIVE VELOCITY IN COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH THIRD-ORDER DERIVATIVES

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Abstract. A formulation of certain barotropic compressible Navier-Stokes equations with third-order derivatives as a viscous Euler system is proposed by using an effective velocity variable. The equations model, for instance, viscous Korteweg or quantum Navier-Stokes flows. The formulation in the new variable allows for the derivation of an entropy identity, which is known as the BD (Bresch-Desjardins) entropy equation. As a consequence of this estimate, a new global-in-time existence result for the one-dimensional quantum Navier-Stokes equations with strictly positive particle densities is proved.

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1. Introduction

Fluid models with third-order derivatives occur, for instance, in the theory of capillarity with diffuse interfaces [17], in viscous shallow-water models [7], and quantum hydrodynamic equations for semiconductors [18]. In this note, we consider a specific class of third-order derivatives including special viscous Korteweg-type models and quantum Navier-Stokes equations. For these models, it is usually far from being trivial to derive suitable a priori (entropy) estimates and to prove the global-in-time existence of solutions, due to the strongly nonlinear third-order derivatives.

Bresch and Desjardins have found a new mathematical entropy, called BD entropy, for compressible Navier-Stokes models and viscous shallow-water equations allowing for density-dependent viscosities and third-order capillary terms [4, 5, 7]. The BD entropy estimate is based on the definition of a new velocity variable involving gradients of the particle density. In this note, we show that the BD entropy is a mathematical entropy for a class of Navier-Stokes equations with third-order derivatives, which are *not* covered by the class of equations studied in [3, 5, 6, 7]. Our class of equations contains Korteweg-type fluid models (not included in [7]) and quantum Navier-Stokes equations [8, 15].

Our main discovery is that, under suitable assumptions, the third-order expression can be *eliminated* by using the new velocity variable. We notice that the third-order terms do *not* vanish in the formulation of [5, 7]. The particle density and the new velocity then solve a viscous Euler system, which allows for the derivation of entropy estimates and a global-in-time existence result in one space dimension.

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More precisely, we consider the barotropic Euler equations for the particle density n and the fluid velocity u,

$$n_t + \mathrm{div}(nu) = 0, \tag{1}$$

$$(nu)_t + \mathrm{div}(nu \otimes u) + \nabla p(n) = nf + \mathrm{div}(S + K) \quad \text{in } \mathbb{R}^d,\ t > 0, \tag{2}$$

supplemented with the initial conditions

$$n(\cdot, 0) = n_I, \quad u(\cdot, 0) = u_I \quad \text{in } \mathbb{R}^d, \tag{3}$$

where $p(n)$ is the (density dependent) pressure, f describes some forces, and $d \geq 1$. The viscous stress tensor S is defined by

$$\mathrm{div}\, S = 2\mathrm{div}(\mu(n)D(u)) + \nabla(\lambda(n)\mathrm{div}\, u), \tag{4}$$

where μ and λ are the Lamé viscosity coefficients and $D(u) = (\nabla u + \nabla u^\top)/2$ is the symmetric part of the velocity gradient. The Korteweg-type stress tensor K is assumed to be given by

$$K = \mu(n)\nabla^2\xi(n) = \mu(n)\xi''(n)|\nabla n|^2 + \mu(n)\xi'(n)\nabla^2 n, \tag{5}$$

where $\nabla^2\xi$ is the Hessian of the (given) scalar function ξ. This definition includes two important examples.

