The quasineutral limit in the quantum drift-diffusion equations*

Ansgar Jüngel† and Ingrid Violet‡

November 8, 2006

Abstract

The quasineutral limit in the transient quantum drift-diffusion equations in one space dimension is rigorously proved. The model consists of a fourth-order parabolic equation for the electron density, including the quantum Bohm potential, coupled to the Poisson equation for the electrostatic potential. The equations are supplemented with Dirichlet-Neumann boundary conditions. For the proof uniform a priori bounds for the solutions of the semi-discretized equations are derived from so-called entropy functionals. The drift term involving the electrostatic potential is estimated by proving a new bound for the electric energy. Since the electrostatic potential is not an admissible test function, an auxiliary test function has to be carefully constructed.

Keywords. Quantum drift-diffusion model, global-in-time existence of weak solutions, entropy estimates, quasi-neutral limit, asymptotic analysis, plasmas, semiconductors.

AMS Classification. 35K55, 35B25, 35B45, 35Q40.

1 Introduction

In charged particle transport, quasineutrality is a commonly used assumption in order to simplify the model equations. Quasineutrality means that the difference between the

*The authors acknowledge partial support from the Project "Hyperbolic and Kinetic Equations" of the European Union, grant HPRN-CT-2002-00282, and from the DAAD-Procope Program. The first author has been supported by the Deutsche Forschungsgemeinschaft, grants JU359/3 (Gerhard-Hess Award) and JU359/5 (Priority Program "Multi-scale Problems"). This research is part of the ESF program "Global and and geometrical aspects of nonlinear partial differential equations (GLOBAL)".

†Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, 55099 Mainz, Germany; e-mail: juengel@mathematik.uni-mainz.de.

‡Laboratoire de Mathématiques Appliquées, CNRS UMR 6620, Université Blaise Pascal (Clermont-Ferrand 2), 63177 Aubière, France; e-mail : violet@math.univ-bpclermont.fr.

concentrations of positive ions and electrons is negligible compared to a reference density. Formally, quasineutral models are obtained in the limit as the ratio of the Debye length to a characteristic length tends to zero. Quasineutral models are used, for instance, in semiconductor theory [31] and plasma physics [33]. Recently, quasineutrality has been studied also in quantum models [2]. An important quantum model are the quantum drift-diffusion equations which are a simple quantum extension of the drift-diffusion model used in both semiconductor and plasma theory (see [7] for a derivation and [8, 24] for reviews on macroscopic quantum models).

In this paper we analyze rigorously the quasineutral limit in the (scaled) quantum drift-diffusion equations in one space dimension for the electron density $n(x,t)$, the positively charged ion (or hole) density $p(x,t)$, and the electrostatic potential $V(x,t)$,

$$n_t - J_{n,x} = 0, \quad J_n = -\frac{\varepsilon^2}{2}(n(\log n)_{xx})_x + (P_n(n))_x - nV_x, \tag{1}$$

$$p_t + J_{p,x} = 0, \quad J_p = \frac{\varepsilon^2}{2}(p(\log p)_{xx})_x - (P_p(p))_x - pV_x, \tag{2}$$

$$\lambda^2 V_{xx} = n - p - C(x), \quad (x,t) \in Q_T = \Omega \times (0,T). \tag{3}$$

Here, J_n and J_p are the current densities and $C(x)$ models fixed charged background ions, usually called the doping concentration. The pressure functions P_n and P_p are typically of the form $P_\alpha(x) = \theta_\alpha x^{q_\alpha}$ $(\alpha = n,p)$ for some $\theta_\alpha > 0$ and $q_\alpha \geq 1$. The parameter ε is the scaled Planck constant and $\lambda > 0$ is the ratio of the Debye length to the characteristic length (e.g., the device diameter). The equations are supplemented with the initial and boundary conditions

$$n = p = 1, \quad n_x = p_x = 0, \quad V = V_D \qquad \text{for } x \in \{0,1\},\ t > 0, \tag{4}$$

$$n(\cdot,0) = n_I, \quad p(\cdot,0) = p_I \qquad \text{in } \Omega, \tag{5}$$

where $V_D(x) = xU$ and $U \in \mathbb{R}$ is the applied potential. In the case that the doping vanishes at the boundary, the Dirichlet boundary conditions for n and p express charge neutrality, whereas the Neumann boundary conditions have been employed in numerical simulations of quantum semiconductor devices [11].

The quantum drift-diffusion model can be derived by the entropy minimization principle from the Wigner-BGK equation in the diffusion limit [7] or from the so-called quantum hydrodynamic equations in the zero-relaxation-time limit [20]. The existence of weak solutions to the stationary equations have been proved in [1]; the transient equations in one space dimension are analyzed in [23] but only for electrons and isothermal pressure $P_n(n) = \theta_n n$. Numerical simulations can be found in [23, 29].

Mathematically, the parabolic equations (1)-(2) are of fourth order. In particular, no maximum principles are available which complicates the analysis [22, 23]. In this context, we mention the so-called Derrida-Lebowitz-Speer-Spohn equation [9], obtained from (1) for zero pressure and zero electric field. This equation has recently attracted a lot of attention in the mathematical literature since it possesses several Lyapunov functionals and there are connections to logarithmic Sobolev inequalities (see [10] and references therein).

The justification of the quasineutral limit in macroscopic models has been first studied in [4] for a nonlinear Poisson equation (the ion density being fixed). The limit in the drift-diffusion equations (i.e. (1)-(3) with $\varepsilon = 0$) has been proved in [13, 21] assuming vanishing or at least not sign-changing doping concentrations. Sign-changing doping profiles have been considered in [36]. The quasineutral limit in the steady state Euler-Poisson equations has been investigated in [26, 27, 28, 34], whereas in [6, 14, 15, 35] the time-dependent case has been analyzed. In [3, 17] the limit in the Vlasov-Poisson system has been shown. To our knowledge, no analytical results on the quasineutral limit in fluid-type *quantum* models are available up to now.

In the quasineutral limit $\lambda \to 0$ we obtain formally from (3) $n = p$ and from (1)-(2)

$$n_t + \frac{\varepsilon^2}{2}(n(\log n)_{xx})_{xx} = \frac{1}{2}(P_n(n) + P_p(n))_{xx}, \quad x \in \Omega,\ t > 0, \tag{6}$$

with initial and boundary conditions

$$n = 1,\ n_x = 0 \quad \text{for } x \in \{0,1\}, \quad n(\cdot, 0) = n_I \quad \text{in } \Omega,\ t > 0. \tag{7}$$

In this paper we make the limit rigorous for vanishing doping profile. First we show the existence of weak solutions to (1)-(4) (for general doping concentrations). In the literature, only results for the unipolar model are available with different boundary conditions [23] or with zero temperature and zero electric field [22]. Therefore, we include a proof for completeness. Moreover, our proof makes clear which quantities are bounded uniformly in the parameter λ (in appropriate norms).

More specifically, we show that the "entropy" $\int (n - \log n)dx$ is nonincreasing with respect to time and that the corresponding entropy production terms provide λ-uniform bounds for $\log n$ and $\log p$ in $L^2(0, T; H^2(\Omega))$ and for n and p in $L^{7/2}(Q_T)$. Also the entropy $\int n(\log n - 1)dx$ is nonincreasing in time, providing the uniform bounds

$$\|n - p\|_{L^2(Q_T)} \le c\lambda, \quad \|V_x\|_{L^2(Q_T)} \le c\lambda^{-1}. \tag{8}$$

These estimates are not sufficient to pass to the limit $\lambda \to 0$ in (1)-(3). Indeed, the sum of (1) and (2) leads to the drift term in weak formulation

$$\int_{Q_T} (n - p)V_x \phi_x dxdt \le \|n - p\|_{L^2(Q_T)} \|V_x\|_{L^2(Q_T)} \|\phi_x\|_{L^\infty(Q_T)} \le c,$$

where ϕ is some (smooth) test function and $c > 0$ a constant independent of λ. Thus, the estimates (8) only show that the above drift term is uniformly bounded; however, we need to prove that it converges to zero as $\lambda \to 0$. The main problem in this limit is that the (negative) electric field V_x is of the order $O(\lambda^{-1})$.

