ANALYSIS OF A PARABOLIC CROSS-DIFFUSION SEMICONDUCTOR MODEL WITH ELECTRON-HOLE SCATTERING

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Abstract. The global-in-time existence of nonnegative solutions to a parabolic strongly coupled system with mixed Dirichlet-Neumann boundary conditions is shown. The system describes the time evolution of the electron and hole densities in a semiconductor when electron-hole scattering is taken into account. The parabolic equations are coupled to the Poisson equation for the electrostatic potential. Written in the quasi-Fermi potential variables, the diffusion matrix of the parabolic system contains strong cross-diffusion terms and is only positive semi-definite such that the problem is formally of degenerate type. The existence proof is based on the study of a fully discretized version of the system, using a backward Euler scheme and a Galerkin method, on estimates for the free energy, and careful weak compactness arguments.

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1. Introduction

This work is a continuation of a series of papers [4, 5, 6, 7, 8] in which the authors study analytical properties of global-in-time solutions $u : \Omega \times [0, \infty) \to \mathbb{R}^n$ of specific parabolic systems of the form

$$\partial_t u - \mathrm{div}(A(u)\nabla u) = f(u) \quad \text{in } Q_T = \Omega \times (0, T), \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain and $T > 0$, together with appropriate boundary and initial conditions. The divergence operator has to be applied to each row of the matrix $A(u)\nabla u$, where $A(u) \in \mathbb{R}^{n\times n}$ and $\nabla u \in \mathbb{R}^{n\times d}$. In the papers [4, 5, 7, 8] the case $n = 2$ has been studied, whereas in [6], $n \in \mathbb{N}$ is arbitrary.

The main feature of (1) is that the diffusion matrix $A(u)$ is *not* symmetric and *not* positive definite and may have "large" non-diagonal elements (strong cross-diffusion case). Moreover, (some of) the components of the solution u represent nonnegative physical quantities, like a particle density or a temperature. Therefore, one is interested in solutions whose components are *nonnegative*. However, due to the strong coupling in (1), usually no

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maximum principle can be applied, and the nonnegativity property has to be derived by other techniques.

We assume that the system (1) has the property that there exists a transformation of variables $u = \rho(w)$ such that the transformed diffusion matrix $B(w) = A(\rho(w))\rho'(w)$ is symmetric and positive (semi-)definite:

$$\partial_t \rho(w) - \operatorname{div}(B(w)\nabla w) = f(\rho(w)). \tag{2}$$

Such systems have been first studied by Alt and Luckhaus in [2], but only for positive definite $B(w)$ and for solutions w which may change sign.

In this paper we consider a system of the form (1) arising in semiconductor device modeling. Compared to our previous work, the system has the *additional* difficulties that

- the transformed system (2) is of degenerate type,
- the coefficients of the matrix $B(w)$ are unbounded, and
- the regularity of the solutions w is very weak (i.e. $\nabla w \notin L^1$).

More precisely, we study the (scaled) equations

$$\partial_t n - \operatorname{div} J_n = R(n,p), \quad \partial_t p - \operatorname{div} J_p = R(n,p), \tag{3}$$

where n and p are the electron and hole densities of the semiconductor crystal, respectively, and the electron and hole current densities are given by

$$J_n = \mu_{nn}(n,p)(\nabla n - n\nabla\psi) + \mu_{np}(n,p)(\nabla p + p\nabla\psi), \tag{4}$$

$$J_p = \mu_{pn}(n,p)(\nabla n - n\nabla\psi) + \mu_{pp}(n,p)(\nabla p + p\nabla\psi), \tag{5}$$

where the mobilities are defined as

$$\mu_{nn}(n,p) = \frac{\mu_n(1+\mu_p n)}{1+\mu_p n + \mu_n p}, \quad \mu_{np}(n,p) = \frac{\mu_n \mu_p n}{1+\mu_p n + \mu_n p},$$
$$\mu_{pp}(n,p) = \frac{\mu_p(1+\mu_n p)}{1+\mu_p n + \mu_n p}, \quad \mu_{pn}(n,p) = \frac{\mu_p \mu_n p}{1+\mu_p n + \mu_n p},$$

and μ_n and μ_p are positive constants. The function ψ in (4)-(5) is the electrostatic potential, coupled to the particle densities via the Poisson equation,

$$\lambda^2 \Delta\psi = n - p - C(x), \tag{6}$$

where $\lambda > 0$ is the Debye length and $C(x)$ is the so-called doping profile which models fixed background charges in the semiconductor crystal. Finally, the Shockley-Read-Hall term $R(n,p)$ models generation-recombination processes,

$$R(n,p) = \frac{n_i^2 - np}{\tau_0 + \tau_n n + \tau_p p},$$

where τ_0, τ_n, and τ_p are the (positive) Shockley-Read-Hall constants, and $n_i > 0$ is the intrinsic density.

Equations (3)-(6) are solved in the bounded semiconductor domain $\Omega \subset \mathbb{R}^d$ ($d \geq 1$). The boundary $\partial\Omega$ consists of two disjoint subsets Γ_D and Γ_N. At the Ohmic contacts Γ_D

we prescribe the particle densities and the electrostatic potential, whereas Γ_N models the union of insulating boundary segments:

$$n = n_D, \quad p = p_D, \quad \psi = \psi_D \qquad \text{on } \Gamma_D,\ t > 0, \tag{7}$$

$$J_n \cdot \nu = J_p \cdot \nu = \nabla\psi \cdot \nu = 0 \qquad \text{on } \Gamma_N,\ t > 0, \tag{8}$$

where ν denotes the exterior unit normal of $\partial\Omega$ which is assumed to exist a.e. Finally, the initial conditions are

$$n(\cdot, 0) = n_0, \quad p(\cdot, 0) = p_0 \qquad \text{in } \Omega. \tag{9}$$

We refer to [3, 11, 16] for the physical background and the mathematical modeling of semiconductors.