- **Korteweg-type fluid model:** Korteweg [21] proposed a constitutive equation for the stress tensor including density gradients, being of the form
$$K = a_1|\nabla n|^2\mathbb{I} + a_2\nabla n \otimes \nabla n + a_3\Delta n\mathbb{I} + a_4\nabla^2 n,$$
where a_i are density-dependent functions and $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{d\times d}$. The standard capillary tensor is given by $a_1 = \kappa(n)/2$, $a_2 = -\kappa(n)$, $a_3 = n\kappa(n)$, and $a_4 = 0$ [14, formula (1.29)]. The expression for K in (5) is obtained after choosing $a_1 = \mu\xi''$, $a_4 = \mu\xi'$, and $a_2 = a_3 = 0$.
- **Quantum Navier-Stokes model:** Harvey [15] suggested a quantum fluid model including the viscosity $\mu(n) = \nu n$, consisting of the mass equation (1) and
$$(nu)_t + \mathrm{div}(nu \otimes u) + \nabla p(n) - 2\varepsilon^2 n\nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big) = nf + 2\nu\mathrm{div}(nD(u)), \tag{6}$$
where $\varepsilon > 0$ is the scaled Planck constant. This equation is obtained from (2) by choosing $\xi(n) = (\varepsilon^2/\nu)\log n$. For $\nu = 0$, system (1) and (6) is called the quantum hydrodynamic model, which is employed in semiconductor simulations [18]. Recently, Brull and Méhats [8] have derived nonlocal quantum Navier-Stokes equations, whose local version can be written in the above form (see Section 5 for details).

Navier-Stokes equations with density-dependent viscosities have been considered in the literature only recently. Liu, Xin, and Yang [23] proved a local-in-time existence result in one space dimension under a structural assumption on the viscosity. Later, the result has been improved to global existence under the mere condition $\mu(n) = n^\theta$ for certain ranges of $\theta > 0$ (and $\lambda(n) = 0$); see the references in [22, 25] and the recent work [27]. Mellet and Vasseur [24] used the BD entropy to achieve a global existence result for positive initial data. The two-dimensional Navier-Stokes equations with a third-order Korteweg stress tensor term and constant viscosity were analyzed in [16]. Bresch, Desjardins, and Lin [7] obtained a global existence result for Korteweg-type Navier-Stokes flows with the viscosities $\mu(n) = n$ or $\mu(n) = n^2$.

This note is organized as follows. In Section 2, we formulate our main results. The differences to the results of Bresch and Desjardins are explained in Section 3. In Section 4, we prove the main results. Finally, we sketch the derivation of the local quantum Navier-Stokes equations in Section 5.

2. Main results

Before we state our main theorems, we recall the results of Bresch and Desjardins [3, 5]. They have considered the system (1)-(2) with S given by (4) and

$$\operatorname{div} K = n\nabla\big(\sigma'(n)\Delta\sigma(n)\big). \tag{7}$$

The classical (free) energy of this system is the sum of the kinetic, internal, and capillary energies,

$$E_{\rm cl} = \int_{\mathbb{R}^3}\Big(\frac{1}{2}n|u|^2 + H(n) + \frac{1}{2}|\nabla\sigma(n)|^2\Big)dx,$$

and it satisfies the following identity:

$$\frac{dE_{\rm cl}}{dt} + \int_{\mathbb{R}^3}\big(2\mu(n)|D(u)|^2 + \lambda(n)(\operatorname{div} u)^2\big)dx = \int_{\mathbb{R}^3} nf\cdot u dx,$$

where H is a primitive of the enthalpy h satisfying $nh'(n) = p'(n)$ and $h(1) = 0$. Notice that H is convex if p is nondecreasing since $H''(n) = p'(n)/n$ for $n > 0$. If μ and λ are linked by the relation $\lambda(n) = 2(n\mu'(n) - \mu(n))$, one can write (1)-(2) as

$$\begin{aligned} & n_t + \operatorname{div}(nu) = 0, \\ & (nw)_t + \operatorname{div}(nu\otimes w) + \nabla p(n) = nf + \operatorname{div}(\mu(n)A(u)) + n\nabla\big(\sigma'(n)\Delta\sigma(n)\big). \end{aligned} \tag{8}$$