Our idea is to derive (instead of (8)) the estimates

$$\|\sqrt{n} - \sqrt{p}\|_{L^2(Q_T)} \le c\lambda, \quad \|(\sqrt{n} + \sqrt{p})V_x\|_{L^2(Q_T)} \le c\lambda^{-8/9}. \tag{9}$$

This gives

$$\int_{Q_T}(n-p)V_x\phi_x dxdt \le \|\sqrt{n}-\sqrt{p}\|_{L^2(Q_T)}\|(\sqrt{n}+\sqrt{p})V_x\|_{L^2(Q_T)}\|\phi_x\|_{L^\infty(Q_T)} \le \lambda^{1/9},$$

and hence, the drift term converges to zero as $\lambda \to 0$. The exponent $8/9$ in (9) is connected with the exponents of some Gagliardo-Nirenberg inequalities (see Lemma 13). The first bound in (9) is a consequence of the estimate using the "entropy" $\int(n-\log n)dx$. The proof of the second bound in (9) is more delicate. It follows from an estimate of the electric energy $\lambda^2\int(V-W)_x^2dx$ if W satisfies the boundary data of V *up to first order*, i.e. $W=V$ and $W_x=V_x$ at $x\in\{0,1\}$. Since $V_x(0,t)$ and $V_x(1,t)$ are only of the order $O(\lambda^{-1})$, W is of the same order and prevents an appropriate estimate. To solve this problem, we approximate W by a function W_δ in such a way that W_δ is of the order $O(1)+O(\delta\lambda^{-1})$ (in the $H^1(\Omega)$ norm). Passing to the limit $\delta\to 0$ then provides the needed estimate in (9).

Our main results are the following theorems.

Theorem 1. *Let $T>0$, $U\in\mathbb{R}$, $C\in L^\infty(\Omega)$, and $0\le n_I, p_I\in L^1(\Omega)$ satisfying*

$$\int_\Omega\big((n_I-\log n_I)+(p_I-\log p_I)\big)dx+\int_\Omega\big(n_I(\log n_I-1)+p_I(\log p_I-1)\big)dx<\infty.$$

Furthermore, let $P_n, P_p\in C^1([0,\infty))$ be nondecreasing and assume that there exist $0<q<7/2$ and $C_P>0$ such that

$$|P_\alpha(x)|\le C_P(1+|x|^q)\quad \textit{for all } x\ge 0,\ \alpha=n,p. \tag{10}$$

Then there exists a weak solution $n,\ p\in L^{7/2}(Q_T)$, $V\in L^\infty(0,T;H^1(\Omega))$ to (1)-(5) such that

$$n,p\ge 0 \textit{ in } Q_T,\quad \log n,\log p\in L^2(0,T;H_0^2(\Omega)),\quad n_t,p_t\in L^1(0,T;H^{-3}(\Omega)).$$

The idea of the proof is to use the exponential transformation $n=e^y$ and $p=e^z$ as in [22] since this automatically gives nonnegative particle densities. First we show the existence of weak solutions to a semi-discrete (elliptic) problem. Appropriate a priori estimates, which are also useful for the quasineutral limit, allow to pass to the limit of vanishing approximation parameter. We stress the fact that, although we employ ideas of [22], the existence theorem is needed since first, there is no existence result for the bipolar quantum drift-diffusion model in the literature; and secondly, the approximation argument is needed in the proof of the quasineutral limit due to the lack of regularity of solutions to (1)-(5).

It is possible to obtain an existence result for more general (non-homogeneous) boundary data but the proof is very technical; we refer to [18] for a related problem providing the needed mathematical tools.

Theorem 2. *Let the assumptions of Theorem 1 hold and let, in addition, $C(x)\equiv 0$, $q\le 7/3$ and $n_I=p_I$ in Ω. Let $(n^{(\lambda)},p^{(\lambda)},V^{(\lambda)})$ be a weak solution (in the sense of Theorem*

1) to (1)-(5). *Then there exists a subsequence of* $(n^{(\lambda)}, p^{(\lambda)}, V^{(\lambda)})$*, which is not relabeled, such that, as* $\lambda \to 0$,

$$\begin{aligned} n^{(\lambda)} \to n, \quad p^{(\lambda)} \to n \qquad & \textit{strongly in } L^3(Q_T), \\ n_t^{(\lambda)} \rightharpoonup n_t, \quad p_t^{(\lambda)} \rightharpoonup n_t \qquad & \textit{weakly in } L^{42/41}(0,T;H^{-3}(\Omega)), \\ \log n^{(\lambda)} \rightharpoonup \log n, \quad \log p^{(\lambda)} \rightharpoonup \log n \qquad & \textit{weakly in } L^2(0,T;H^2(\Omega)), \end{aligned}$$

and the limit function n *solves* (6)-(7).

Our assumptions avoid boundary and initial layers. We refer to [21] for the treatment of boundary layers and to [13] for the analysis of initial layers in the drift-diffusion model (cf. [27] and Remark 14).

Taking the difference of equations (1) and (2) provides in the limit $\lambda \to 0$ formally an equation for the electrostatic potential,

$$-((n+p)V_x)_x = (P_n(n) - P_p(n))_{xx} \quad \text{in } \Omega, \quad V(0,t) = 0, \quad V(1,t) = U.$$

However, since V_x is of the order $O(\lambda^{-1})$ we cannot justify this limit equation rigorously. In the drift-diffusion equations, this is possible under certain assumptions (see [21]).

If uniqueness of solutions holds for the problem (6)-(7), the whole sequence $(n^{(\lambda)}, p^{(\lambda)}, V^{(\lambda)})$ converges. However, there is no general uniqueness result for the limit problem. For a uniqueness theorem in the case of vanishing pressure under additional assumptions, we refer to [10].

The proof of Theorems 1 and 2 uses in several places the fact that we consider the one-dimensional equations. An existence proof for the multi-dimensional equations for vanishing pressure and vanishing electric field has been shown in [16] but only using periodic boundary conditions. The treatment of the quantum drift-diffusion model in several dimensions with physically motivated boundary conditions is currently not known.

Another interesting limit is the semiclassical limit $\varepsilon \to 0$. For a result in the stationary equations we refer to [1]. In [5] the limit has been shown in the transient case with homogeneous Neumann boundary conditions. Our a priori estimates seem to be not sufficient to perform the limit for the boundary conditions (4) (see Remark 6).

The paper is organized as follows. In section 2 we derive some a priori estimates needed for the existence result and we prove Theorem 1. The estimates are also useful for the quasineutral limit. Section 3 is devoted to the derivation of additional estimates independent of λ and the proof of Theorem 2.

Finally, for convenience of the reader, we recall the Gagliardo-Nirenberg inequality [19] which are employed several times throughout this paper.

Lemma 3. *Let* m, $k \in \mathbb{N}_0$ *with* $0 \le k \le m$, $0 \le \theta < 1$, *and* $1 \le p$, q, $r \le \infty$, *and let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with smooth boundary. If both*

$$k - \frac{d}{p} \le \theta\Big(m - \frac{d}{q}\Big) + (1-\theta)\Big(-\frac{d}{r}\Big) \quad \textit{and} \quad \frac{1}{p} \le \frac{\theta}{q} + \frac{1-\theta}{r},$$

then any function $f \in W^{m,q}(\Omega) \cap L^r(\Omega)$ belongs to $W^{k,p}(\Omega)$, and there exists a constant $C > 0$ independent of f such that

$$\|f\|_{W^{k,p}} \leq C\|f\|_{W^{m,q}}^{\theta}\|f\|_{L^r}^{1-\theta}. \tag{11}$$

2 Existence of solutions

2.1 A priori estimates

We divide the time interval $(0, T]$ for some $T > 0$ in N subintervals $(t_{k-1}, t_k]$ with $t_k = \tau k$, $k = 0, \ldots, N$, and $\tau = T/N$ is the time step. For given $k \in \{1, \ldots, N\}$ and y_{k-1}, $z_{k-1} \in H_0^2(\Omega)$ we solve the semi-discrete system

$$\begin{aligned}
\frac{1}{\tau}(e^{y_k} - e^{y_{k-1}}) + \frac{\varepsilon^2}{2}(e^{y_k} y_{k,xx})_{xx} &= \big((P_n(e^{y_k}))_x - e^{y_k} V_{k,x}\big)_x, &(12)\\
\frac{1}{\tau}(e^{z_k} - e^{z_{k-1}}) + \frac{\varepsilon^2}{2}(e^{z_k} z_{k,xx})_{xx} &= \big((P_p(e^{z_k}))_x + e^{z_k} V_{k,x}\big)_x, &(13)\\
\lambda^2 V_{k,xx} &= e^{y_k} - e^{z_k} - C(x) \quad \text{in } \Omega, &(14)
\end{aligned}$$

for y_k, $z_k \in H_0^2(\Omega)$, $V_k - V_D \in H^1(\Omega)$, where $V_D(x) = xU$, $x \in \Omega$. We introduce the piecewise constant functions

$$y^{(N)}(x,t) = y_k(x), \quad z^{(N)}(x,t) = z_k(x), \quad V^{(N)}(x,t) = V_k(x) \quad \text{for } x \in \Omega,\ t \in (t_{k-1}, t_k], \tag{15}$$

where $k = 1, \ldots, N$. First we show that the entropy

$$E_k^{(1)} = \int_\Omega ((e^{y_k} - y_k) + (e^{z_k} - z_k))\, dx$$

is non-increasing. Let y_k, $z_k \in H_0^2(\Omega)$, $V_k - V_D \in H_0^1(\Omega)$ be a solution to (12)-(14).