Furthermore, if $\sigma = 2\mu$, there is another energy identity, involving the BD entropy

$$E_{\rm BD} = \int_{\mathbb{R}^3}\Big(\frac{1}{2}n|w|^2 + H(n) + \frac{1}{2}|\nabla\mu(n)|^2\Big)dx, \tag{9}$$

reading as

$$\frac{dE_{\rm BD}}{dt} + \int_{\mathbb{R}^3}\big(2\mu(n)|A(u)|^2 + \sigma'(n)(\Delta\sigma(n))^2\big)dx = \int_{\mathbb{R}^3} nf\cdot w dx, \tag{10}$$

where

$$w = u + \nabla\phi(n) \quad \text{with } n\phi'(n) = 2\mu'(n)$$

is the new velocity variable and $A(u) = (\nabla u - \nabla u^\top)/2$ is the antisymmetric part of the velocity gradient. A similar identity holds when σ vanishes [3]. It gives additional a priori estimates which have been exploited for the analysis of the Navier-Stokes equations [6, 7, 22, 24].

For the system (1)-(2) with the third-order term (5), the classical energy seems not to lead to suitable a priori estimates. However, the BD entropy allows for an estimate and moreover, the system (1)-(2) can be *completely* written in terms of w as a *viscous Euler system.* More precisely, the following result holds.

Theorem 2.1 (Viscous Euler formulation)**.** *Let* (n, u) *be a smooth solution to* (1)-(3) *and let*

$$\mu'(n) = n\xi'(n), \quad \lambda(n) = n\mu'(n) - \mu(n) \quad \textit{for all } n > 0 \tag{11}$$

and (4)-(5) *hold. Then* (n, w) *with* $w = u + \nabla\xi(n)$ *is a smooth solution to the* viscous Euler system

$$n_t + \operatorname{div}(nw) = \Delta\mu(n), \tag{12}$$

$$(nw)_t + \operatorname{div}(nw \otimes w) + \nabla p(n) = nf + \Delta(\mu(n)w) \quad \text{in } \mathbb{R}^d,\ t > 0, \tag{13}$$

$$n(\cdot, 0) = n_I, \quad w(\cdot, 0) = w_I := u_I + \nabla\xi(n_I) \quad \text{in } \mathbb{R}^d. \tag{14}$$

Moreover, if (n, w) *is a smooth solution to* (12)-(14)*, then* (n, u) *with* $u = w - \nabla\xi(n)$ *solves* (1)-(3)*. Finally, the following energy identity holds:*

$$\frac{d}{dt}\int_{\mathbb{R}^d}\Big(\frac{1}{2}n|w|^2 + H(n)\Big)dx + \int_{\mathbb{R}^d}\big(\mu(n)|\nabla w|^2 + \xi'(n)p'(n)|\nabla n|^2\big)dx = \int_{\mathbb{R}^d} nf \cdot w dx. \tag{15}$$

We stress the fact that this formulation is *not* included in the papers [5, 7] since our viscous stress tensor (5) is different. In particular, the special structure of (5) allows us to *eliminate* the third-order terms, leading to diffusive expressions in $\mu(n)$ and $\mu(n)w$. Notice that we need a slightly different relation between μ and λ compared to the relation of Bresch and Desjardins. The reason is that we are able to formulate the momentum equation completely in terms of w, which eliminates the term $2\mu(n)|A(u)|^2$ in (10).

The formulation in the velocity w enables us to prove a global existence result for the one-dimensional quantum Navier-Stokes model in the case $\varepsilon = \nu$.

Theorem 2.2 (Quantum Navier-Stokes model)**.** *Let* $d = 1$ *and* $n_I \in W^{1,\infty}(\mathbb{R})$*,* $u_I \in L^\infty(\mathbb{R})$ *such that* $n_I \geq \delta > 0$ *in* $\mathbb{R}$*. Assume that* $\mu(n) = n$ *for* $n \geq 0$*, (11),* $\varepsilon = \nu$*, and* $f = 0$*. Then there exists a smooth bounded solution* (n, u) *to* (1)-(2) *satisfying* $n(x, t) \geq c(\delta, t) > 0$ *for* $(x, t) \in \mathbb{R} \times [0, \infty)$*.*

The above theorem follows from the results of [10, 12] (see Section 4). The global existence of weak solutions to the multi-dimensional quantum Navier-Stokes equations on a torus with nonnegative particle densities and $\varepsilon < \nu$ was proved in [19]. Dong [13] generalized this result to $\varepsilon = \nu$.