Lemma 4. *There exists a constant $c(\lambda) > 0$ which is independent of λ if $C(x) \equiv 0$ such that*

$$E_k^{(1)} + \frac{\varepsilon^2}{2}\sum_{j=1}^{k}\tau\int_\Omega (y_{j,xx}^2 + z_{j,xx}^2)dx + \frac{1}{\lambda^2}\sum_{j=1}^{k}\tau\int_\Omega (e^{y_j} - e^{z_j})(y_j - z_j)dx \leq c(\lambda)E_0^{(1)}. \tag{16}$$

Proof. We employ $1 - e^{-y_k} \in H_0^2(\Omega)$ as a test function in the weak formulation of (12) to obtain

$$\begin{aligned}
\frac{1}{\tau}\int_\Omega (e^{y_k} - e^{y_{k-1}})(1 - e^{-y_k})dx + \frac{\varepsilon^2}{2}\int_\Omega (y_{k,xx}^2 - y_{k,x}^2 y_{k,xx})dx&\\
= -\int_\Omega (P_n'(e^{y_k})y_{k,x}^2 - V_{k,x}y_{k,x})dx.& \qquad (17)
\end{aligned}$$

With the elementary inequality $e^x \geq 1 + x$ for $x \in \mathbb{R}$ we can write

$$\begin{aligned}(e^{y_k} - e^{y_{k-1}})(1 - e^{-y_k}) &= e^{y_k} - e^{y_{k-1}} + e^{y_{k-1} - y_k} - 1 \\ &\geq (e^{y_k} - y_k) - (e^{y_{k-1}} - y_{k-1}).\end{aligned}$$

Since $y_{k,x} = 0$ on the boundary, the second integral on the left-hand side of (17) becomes

$$\frac{\varepsilon^2}{2} \int_\Omega \Big(y_{k,xx}^2 - \frac{1}{3} (y_{k,x}^3)_x \Big) dx = \frac{\varepsilon^2}{2} \int_\Omega y_{k,xx}^2 dx.$$

Thus, it follows from (17), taking into account that $P_n'(x) \geq 0$ by assumption,

$$\frac{1}{\tau} \int_\Omega (e^{y_k} - y_k) dx + \frac{\varepsilon^2}{2} \int_\Omega y_{k,xx}^2 dx \leq \frac{1}{\tau} \int_\Omega (e^{y_{k-1}} - y_{k-1}) dx + \int_\Omega V_{k,x} y_{k,x} dx.$$

We obtain a similar equation for z_k. Then, adding both inequalities and using the Poisson equation (14), we arrive at

$$\begin{aligned}&\frac{1}{\tau} E_k^{(1)} + \frac{\varepsilon^2}{2} \int_\Omega (y_{k,xx}^2 + z_{k,xx}^2) dx \leq \frac{1}{\tau} E_{k-1}^{(1)} + \int_\Omega V_{k,x}(y_{k,x} - z_{k,x}) dx \\ &= \frac{1}{\tau} E_{k-1}^{(1)} - \frac{1}{\lambda^2} \int_\Omega (e^{y_k} - e^{z_k} - C(x))(y_k - z_k) dx \\ &\leq \frac{1}{\tau} E_{k-1}^{(1)} - \frac{1}{\lambda^2} \int_\Omega (e^{y_k} - e^{z_k})(y_k - z_k) dx + \frac{1}{\lambda^2} \|C\|_{L^\infty(\Omega)} \int_\Omega (|y_k| + |z_k|) dx.\end{aligned}$$

Since $|x| \leq e^x - x$ for all $x \in \mathbb{R}$, this yields

$$E_k^{(1)} + \tau \frac{\varepsilon^2}{2} \int_\Omega (y_{k,xx}^2 + z_{k,xx}^2) dx + \frac{\tau}{\lambda^2} \int_\Omega (e^{y_k} - e^{z_k})(y_k - z_k) dx \leq E_{k-1}^{(1)} + \frac{\tau}{\lambda^2} \|C\|_{L^\infty(\Omega)} E_k^{(1)}.$$

Hence, choosing $\tau > 0$ small enough, we obtain (16). $\square$

An immediate consequence of the entropy estimate (16) (and the Poincaré inequality) are the following uniform bounds for the functions $y^{(N)}$ and $z^{(N)}$ (see (15)):

$$\|y^{(N)}\|_{L^\infty(0,T;L^1(\Omega))} + \|z^{(N)}\|_{L^\infty(0,T;L^1(\Omega))} \leq c(\lambda), \tag{18}$$

$$\|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))} + \|e^{z^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))} \leq c(\lambda), \tag{19}$$

$$\|y^{(N)}\|_{L^2(0,T;H^2(\Omega))} + \|z^{(N)}\|_{L^2(0,T;H^2(\Omega))} \leq c(\lambda). \tag{20}$$

Again, if $C(x) \equiv 0$, the constant $c(\lambda)$ does not depend on λ. From these estimates we are able to deduce more uniform bounds.

Lemma 5. *There exists a constant $c(\lambda) > 0$ which does not depend on λ if $C(x) \equiv 0$ such that*

$$\|e^{y^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))} + \|e^{z^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))} \leq c(\lambda), \tag{21}$$

$$\|e^{y^{(N)}}\|_{L^{7/2}(Q_T)} + \|e^{z^{(N)}}\|_{L^{7/2}(Q_T)} \leq c(\lambda), \tag{22}$$

where we recall that $Q_T = \Omega \times (0, T)$.

Proof. We employ the Gagliardo-Nirenberg inequality and the estimates (18), (20) to find

$$\begin{aligned}\|y^{(N)}\|_{L^{5/2}(0,T;W^{1,\infty}(\Omega))} &\le \|y_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(\Omega))} \\ &\le \|y^{(N)}\|^{1/5}_{L^\infty(0,T;L^1(\Omega))}\|y^{(N)}\|^{4/5}_{L^2(0,T;H^2(\Omega))} \le c(\lambda).\end{aligned}$$

Therefore, with (18) and (19),

$$\begin{aligned}\|e^{y^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))} &\le c\left(\|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^1(\Omega))} + \|(e^{y^{(N)}})_x\|_{L^{5/2}(0,T;L^1(\Omega))}\right) \\ &\le c\|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^1(\Omega))} + c\|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))}\|y_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(\Omega))} \\ &\le c(\lambda).\end{aligned}$$

This shows (21). In order to prove (22) we use again the Gagliardo-Nirenberg inequality:

$$\begin{aligned}\|e^{y^{(N)}}\|^{7/2}_{L^{7/2}(Q_T)} &\le c\int_0^T \|e^{y^{(N)}}\|_{L^1(\Omega)}\|e^{y^{(N)}}\|^{5/2}_{W^{1,1}(\Omega)}dt \\ &\le c\|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))}\|e^{y^{(N)}}\|^{5/2}_{L^{5/2}(0,T;W^{1,1}(\Omega))} \le c(\lambda).\end{aligned}$$

The bounds for $e^{z^{(N)}}$ are derived in a similar way. □

Remark 6. The constants in (21)-(22) depend on ε since the estimates for $y^{(N)}$ and $z^{(N)}$ in $L^2(0,T;H^2(\Omega))$ do so. Hence, most of the subsequent bounds also depend on ε.

Lemma 7. *There exists a constant $c(\lambda) > 0$ depending on λ such that*

$$\|V^{(N)}\|_{L^2(0,T;H^1(\Omega))} \le c(\lambda). \tag{23}$$

Proof. By elliptic estimates,

$$\begin{aligned}\lambda^2\|V_x^{(N)}\|_{L^2(Q_T)} &\le c\left(\|e^{y^{(N)}} - e^{z^{(N)}} - C(x)\|_{L^2(0,T;H^{-1}(\Omega))} + 1\right) \\ &\le c\left(\|e^{y^{(N)}} - e^{z^{(N)}} - C(x)\|_{L^2(0,T;L^1(\Omega))} + 1\right) \le c(\lambda),\end{aligned}$$

since $L^1(\Omega)$ injects continuously into $H^{-1}(\Omega)$ in one space dimension. □

Finally, we need an estimate for the discrete time derivative. For this, we introduce the shift operator

$$(\sigma_N e^{y^{(N)}})(x,t) = e^{y_{k-1}(x)}, \quad (\sigma_N e^{z^{(N)}})(x,t) = e^{z_{k-1}(x)} \quad \text{for } x \in \Omega,\ t \in (t_{k-1}, t_k]. \tag{24}$$

Lemma 8. *There exists a constant $c(\lambda) > 0$ depending on λ such that for $s = \min\{7/2q, 14/11\} > 1$,*

$$\|e^{y^{(N)}} - \sigma_N e^{y^{(N)}}\|_{L^s(0,T;H^{-3}(\Omega))} + \|e^{z^{(N)}} - \sigma_N e^{z^{(N)}}\|_{L^s(0,T;H^{-3}(\Omega))} \le \tau c(\lambda). \tag{25}$$

Proof. We estimate the semi-discrete equation (12) in the norm of $L^s(0,T;H^{-3}(\Omega))$. This gives

$$\tau^{-1}\|e^{y^{(N)}} - \sigma_N e^{y^{(N)}}\|_{L^s(0,T;H^{-3}(\Omega))} \le \varepsilon^2 \|e^{y^{(N)}} y^{(N)}_{xx}\|_{L^s(0,T;H^{-1}(\Omega))} + \|P_n(e^{y^{(N)}})\|_{L^s(0,T;H^{-1}(\Omega))} + \|e^{y^{(N)}} V^{(N)}_x\|_{L^s(0,T;H^{-1}(\Omega))}.$$

The first term on the right-hand side is bounded by Hölder's inequality and (20), (22):

$$\begin{aligned}\|e^{y^{(N)}} y^{(N)}_{xx}\|_{L^s(0,T;H^{-1}(\Omega))} &\le c\|e^{y^{(N)}} y^{(N)}_{xx}\|_{L^s(0,T;L^s(\Omega))} \le c\|e^{y^{(N)}}\|_{L^{2s/(2-s)}(Q_T)} \|y^{(N)}_{xx}\|_{L^2(Q_T)} \\ &\le c\|e^{y^{(N)}}\|_{L^{7/2}(Q_T)} \|y^{(N)}_{xx}\|_{L^2(Q_T)} \le c(\lambda),\end{aligned}$$

since $2s/(2-s) \le 7/2$ is equivalent to $s \le 14/11$. For the second term on the above right-hand side we employ the growth condition on the pressure functions and (22):

$$\|P_n(e^{y^{(N)}})\|_{L^s(0,T;H^{-1}(\Omega))} \le c\|P_n(e^{y^{(N)}})\|_{L^s(0,T;L^s(\Omega))} \le c\left(1 + \|e^{y^{(N)}}\|^q_{L^{7/2}(Q_T)}\right) \le c(\lambda).$$

Finally, the last term on the right-hand side can be estimated by using (22) and (23):

$$\begin{aligned}\|e^{y^{(N)}} V^{(N)}_x\|_{L^s(0,T;H^{-1}(\Omega))} &\le c\|e^{y^{(N)}} V^{(N)}_x\|_{L^s(0,T;L^s(\Omega))} \\ &\le c\|e^{y^{(N)}}\|_{L^{2s/(2-s)}(Q_T)} \|V^{(N)}_x\|_{L^2(Q_T)} \le c(\lambda).\end{aligned}$$

Putting together the three inequalities gives (25). The proof for $z^{(N)}$ is analogous. □

2.2 Proof of Theorem 1

First we show that the semi-discrete problem (12)-(14) admits a solution.

Lemma 9. *Under the hypotheses of Theorem 1 there exists a sequence* $(y_k, z_k, V_k) \in (H^2_0(\Omega))^2 \times H^2(\Omega)$ *with* $V_k(0) = 0$ *and* $V_k(1) = U$ *satisfying* (12)-(14).

Proof. Let y_{k-1}, $z_{k-1} \in H^2_0(\Omega)$ be given. Let v, $w \in H^1(\Omega)$ and solve first

$$\lambda^2 V_{k,xx} = e^v - e^w - C(x) \quad \text{in } \Omega, \quad V_k(0) = 0,\ V_k(1) = U.$$

This problem admits a unique solution $V_k \in H^2(\Omega)$. Then we solve in $H^2_0(\Omega)$ the linear problems

$$\begin{aligned}\frac{\sigma}{\tau}(e^v - e^{y_{k-1}}) + \frac{\varepsilon^2}{2}(e^v y_{k,xx})_{xx} &= \sigma\big((P_n(e^v))_x - e^v V_{k,x}\big)_x, \\ \frac{\sigma}{\tau}(e^w - e^{z_{k-1}}) + \frac{\varepsilon^2}{2}(e^w z_{k,xx})_{xx} &= \sigma\big((P_p(e^w))_x + e^w V_{k,x}\big)_x,\end{aligned}$$

where $\sigma \in [0,1]$. There exists a unique solution $(y_k, z_k) \in (H^2_0(\Omega))^2$. This defines the fixed-point operator $S : (H^1(\Omega))^2 \times [0,1] \to (H^1(\Omega))^2$, $(v, w, \sigma) \mapsto (y_k, z_k)$. Then S is well defined and satisfies $S(v,w,0)=(0,0)$. Furthermore, it is not difficult to check that S is continuous and, in view of the compact embedding $H_0^2(\Omega)\hookrightarrow H^1(\Omega)$, also compact. It remains to show that there is a uniform bound for all fixed points of $S(\cdot,\cdot,\sigma)$. The estimates of section 2.1 establish the case $\sigma=1$. The estimates for $\sigma<1$ are similar (and, in fact, independent of σ). This provides the wanted bound in $H^1(\Omega)$ and the Leray-Schauder fixed-point theorem can be applied to yield the existence of a solution to (12)-(14). $\square$

Now we are able to prove Theorem 1. For this, we have to perform the limit $\tau\to 0$ in (12)-(14). Actually, the uniform bounds (21) and (25) and the compact embedding $W^{1,1}(\Omega)\hookrightarrow L^1(\Omega)$ allow to apply Theorem 5 of [32] (Aubin's lemma) yielding the existence of a subsequence of $e^{y^{(N)}}$ and $e^{z^{(N)}}$ (not relabeled) such that $e^{y^{(N)}}\to v$, $e^{z^{(N)}}\to w$ strongly in $L^1(Q_T)$ as $N\to\infty$ or, equivalently, $\tau\to 0$. Moreover, again for a subsequence which is not relabeled,

$$y^{(N)}\rightharpoonup y,\ z^{(N)}\rightharpoonup z\quad\text{weakly in } L^2(0,T;H^2(\Omega)) \tag{26}$$

as $\tau\to 0$. The bounds (18) and (19) allow to use the same arguments as in the proof of Theorem 1.2 in [18] showing that $v=e^y$ and $w=e^z$. Since, by (22), $(e^{y^{(N)}})$ is bounded in $L^{7/2}(Q_T)$ and $e^{y^{(N)}}\to e^y$ a.e., the result in [25, Ch. 1.3 and p. 144] yields

$$e^{y^{(N)}}\to e^y\quad\text{strongly in } L^2(Q_T). \tag{27}$$

Moreover, the same bound and hypothesis (10) imply that $(P_n(e^{y^{(N)}}))$ is bounded in $L^s(0,T;L^s(\Omega))$ for $s=7/2q>1$ and hence, by the same argument as before,

$$P_n(e^{y^{(N)}})\to P_n(e^y)\quad\text{strongly in } L^1(Q_T). \tag{28}$$

Finally, the bound (25) gives, up to a subsequence,

$$\frac{1}{\tau}(e^{y^{(N)}}-\sigma_N e^{y^{(N)}})\rightharpoonup (e^y)_t\quad\text{weakly in } L^s(0,T;H^{-3}(\Omega). \tag{29}$$

The same limits hold for $z^{(N)}$. Moreover, by (23),

$$V^{(N)}\rightharpoonup V\quad\text{weakly in } L^2(0,T;H^1(\Omega)). \tag{30}$$

The limits (26)-(30) allow to pass to the limit $\tau\to 0$ in the weak formulation of (12),

$$\begin{aligned}\int_0^T\int_\Omega \frac{1}{\tau}(e^{y^{(N)}}-\sigma_N e^{y^{(N)}})\phi\,dxdt+\frac{\varepsilon^2}{2}\int_0^T\int_\Omega e^{y^{(N)}}y_{xx}^{(N)}\phi_{xx}dxdt\\ =\int_0^T\int_\Omega\left(P_n(e^{y^{(N)}})\phi_{xx}+e^{y^{(N)}}V_x^{(N)}\phi_x\right)dxdt\end{aligned}$$

for all $\phi\in L^\infty(0,T;H^3(\Omega)\cap H_0^2(\Omega))$. The limit functions satisfy $y,\ z\in L^2(0,T;H_0^2(\Omega))$, $V-V_D\in L^\infty(0,T;H_0^1(\Omega))$, which shows that the boundary conditions are satisfied. Furthermore, the initial conditions hold in the sense of $H^{-3}(\Omega)$. This finishes the proof of Theorem 1.