3. Remarks

In this section, we comment the theorems of Section 2 and detail the differences to the results of Bresch and Desjardins.

The result of Theorem 2.1 is also valid using the more general expression

$$\operatorname{div} K = \operatorname{div}(\mu(n)\nabla^2\xi(n)) + n\nabla(\mu'(n)\Delta\mu(n)).$$

Indeed, a computation similar to the proof of Theorem 2.1 shows that

$$\frac{d}{dt}\int_{\mathbb{R}^d}\Big(\frac{1}{2}n|w|^2+H(n)\Big)dx+\int_{\mathbb{R}^d}\big(\mu'(n)(\Delta\mu(n))^2+\mu(n)|\nabla w|^2+\xi'(n)p'(n)|\nabla n|^2\big)dx = \int_{\mathbb{R}^d} nf\cdot w dx.$$

We expect that the existence result of Theorem 2.2 can be generalized to adiabatic pressures $p(n) = n^\alpha$ with $\alpha > 1$, more general functions μ, or bounded domains with suitable boundary conditions. On the other hand, the global existence in several space dimensions, with strictly positive particle density, is less obvious. An existence proof with nonnegative particle density is shown in [19] with $\mu(n) = \nu n$ and $\varepsilon < \nu$.

One may ask the question under which conditions the viscous stress tensor (5) and (7) coincide. We claim that this is the case if and only if

$$\sigma(n) = 2c_0\sqrt{n} + c_1, \quad \mu(n) = \sqrt{c_0^2 n^2 + c_2}, \quad n\xi'(n) = \mu'(n), \quad c_i \in \mathbb{R}.$$

To prove this statement, we identify the coefficients in

$$\begin{aligned}
\operatorname{div}(\mu(n)\nabla^2\xi(n)) &= (\mu\xi'')'(n)|\nabla n|^2\nabla n + (\mu\xi'')(n)\Delta n\nabla n \\
&\quad + \big((\mu\xi')' + \mu\xi''\big)(n)\nabla^2\nabla n + (\mu\xi')(n)\nabla\Delta n, \\
n\nabla(\sigma'(n)\Delta\sigma(n)) &= n(\sigma'\sigma'')'(n)|\nabla n|^2\nabla n + 2n(\sigma'\sigma'')(n)\Delta n\nabla n \\
&\quad + 2n(\sigma'\sigma'')(n)\nabla^2\nabla n + n(\sigma'(n))^2\nabla\Delta n
\end{aligned}$$

to find that $\mu\xi' = n(\sigma')^2$ and $(\mu\xi')' + \mu\xi'' = 2n\sigma'\sigma''$, which implies that $\mu\xi'' = -(\sigma')^2$. Furthermore, we have $\mu\xi'' = 2n\sigma'\sigma''$, and hence, $2n\sigma'\sigma'' = -(\sigma')^2$. The general solution to this differential equation is given by $\sigma(n) = 2c_0\sqrt{n} + c_1$. We infer that $(\mu\xi')(n) = n(\sigma'(n))^2 = c_0^2$. Then, using $n\xi' = \mu'$, it follows that $(\mu'\mu)(n) = n\mu(n)\xi'(n) = c_0^2 n$ and $\mu(n) = \sqrt{c_0^2 n^2 + c_2}$.

Our results coincide with those of Bresch and Desjardins if $n\mu'(n) - \mu(n) = 0$ and $\phi(n) = 2\xi(n)$ for $n > 0$, which implies that $\mu(n) = c_0 n$ and $\phi(n) = 2c_0 \log n$. This corresponds to the quantum Navier-Stokes equations.