3 The quasi-neutral limit

3.1 A priori estimates

For the quasi-neutral limit $\lambda \to 0$ we need additional estimates. We recall that the condition $C(x) \equiv 0$ implies that the uniform bounds (16)-(22) are independent of λ.

Lemma 10. *There exists a constant $c > 0$ independent of λ such that*

$$\|e^{y^{(N)}/2} - e^{z^{(N)}/2}\|_{L^2(Q_T)} \leq c\lambda. \tag{31}$$

Proof. The entropy estimate (16) gives

$$\int_{Q_T} (e^{y^{(N)}} - e^{z^{(N)}})(y^{(N)} - z^{(N)})dxdt \leq c\lambda^2.$$

Then the assertion follows if we can show that

$$2(\sqrt{x} - \sqrt{y})^2 \leq (x - y)(\log x - \log y) \quad \text{for all } x, y \geq 0. \tag{32}$$

This inequality can be seen as follows. It is sufficient to consider $x \geq y > 0$. Then (32) is equivalent to

$$2(\sqrt{x} - \sqrt{y}) \leq (\sqrt{x} + \sqrt{y}) \log \frac{x}{y}$$

and

$$2\frac{\sqrt{x/y} - 1}{\sqrt{x/y} + 1} = 2\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \leq 2 \log \sqrt{\frac{x}{y}}.$$

Thus we only need to prove that

$$\frac{z-1}{z+1} \leq \log z \quad \text{for all } z \geq 1.$$

But this is a consequence of $\log z \geq z - 1 \geq (z-1)/(z+1)$ for $z \geq 1$, thus proving the lemma. □

The following estimates are derived from the boundedness of the entropy

$$E_k^{(2)} = \int_\Omega (e^{y_k}(y_k - 1) + e^{z_k}(z_k - 1) + 2)\, dx > 0.$$

Lemma 11. *The following estimate holds:*

$$E_k^{(2)} + \frac{\varepsilon^2}{2} \sum_{j=1}^{k} \tau \int_\Omega (e^{y_j} y_{j,xx}^2 + e^{z_j} z_{j,xx}^2)dx + \frac{1}{\lambda^2} \sum_{j=1}^{k} \tau \int_\Omega (e^{y_j} - e^{z_j})^2 dx \leq E_0^{(2)}. \tag{33}$$

Proof. We employ the test function $y_k \in H_0^2(\Omega)$ in the weak formulation of (12) to obtain

$$\frac{1}{\tau}\int_\Omega (e^{y_k} - e^{y_{k-1}})y_k dx + \frac{\varepsilon^2}{2}\int_\Omega e^{y_k} y_{k,xx}^2 dx = -\int_\Omega (P_n'(e^{y_k})e^{y_k}y_{k,x}^2 - V_{k,x}e^{y_k}y_{k,x})dx. \quad (34)$$

The convexity of $x \mapsto e^x$ implies that $e^x - e^y - e^y(x-y) \geq 0$ and hence,

$$\begin{aligned}(e^{y_k} - e^{y_{k-1}})y_k &\geq (e^{y_k} - e^{y_{k-1}})y_k + e^{y_{k-1}}(y_k - y_{k-1}) - e^{y_k} + e^{y_{k-1}} \\ &= e^{y_k}(y_k - 1) - e^{y_{k-1}}(y_{k-1} - 1).\end{aligned}$$

Thus it follows

$$\frac{1}{\tau}\int_\Omega e^{y_k}(y_k - 1)dx + \frac{\varepsilon^2}{2}\int_\Omega e^{y_k}y_{k,xx}^2 dx \leq \frac{1}{\tau}\int_\Omega e^{y_{k-1}}(y_{k-1} - 1)dx + \int_\Omega V_{k,x}(e^{y_k})_x dx.$$

A similar inequality holds for z_k. Adding both inequalities and then employing the Poisson equation (14) gives

$$\frac{1}{\tau}E_k^{(2)} + \frac{\varepsilon^2}{2}\int_\Omega (e^{y_k}y_{k,xx}^2 + e^{z_k}z_{k,xx}^2)dx \leq \frac{1}{\tau}E_{k-1}^{(2)} - \frac{1}{\lambda^2}\int_\Omega (e^{y_k} - e^{z_k})^2 dx.$$

This gives the assertion. $\square$

From Lemma 11 immediately follows that

$$\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^2(Q_T)} \leq c\lambda$$

and therefore, for sufficiently small $\lambda > 0$,

$$\|V_x^{(N)}\|_{L^2(Q_T)} \leq c(1 + \lambda^{-2}\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^2(Q_T)}) \leq c\lambda^{-1}.$$

In the L^3 norm the exponent in λ is smaller as shown in the following lemma.

Lemma 12. *There exists a constant $c > 0$ independent of λ such that*

$$\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^3(Q_T)} \leq c\lambda^{2/9}, \quad (35)$$
$$\|V^{(N)}\|_{L^3(0,T;W^{2,3}(\Omega))} \leq c\lambda^{-16/9}. \quad (36)$$

Proof. By Hölder's inequality,

$$\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^3(Q_T)} \leq \|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^2(Q_T)}^{2/9}\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^{7/2}(Q_T)}^{7/9} \leq c\lambda^{2/9},$$

employing (22) and (33), which shows (35). The estimate (36) is a consequence from (35):

$$\|V_{xx}^{(N)}\|_{L^3(Q_T)} = \lambda^{-2}\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^3(Q_T)} \leq c\lambda^{-16/9}.$$

This finishes the proof. $\square$

The following lemma is our key result.

Lemma 13. *There exists a constant $c(\varepsilon) > 0$ independent of λ such that, for sufficiently small $\lambda > 0$,*

$$\|(e^{y^{(N)}/2} + e^{z^{(N)}/2})V_x^{(N)}\|_{L^2(Q_T)} \leq c\lambda^{-8/9}. \tag{37}$$

Proof. The key idea is to define a special extension $W_k(x)$ of the boundary data such that $W_k - V_k \in H_0^2(\Omega)$ becomes an admissible test function in the weak formulation of (12)-(13). The problem is that $V_{k,x}(0)$ and $V_{k,x}(1)$ are unbounded as $\lambda \to 0$. Therefore, we need to take special care in the definition of W_k. We define

$$W_k(x) = \begin{cases} \delta(V_{k,x}(0) - U)\left(\frac{x}{\delta}\right)^3 + 2\delta(U - V_{k,x}(0))\left(\frac{x}{\delta}\right)^2 + \delta V_{k,x}(0)\frac{x}{\delta} & : \ x \in [0,\delta] \\ xU & : \ x \in [\delta, 1-\delta] \\ \delta(U - V_{k,x}(1))\left(\frac{1-x}{\delta}\right)^3 & \\ \quad + 2\delta(V_{k,x}(1) - U)\left(\frac{1-x}{\delta}\right)^2 - \delta V_{k,x}(1)\frac{1-x}{\delta} + U & : \ x \in [1-\delta, 1]. \end{cases}$$

This function is continuously differentiable, is an element of $H^2(\Omega)$ and satisfies

$$W_k(0) = 0, \quad W_k(1) = U, \quad W_{k,x}(0) = V_{k,x}(0), \quad W_{k,x}(1) = V_{k,x}(1).$$

Let $W^{(N)}(\cdot, t) = W_k$ if $t \in (t_{k-1}, t_k]$. We claim that, for sufficiently small $\lambda > 0$,

$$\|W_x^{(N)}\|_{L^3(Q_T)} \leq c\delta\lambda^{-16/9}, \tag{38}$$

$$\|W_{xx}^{(N)}\|_{L^3(Q_T)} \leq c\lambda^{-16/9}. \tag{39}$$

Indeed, by elliptic estimates and (35), we have

$$\begin{aligned} \|W_x^{(N)}\|_{L^3(Q_T)} &\leq c(1 + \delta\|V_x^{(N)}(0,\cdot)\|_{L^3(0,T)} + \delta\|V_x^{(N)}(1,\cdot)\|_{L^3(0,T)}) \\ &\leq c(1 + \delta\|V^{(N)}\|_{L^3(0,T;W^{1,\infty}(\Omega))}) \leq c(1 + \delta\|V^{(N)}\|_{L^3(0,T;W^{2,1}(\Omega))}) \\ &\leq c(1 + \delta\lambda^{-2}\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^3(0,T;L^1(\Omega))}) \leq c\delta\lambda^{-16/9}. \end{aligned}$$

This shows (38). In order to prove (39), we use (36):

$$\begin{aligned} \|W_{xx}^{(N)}\|_{L^3(Q_T)} &\leq c(1 + \|V_x^{(N)}(0,\cdot)\|_{L^3(0,T)} + \|V_x^{(N)}(1,\cdot)\|_{L^3(0,T)}) \\ &\leq c(1 + \|V^{(N)}\|_{L^3(0,T;W^{1,\infty}(\Omega))}) \leq c\lambda^{-16/9}. \end{aligned}$$

Now we employ $W_k - V_k \in H_0^2(\Omega)$ as a test function in (12)-(13) and take the difference of the resulting equations to obtain

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega ((e^{y_k} - e^{z_k}) - (e^{y_{k-1}} - e^{z_{k-1}}))(W_k - V_k)dx \\ &\qquad + \frac{\varepsilon^2}{2}\int_\Omega (e^{y_k}y_{k,xx} - e^{z_k}z_{k,xx})(W_k - V_k)_{xx}dx \\ &= \int_\Omega (P_n(e^{y_k}) - P_p(e^{z_k}))(W_k - V_k)_{xx}dx + \int_\Omega (e^{y_k} + e^{z_k})V_{k,x}(W_k - V_k)_x dx. \end{aligned} \tag{40}$$

The first integral on the left-hand side can be estimated by means of the Poisson equation (14) and Young's inequality:

$$
\begin{aligned}
&\frac{1}{\tau}\int_\Omega \left((e^{y_k}-e^{z_k})-(e^{y_{k-1}}-e^{z_{k-1}})\right)(W_k-V_k)dx = \frac{\lambda^2}{\tau}\int_\Omega (V_k-V_{k-1})_x(V_k-W_k)_x dx \\
&= \frac{\lambda^2}{\tau}\int_\Omega (V_k-W_k)_x^2 dx - \frac{\lambda^2}{\tau}\int_\Omega (V_{k-1}-W_{k-1})_x(V_k-W_k)_x dx \\
&\quad + \frac{\lambda^2}{\tau}\int_\Omega (W_k-W_{k-1})_x(V_k-W_k)_x dx \\
&\ge \frac{\lambda^2}{2\tau}\int_\Omega (V_k-W_k)_x^2 dx - \frac{\lambda^2}{2\tau}\int_\Omega (V_{k-1}-W_{k-1})_x^2 dx + \frac{\lambda^2}{\tau}\int_\Omega (W_k-W_{k-1})_x(V_k-W_k)_x dx.
\end{aligned}
$$

Applying Young's inequality to the last integral in (40) gives

$$
\int_\Omega (e^{y_k}+e^{z_k})V_{k,x}(W_k-V_k)_x dx \le -\frac{1}{2}\int_\Omega (e^{y_k}+e^{z_k})V_{k,x}^2 dx + \frac{1}{2}\int_\Omega (e^{y_k}+e^{z_k})W_{k,x}^2 dx.
$$

Thus, summation over k in (40) yields

$$
\begin{aligned}
&\frac{\lambda^2}{2}\int_\Omega (V_N-W_N)_x^2 dx + \frac{1}{2}\sum_{k=1}^N \tau\int_\Omega (e^{y_k}+e^{z_k})V_{k,x}^2 dx \\
&\le \frac{\lambda^2}{2}\int_\Omega (V_0-W_0)_x^2 dx - \lambda^2\sum_{k=1}^N \int_\Omega (W_k-W_{k-1})_x(V_k-W_k)_x dx \\
&\quad + \frac{1}{2}\sum_{k=1}^N \tau\int_\Omega (e^{y_k}+e^{z_k})W_{k,x}^2 dx + \frac{\varepsilon^2}{2}\sum_{k=1}^N \tau\int_\Omega (e^{y_k}y_{k,xx}+e^{z_k}z_{k,xx})(V_k-W_k)_{xx} dx \\
&\quad - \sum_{k=1}^N \tau\int_\Omega (P_n(e^{y_k})-P_p(e^{z_k}))(V_k-W_k)_{xx} dx \\
&= I_1+\cdots+I_5.
\end{aligned}
$$

In the following, we write the integrals $I_1,\ldots,I_5$ in terms of $y^{(N)}$, $z^{(N)}$, $V^{(N)}$, and $W^{(N)}$.

For the first integral I_1 we notice that our assumption on the initial data gives $\lambda^2 V_{0,xx} = e^{y_0}-e^{z_0}=0$ in Ω which, together with the boundary conditions $V_0(0)=0$, $V_0(1)=U$, shows that V_0 is a linear function and in particular independent of λ. Thus, also $W_{0,x}$ does not depend on λ and

$$
I_1 \le \lambda^2\|V_x^{(N)}(\cdot,0)\|_{L^2(\Omega)}^2 + \lambda^2\|W_x^{(N)}(\cdot,0)\|_{L^2(\Omega)}^2 \le c.
$$

For I_2 we use (38):

$$
I_2 \le \frac{2\lambda^2}{\tau}\|W_x^{(N)}\|_{L^2(Q_T)}\left(\|V_x^{(N)}\|_{L^2(Q_T)}+\|W_x^{(N)}\|_{L^2(Q_T)}\right) \le \frac{c\delta\lambda^{2/9}}{\tau}(\lambda^{-1}+\delta\lambda^{-16/9}) \le \frac{c\delta}{\tau\lambda^{7/9}},
$$

choosing $\delta \le \lambda^{7/9}$. Taking into account (22) and (38) gives

$$I_3 \le \frac{1}{2}\left(\|e^{y^{(N)}}\|_{L^3(Q_T)} + \|e^{z^{(N)}}\|_{L^3(Q_T)}\right)\|W_x^{(N)}\|^2_{L^3(Q_T)} \le c\delta^2\lambda^{-32/9},$$

and an application of Hölder's inequality and (22), (33), (36), and (39) yield

$$\begin{aligned} I_4 &\le \frac{\varepsilon^2}{2}\left(\|e^{y^{(N)}/2}y_{xx}^{(N)}\|_{L^2(Q_T)}\|e^{y^{(N)}/2}\|_{L^6(Q_T)} + \|e^{z^{(N)}/2}z_{xx}^{(N)}\|_{L^2(Q_T)}\|e^{y^{(N)}/2}\|_{L^6(Q_T)}\right) \\ &\quad \times \left(\|V_{xx}^{(N)}\|_{L^3(Q_T)} + \|W_{xx}^{(N)}\|_{L^3(Q_T)}\right) \le c\lambda^{-16/9}. \end{aligned}$$

We proceed with the integral I_5 which we estimate using the growth condition on P_n and P_p and (36), (39):

$$\begin{aligned} I_5 &\le \left(\|P_n(e^{y^{(N)}})\|_{L^{3/2}(Q_T)} + \|P_n(e^{y^{(N)}})\|_{L^{3/2}(Q_T)}\right)\left(\|V_{xx}^{(N)}\|_{L^3(Q_T)} + \|W_{xx}^{(N)}\|_{L^3(Q_T)}\right) \\ &\le c\left(1 + \|e^{y^{(N)}}\|^q_{L^{3q/2}(Q_T)} + \|e^{z^{(N)}}\|^q_{L^{3q/2}(Q_T)}\right)\lambda^{-16/9} \le c\lambda^{-16/9}, \end{aligned}$$

since $3q/2 \le 7/2$ is equivalent to our assumption $q \le 7/3$.

The above estimates yield, for sufficiently small $\lambda > 0$,

$$\int_{Q_T}(e^{y^{(N)}} + e^{z^{(N)}})(V^{(N)})_x^2 dxdt \le c\left(1 + \lambda^{-16/9} + \delta^2\lambda^{-32/9} + \delta\tau^{-1}\lambda^{-7/9}\right)$$

Letting $\delta \to 0$ then gives the assertion. □

Remark 14. In order to avoid an initial time layer we have assumed that $n_I = p_I$. The above proof shows that it is enough to require that $\|n_I - p_I\|_{H^{-1}(\Omega)}$ is of the order $O(\lambda^{1/9})$. Indeed, the estimate

$$\lambda^2\|V_{0,x}\|^2_{L^2(\Omega)} \le c(1 + \lambda^{-2}\|e^{y_0} - e^{z_0}\|^2_{H^{-1}(\Omega)}) \le c\lambda^{-16/9}$$

shows that $I_1 \le c\lambda^{-16/9}$ holds.