The effective velocity $w = u + \nabla\xi(n)$ also appears in related models. Brenner [2] suggested the modified Navier-Stokes model

$$n_t + \operatorname{div}(nw) = 0, \quad (nu)_t + \operatorname{div}(nu \otimes w) + \nabla p(n) = \operatorname{div} S, \tag{16}$$

which is similar to (8). Brenner interpretes u and w as the volume and mass velocities, respectively, which are related by the constitutive equation $u - w = \nu\nabla\log n$ with the phenomenological constant $\nu > 0$. The velocity $w = u - \nu\nabla\log n$ can be employed in the viscous quantum hydrodynamic model [20, p. 453],

$$\begin{aligned}
& n_t + \operatorname{div}(nu) = \nu\Delta n, \\
& (nu)_t + \operatorname{div}(nu \otimes u) + \nabla p(n) = nf + \nu\Delta(nu) + 2\varepsilon^2\nabla\Big(\frac{\Delta\sqrt{n}}{\sqrt{n}}\Big).
\end{aligned}$$

In [26, p. 328], the expression $\nu\nabla\log n$ is referred to as the quantum diffusive velocity. In the new velocity formulation (n, w), the factor of the third-order quantum term does not vanish but it increases to $2\varepsilon^2 + 2\nu^2$. We report that related variables have been used too. The velocity $w = u - D\nabla n$, where D is a diffusion coefficient, has been employed in the analysis of the interfacial tension in the mixture of incompressible liquids [17, formula (3.6)]. Furthermore, an Euler-Korteweg model has been reformulated in [1] by using the complex variable $w = u + i\sqrt{n\kappa}\nabla\log n$, where $i^2 = -1$ and $\kappa = \kappa(n)$ is the capillary function. It turns out that in the new variable, the momentum equation becomes a variable-coefficient Schrödinger equation.

4. Proof of Theorems 2.1 and 2.2

First, we prove Theorem 2.1. Let (n, u) be a smooth solution to (1)-(3) and let $w = u + \nabla\xi(n)$. Equation (12) follows directly from

$$\operatorname{div}(nw) = \operatorname{div}(nu) + \operatorname{div}(n\xi'(n)\nabla n) = \operatorname{div}(nu) + \operatorname{div}(\mu'(n)\nabla n).$$

In order to reformulate the momentum equation, we need some auxiliary results. First, we compute, similarly as in [5], $\nabla\xi(n)_t = -\nabla(\xi'(n)\mathrm{div}(nu))$ and, using $\mu'(n) = n\xi'(n)$,

$$
\begin{aligned}
(n\nabla\xi(n))_t &= -\nabla\xi(n)\mathrm{div}(nu) - n\nabla\big(\xi'(n)\mathrm{div}(nu)\big) = -\nabla\big(n\xi'(n)\mathrm{div}(nu)\big) \\
&= -\nabla\big(\mu'(n)\mathrm{div}(nu)\big) = -\nabla\mathrm{div}(\mu(n)u) + \nabla\big((\mu(n) - n\mu'(n))\mathrm{div}\,u\big). \qquad (17)
\end{aligned}
$$

Second, we observe that

$$
\mathrm{div}\big(n\nabla\xi(n)\otimes\nabla\xi(n)\big) = \Delta(\mu(n)\nabla\xi(n)) - \mathrm{div}(\mu(n)\nabla^2\xi(n)), \qquad (18)
$$

$$
\mathrm{div}\big(n\nabla\xi(n)\otimes u + nu\otimes\nabla\xi(n)\big) = \Delta(\mu(n)u) - 2\mathrm{div}(\mu(n)D(u)) + \nabla\mathrm{div}(\mu(n)u). \qquad (19)
$$

Adding these two identities, using (17), and employing $\lambda(n) = n\mu'(n) - \mu(n)$, we arrive to

$$
\begin{aligned}
(nw)_t + \mathrm{div}(nw\otimes w) - \Delta(\mu(n)w) &= (nu)_t + \mathrm{div}(nu\otimes u) - 2\mathrm{div}(\mu(n)D(u)) - \nabla(\lambda(n)\mathrm{div}\,u) \\
&\quad + \nabla\big((\lambda(n) + \mu(n) - n\mu'(n))\mathrm{div}\,u\big) - \mathrm{div}(\mu(n)\nabla^2\xi(n)) \\
&= -\nabla p(n) + nf, \qquad (20)
\end{aligned}
$$

which proves (13). For the derivation of the energy identity, we differentiate

$$
\frac{d}{dt}\int_{\mathbb{R}^d}\Big(\frac{n}{2}|w|^2 + H(n)\Big)dx = \int_{\mathbb{R}^d}\Big(n_t\Big(-\frac{1}{2}|w|^2 + H'(n)\Big) + (nw)_t\cdot w\Big)dx. \qquad (21)
$$

Inserting (12)-(13) and integrating by parts, we obtain after a straightforward calculation

$$
\frac{d}{dt}\int_{\mathbb{R}^d}\Big(\frac{n}{2}|w|^2 + H(n)\Big)dx = -\int_{\mathbb{R}^d}\big(\mu(n)|\nabla w|^2 + \mu'(n)H''(n)|\nabla n|^2\big)dx. \qquad (22)
$$

By the definitions of μ and H, $\mu'(n)H''(n) = n\xi'(n)h'(n) = \xi'(n)p'(n)$, which proves the claim.

Finally, it remains to prove Theorem 2.2. We give only a sketch of the proof since the techniques are well known. The local existence of smooth solutions to (12)-(14) follows from the results of DiPerna [12]. Introduce the Riemann invariants $w = ne^u$ and $z = ne^{-u}$. By the theory of positive invariant regions from [10], we find that $\{(w,z) : w \le \text{const.},\ z \le \text{const.}\}$ is an invariant region of (12)-(13). This implies L^∞ bounds for (w,z) and also for (n,u). Furthermore, there exists a positive lower bound for n since the initial density is strictly positive; see [9, 12]. The L^∞ bounds together with the lower bound on n give the global existence of solutions.

5. Derivation of the quantum Navier-Stokes model

By employing a Chapman-Enskog expansion in the kinetic Wigner-BGK equation and a maximum quantum entropy closure, Brull and Méhats [8] derived the following nonlocal macroscopic model:

$$
\begin{aligned}
& n_t + \mathrm{div}(nu) = 0, \\
& (nu)_t + \mathrm{div}(nB\otimes u) + n\nabla B(u - B) + n\nabla(V - A) = \nu S, \qquad (23)
\end{aligned}
$$

where $\nu > 0$ is the relaxation time of the BGK collision operator in the Wigner equation, and (n,u) and (A,B) are related by a nonlocal expression. We do not need the precise definition of this relation but only an expansion of A and B in powers of ε^2 [11, Lemma 3.1]:

$$
A = -\log n + \frac{\varepsilon^2}{6}\frac{\Delta\sqrt{n}}{\sqrt{n}} - \frac{\varepsilon^2}{24}|\mathrm{curl}(u)|^2 + O(\varepsilon^4), \quad nB = nu + \frac{\varepsilon^2}{12}\mathrm{curl}(n\,\mathrm{curl}\,u) + O(\varepsilon^4). \qquad (24)
$$

The local model is derived under the assumption that the flow is nearly irrotational, $\mathrm{curl}\,u(\cdot,0) = O(\varepsilon)$, which implies that $\mathrm{curl}\,u(\cdot,t) = O(\varepsilon)$ for all $t > 0$. Then, inserting the expansions (24) and

$S = 2\mathrm{div}(nD(u)) + O(\varepsilon^2)$ [8, Remark 1] into formula (49) of [11], which is equivalent to (23), we arrive to (1) and (6).

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