Remark 15. The assumption $q \le 7/3$ can be improved to $q < 5/2$ by more technical effort. Indeed, this condition is only needed in the computation of the integral I_5. In order to show how I_5 can be estimated assuming only $q < 5/2$, we proceed as follows.

By the same arguments as in the proof of Lemma 12, we can derive

$$\|e^{y^{(N)}} - e^{z^{(N)}}\|_{L^r(Q_T)} \le c\lambda^\theta, \quad \|V^{(N)}\|_{L^r(0,T;W^{2,r}(\Omega))} \le c\lambda^{\theta-2}$$

for $2 < r < 7/2$ and $\theta = 2(7-2r)/3r \in (0,1)$. Then

$$I_5 \le c\left(1 + \|e^{y^{(N)}}\|^q_{L^{qr/(r-1)}(Q_T)} + \|e^{z^{(N)}}\|^q_{L^{qr/(r-1)}(Q_T)}\right)\lambda^{\theta-2} \le c\lambda^{\theta-2},$$

since $qr/(r-1) \le 7/2$ is equivalent to $q < 5/2$. This yields

$$\|(e^{y^{(N)}} + e^{z^{(N)}})V_x^{(N)}\|_{L^2(Q_T)} \le c\lambda^{\theta/2-1},$$

which is sufficient for the proof of Theorem 2. However, the proof of Lemma 16 below becomes more involved. Therefore, and since the improvement is only marginal, we have assumed the stronger condition $q \le 7/3$.

Lemma 16. *There exists a constant $c > 0$ independent of λ such that for $s = 42/41$,*

$$\|e^{y^{(N)}} + e^{z^{(N)}} - \sigma_N(e^{y^{(N)}} + e^{z^{(N)}})\|_{L^s(0,T;H^{-3}(\Omega))} \leq c\tau.$$

Recall that σ_N is the shift operator defined in (24).

Proof. We estimate the sum of equations (12) and (13):

$$\begin{aligned}
&\frac{1}{\tau}\|e^{y^{(N)}} + e^{z^{(N)}} - \sigma_N(e^{y^{(N)}} + e^{z^{(N)}})\|_{L^s(0,T;H^{-3}(\Omega))} \\
&\quad \leq \frac{\varepsilon^2}{2}\|e^{y^{(N)}}y^{(N)}_{xx} + e^{z^{(N)}}z^{(N)}_{xx}\|_{L^s(Q_T)} + \|P_n(e^{y^{(N)}}) + P_p(e^{z^{(N)}})\|_{L^s(Q_T)} \\
&\qquad + \|(e^{y^{(N)}} - e^{z})V^{(N)}_x\|_{L^s(Q_T)}.
\end{aligned}$$

The first term on the right-hand side is bounded by (20) and (22):

$$\|e^{y^{(N)}}y^{(N)}_{xx}\|_{L^s(Q_T)} \leq \|e^{y^{(N)}}\|_{L^{2s/(2-s)}(Q_T)}\|y^{(N)}_{xx}\|_{L^2(Q_T)} \leq \|e^{y^{(N)}}\|_{L^{21/10}(Q_T)}\|y^{(N)}_{xx}\|_{L^2(Q_T)} \leq c,$$

and similarly for the expression for $z^{(N)}$. Taking into account the growth assumption on P_n and (22) we find

$$\|P_n(e^{y^{(N)}})\|_{L^s(Q_T)} \leq c(1 + \|e^{y^{(N)}}\|^q_{L^q(Q_T)}) \leq c,$$

and analogously for $z^{(N)}$. For the drift term we need Lemma 10 and (22):

$$\|e^{y^{(N)}/2} - e^{z^{(N)}/2}\|_{L^{21/10}(Q_T)} \leq \|e^{y^{(N)}/2} - e^{z^{(N)}/2}\|^{8/9}_{L^2(Q_T)}\|e^{y^{(N)}/2} - e^{z^{(N)}/2}\|^{1/9}_{L^{7/2}(Q_T)} \leq c\lambda^{8/9}.$$

This yields, together with Lemma 13,

$$\begin{aligned}
\|(e^{y^{(N)}} - e^{z^{(N)}})V^{(N)}_x\|_{L^s(Q_T)} &\leq \|e^{y^{(N)}/2} - e^{z^{(N)}/2}\|_{L^{21/10}(Q_T)}\|(e^{y^{(N)}/2} + e^{z^{(N)}/2})V^{(N)}_x\|_{L^2(Q_T)} \\
&\leq c\lambda^{8/9}\lambda^{-8/9} = c.
\end{aligned}$$

Putting together the above bounds gives the assertion. □

3.2 Proof of Theorem 2

The results of section 2.2 allow to pass to the limit $\tau \to 0$ in the uniform estimates of the previous section. This yields weak solutions $y^{(\lambda)}$, $z^{(\lambda)}$, and $V^{(\lambda)}$ satisfying the equations

$$(e^{y^{(\lambda)}})_t + \frac{\varepsilon^2}{2}(e^{y^{(\lambda)}}y^{(\lambda)}_{xx})_{xx} = \left((P_n(e^{y^{(\lambda)}}))_x - e^{y^{(\lambda)}}V^{(\lambda)}\right)_x, \tag{41}$$

$$(e^{z^{(\lambda)}})_t + \frac{\varepsilon^2}{2}(e^{z^{(\lambda)}}z^{(\lambda)}_{xx})_{xx} = \left((P_p(e^{z^{(\lambda)}}))_x + e^{z^{(\lambda)}}V^{(\lambda)}\right)_x, \tag{42}$$

the boundary and initial conditions (4)-(5) and the following uniform bounds:

$$\begin{aligned}
\|e^{y^{(\lambda)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))} + \|e^{z^{(\lambda)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))} &\le c, \\
\|(e^{y^{(\lambda)}} + e^{z^{(\lambda)}})_t\|_{L^{42/41}(0,T;H^{-3}(\Omega))} &\le c, \\
\|y^{(\lambda)}\|_{L^2(0,T;H^2(\Omega))} + \|z^{(\lambda)}\|_{L^2(0,T;H^2(\Omega))} &\le c, \\
\|e^{y^{(\lambda)}}\|_{L^{7/2}(Q_T)} + \|e^{z^{(\lambda)}}\|_{L^{7/2}(Q_T)} &\le c
\end{aligned}$$

as well as, by Lemmas 10 and 13,

$$\|e^{y^{(\lambda)}/2} - e^{z^{(\lambda)}/2}\|_{L^2(Q_T)} \le c\lambda, \qquad \|(e^{y^{(\lambda)}/2} + e^{z^{(\lambda)}/2})V_x^{(\lambda)}\|_{L^2(Q_T)} \le c\lambda^{-8/9}. \tag{43}$$

Thus, Aubin's lemma and the arguments of section 2.2 show the existence of a subsequence (not relabeled) such that, as $\lambda \to 0$,

$$\begin{aligned}
e^{y^{(\lambda)}} \to e^y, \quad e^{z^{(\lambda)}} \to e^y \quad & \text{strongly in } L^3(Q_T) \text{ and weakly in } L^{7/2}(Q_T), \\
y^{(\lambda)} \rightharpoonup y, \quad z^{(\lambda)} \rightharpoonup y \quad & \text{weakly in } L^2(0,T;H^2(\Omega)), \\
(e^{y^{(\lambda)}} + e^{z^{(\lambda)}})_t \rightharpoonup 2(e^y)_t \quad & \text{weakly in } L^{42/41}(0,T;H^{-3}(\Omega)).
\end{aligned}$$

These convergence results imply for all sufficiently smooth ϕ, as $\lambda \to 0$,

$$\begin{aligned}
\int_0^T \langle (e^{y^{(\lambda)}} + e^{z^{(\lambda)}})_t, \phi\rangle_{H^{-3},H^3} dt &\to 2\int_0^T \langle (e^y)_t, \phi\rangle_{H^{-3},H^3} dt, \\
\int_{Q_T} (e^{y^{(\lambda)}} y_{xx}^{(\lambda)} + e^{z^{(\lambda)}} z_{xx}^{(\lambda)})\phi_{xx} dx dt &\to 2\int_{Q_T} e^y y_{xx}\phi_{xx} dx dt, \\
\int_{Q_T} (P_n(e^{y^{(\lambda)}}) + P_p(e^{z^{(\lambda)}}))\phi_{xx} dx dt &\to \int_{Q_T} (P_n(e^y) + P_p(e^y))\phi_{xx} dx dt.
\end{aligned}$$

The delicate integral is the expression containg the drift term. Here we need (43):

$$\begin{aligned}
&\int_{Q_T} (e^{y^{(\lambda)}} - e^{z^{(\lambda)}})V_x^{(\lambda)}\phi_x dx dt \\
&\le \|e^{y^{(\lambda)}/2} - e^{z^{(\lambda)}/2}\|_{L^2(Q_T)} \|(e^{y^{(\lambda)}/2} + e^{z^{(\lambda)}/2})V_x^{(\lambda)}\|_{L^2(Q_T)} \|\phi_x\|_{L^\infty(Q_T)} \\
&\le c\lambda \cdot \lambda^{-8/9} \le c\lambda^{1/9} \to 0.
\end{aligned}$$

These results allow to pass to the limit in the sum of the equations (41) and (42),

$$\begin{aligned}
&\int_0^T \langle (e^{y^{(\lambda)}} + e^{z^{(\lambda)}})_t, \phi\rangle_{H^{-3},H^3} dt + \frac{\varepsilon^2}{2}\int_{Q_T} (e^{y^{(\lambda)}} y_{xx}^{(\lambda)} + e^{z^{(\lambda)}} z_{xx}^{(\lambda)})\phi_{xx} dx dt \\
&= \int_{Q_T} (P_n(e^{y^{(\lambda)}}) + P_p(e^{z^{(\lambda)}}))\phi_{xx} dx dt + \int_{Q_T} (e^{y^{(\lambda)}} - e^{z^{(\lambda)}})V_x^{(\lambda)}\phi_x dx dt,
\end{aligned}$$

which proves Theorem 2.

References

[1] N. Ben Abdallah and A. Unterreiter. On the stationary quantum drift-diffusion model. *Z. Angew. Math. Phys.* 49 (1998), 251-275.

[2] L. Bonilla, V. Kochelap, and C. Velasco. Pattern formation and pattern stability under bistable electro-optical absorption in quantum wells I. *J. Phys. C* 11 (1999), 6395-6411.

[3] Y. Brenier and E. Grenier. Limite singulière du système de Vlasov-Poisson dans le régime de quasi neutralité: Le cas indépendant du temps. *C. R. Acad. Sci., Paris, Sér. I* 318 (1994), 121-124.

[4] H. Brezis, F. Golse and R. Sentis. Analyse asymptotique de l'équation de Poisson couplée à la relation de Boltzmann. Quasi-neutralité des plasmas. *C. R. Acad. Sci. Paris* 321 (1995), 953-959.

[5] L. Chen and Q. Ju. Existence of weak solution and semiclassical limit for quantum drift-diffusion model. Preprint, Tsinghua University, Beijing, China, 2005.

[6] S. Cordier and E. Grenier. Quasineutral limit of an Euler-Poisson system arising from plasma physics. *Commun. Part. Diff. Eqs.* 25 (2000), 1099-1113.

[7] P. Degond, F. Méhats, and C. Ringhofer. Quantum energy-transport and drift-diffusion models. *J. Stat. Phys.* 118 (2005), 625-665.

[8] P. Degond, F. Méhats, and C. Ringhofer. Quantum hydrodynamic models derived from the entropy principle. To appear in *Contemp. Math.*, 2005.

[9] B. Derrida, J. Lebowitz, E. Speer, and H. Spohn. Fluctuations of a stationary nonequilibrium interface. *Phys. Rev. Lett.* 67 (1991), 165-168.

[10] J. Dolbeault, I. Gentil, and A. Jüngel. A nonlinear fourth-order parabolic equation and related logarithmic Sobolev inequalities. *Commun. Math. Sci.* 4 (2006), 275-290.

[11] C. Gardner. The quantum hydrodynamic model for semiconductor devices. *SIAM J. Appl. Math.* 54 (1994), 409-427.

[12] I. Gasser. The initial time layer problem and the quasineutral limit in a nonlinear drift diffusion model for semiconductors. *NoDEA, Nonlin. Diff. Eqs. Appl.* 8 (2001), 237-249.

[13] I. Gasser, C.D. Levermore, P. Markowich, and C. Schmeiser. The initial time layer problem and the quasineutral limit in the semiconductor drift-diffusion model. *Eur. J. Appl. Math.* 12 (2001), 497-512.

[14] I. Gasser and P. Marcati. The combined relaxation and vanishing Debye length limit in the hydrodynamic model for semiconductors. *Math. Meth. Appl. Sci.* 24 (2001), 81-92.

[15] I. Gasser and P. Marcati. A quasi-neutral limit in a hydrodynamic model for charged fluids. *Monatsh. Math.* 138 (2003), 189-208.

[16] U. Gianazza, G. Savaré, and G. Toscani. A fourth-order nonlinear PDE as gradient flow of the Fisher information in Wasserstein spaces. Preprint, Università di Pavia, Italy, 2006.

[17] E. Grenier. Oscillations in quasineutral plasmas. *Commun. Partial Diff. Eqs.* 21 (1996), 363-394.

[18] M. Gualdani, A. Jüngel, and G. Toscani. A nonlinear fourth-order parabolic equation with non-homogeneous boundary conditions. *SIAM J. Math. Anal.* 37 (2006), 1761-1779.

[19] D. Henry. How to remember the Sobolev inequalities. Differential equations (São Paolo, 1981), Lecture Notes in Math. 957, pp. 97-109, Springer, Berlin, 1982.

[20] A. Jüngel, H.-L. Li, and A. Matsumura. The relaxation-time limit in the quantum hydrodynamic equations for semiconductors. *J. Diff. Eqs.* 225 (2006), 440-464.

[21] A. Jüngel and Y.-J. Peng. A hierarchy of hydrodynamic models for plasmas: Quasi-neutral limits in the drift-diffusion equations. *Asympt. Anal.* 28 (2001), 49-73.

[22] A. Jüngel and R. Pinnau. Global non-negative solutions of a nonlinear fourth-oder parabolic equation for quantum systems. *SIAM J. Math. Anal.* 32 (2000), 760-777.

[23] A. Jüngel and R. Pinnau. A positivity-preserving numerical scheme for a nonlinear fourth-order parabolic equation. *SIAM J. Num. Anal.* 39 (2001), 385-406.

[24] A. Jüngel and J.-P. Milišić. Macroscopic quantum models with and without collisions. To appear in *Proceedings of the Sixth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics*, Kyoto, Japan. *Transp. Theory Stat. Phys.*, 2005.

[25] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Paris, 1969.

[26] Y.-J. Peng. Some asymptotic analysis in steady-state Euler-Poisson equations for potential flow. *Asympt. Anal.* 36 (2003), 75-92.

[27] Y.-J. Peng and J.-G. Wang. Boundary layers and quasi-neutral limit in steady state Euler-Poisson equations for potential flows. *Nonlinearity* 17 (2004), 835-849.

[28] Y.-J. Peng and J.-G. Wang. Quasineutral limit and boundary layers in Euler-Poisson system. In: N. Kenmochi (ed.) et al., *Proceedings of the international conference on nonlinear partial differential equations and their applications, Shanghai, China, November 23-27, 2003*, Gakkotosho, Tokio (2004), 239-252.

[29] R. Pinnau. A Scharfetter-Gummel type discretization of the quantum drift diffusion model. *Proc. Appl. Math. Mech.* 2 (2003), 37-40.

[30] C. Schmeiser and S. Wang. Quasineutral limit of the drift-diffusion model for semiconductors with general initial data. *Math. Models Meth. Appl. Sci.* 13 (2003), 463-470.

[31] W. Shockley. The theory of p-n junctions in semiconductors and p-n junction transistors. *Bell. Syst. Tech. J.* 27 (1949), 435-489.

[32] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl., IV. Ser.* 146 (1987), 65-96.

[33] A. Sitenko and V. Malnev. *Plasma Physics Theory.* Chapman and Hall, London, 1995.

[34] M. Slemrod and N. Sternberg. Quasi-neutral limit for Euler-Poisson system. *J. Nonlinear Sci.* 11 (2001), 193-209.

[35] S. Wang. Quasineutral limit of Euler-Poisson system with and without viscosity. *Commun. Partial Diff. Eqs.* 29 (2004), 419-456.

[36] S. Wang, Z. Xin, and P. Markowich. Quasineutral limit of the drift diffusion models for semiconductors: The case of general sign-changing doping profile. *SIAM J. Math. Anal.* 37 (2006), 1854-1889.