The system (3)-(6) models the carrier transport through a semiconductor device which is strongly affected by electron-hole scattering. This happens, for instance, in high-injection situations [13, 17]. The model (3)-(6) can be formally derived from the semiconductor Boltzmann equation with a collision operator taking into account electron-hole scattering [18]. Notice that, setting $\mu_{nn}(n,p) = \mu_n$, $\mu_{pp}(n,p) = \mu_p$ and $\mu_{np}(n,p) = \mu_{pn}(n,p) = 0$, the above system reduces to the standard drift-diffusion equations [16]. In the following, we set the physical constants equal to one, i.e. $\mu_n = \mu_p = n_i = \tau_0 = \tau_n = \tau_p = \lambda = 1$, since they do not affect our analysis.

The diffusion matrix $A(u)$ with $u = (n,p)^\top$,

$$A(u) = \begin{pmatrix} \mu_{nn}(n,p) & \mu_{np}(n,p) \\ \mu_{pn}(n,p) & \mu_{pp}(n,p) \end{pmatrix},$$

is *not* symmetric and generally *not* positive definite. It is well known in the case of the standard drift-diffusion equations that the system becomes "symmetric" in the quasi-Fermi potentials

$$\phi_n = \ln n - \psi, \quad \phi_p = \ln p + \psi.$$

Also here, in the variables $w = (\phi_n, \phi_p)^\top$, the transformed diffusion matrix (see (2))

$$B(n,p) = \begin{pmatrix} n\mu_{nn}(n,p) & p\mu_{np}(n,p) \\ n\mu_{pn}(n,p) & p\mu_{pp}(n,p) \end{pmatrix}$$

becomes symmetric and positive (semi-)definite:

$$x^\top B x = \frac{nx_1^2 + px_2^2 + (nx_1 + px_2)^2}{1+n+p} \geq \frac{\min\{n,p\}}{1+n+p}\|x\|$$

for all $x = (x_1, x_2)^\top \in \mathbb{R}^2$. Clearly, the system degenerates when $n = 0$ or $p = 0$. In the new variables, we rewrite the system (3)-(5) in the compact form

$$\partial_t \begin{pmatrix} n \\ p \end{pmatrix} - \operatorname{div}\left(B(n,p)\nabla \begin{pmatrix} \phi_n \\ \phi_p \end{pmatrix} \right) = \begin{pmatrix} R(n,p) \\ R(n,p) \end{pmatrix}, \quad x \in \Omega,\ t > 0. \tag{10}$$

The existence of weak solutions to the stationary version of (10) with boundary conditions (7)-(8) close to the thermal equilibrium state $J_n = J_p = 0$ has been proved in [10]. In [14] the existence of global-in-time weak solutions of the transient model is shown. The proof relies on quite involved approximations of the cross-diffusion terms by finite differences and L^1 compactness arguments. The advantage of the finite-difference approximation is that the diffusion matrix of the approximate system becomes diagonal (also see [4]). In this paper we show that it is possible to treat the full system directly which *simplifies* the proof *significantly*. Moreover, we give a much simpler argument to obtain strong L^1 convergence of the approximate solutions. More precisely, we show the following result.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ with $\partial\Omega = \Gamma_D \cup \Gamma_N \in C^{s-1,1}$ $(s \geq d/2+1)$ such that $\Gamma_D \cap \Gamma_N = \emptyset$, $\mathrm{meas}(\Gamma_D) > 0$, and Γ_N is open and closed in $\partial\Omega$. Furthermore, let $T > 0$, $c > 0$, $C \in L^1(\Omega)$, and*

$$n_D,\ p_D \geq c > 0 \text{ in } Q_T = \Omega \times (0,T), \quad n_D,\ p_D,\ \psi_D \in W^{1,\infty}(\Omega),$$
$$n_0,\ p_0 \geq c > 0 \text{ in } \Omega, \quad n_0,\ p_0 \in L^\infty(\Omega).$$

Then there exists a solution (n,p,ψ) of (3)-(9) satisfying

$$\begin{aligned}
&n,\, p \geq 0 \quad \text{in } Q_T, \quad \psi - \psi_D \in L^\infty(0,T; H^1_D(\Omega)), \\
&n,\, p \in W^{1,\sigma}(0,T; (H^s_D(\Omega))') \cap L^\infty(0,T; L^1(\Omega)) \text{ for all } 1 < \sigma < 2, \\
&\sqrt{n} - \sqrt{n_D},\ \sqrt{p} - \sqrt{p_D} \in L^2(0,T; W^{1,1}_D(\Omega)).
\end{aligned}$$

The initial conditions (9) are satisfied in the sense of $C^0([0,T]; (H^s_D(\Omega))')$, and the functions n, p, and ψ solve (3)-(6) in the following sense:

$$\int_0^T \langle \partial_t n, \xi \rangle dt + \int_{Q_T} \frac{2(1+n)\sqrt{n}\nabla\sqrt{n} + 2n\sqrt{p}\nabla\sqrt{p} - n(1+n-p)\nabla\psi}{1+n+p} \cdot \nabla\xi dxdt = \int_{Q_T} \frac{1-np}{1+n+p}\xi dxdt, \tag{11}$$

$$\int_0^T \langle \partial_t p, \xi \rangle dt + \int_{Q_T} \frac{2(1+p)\sqrt{p}\nabla\sqrt{p} + 2p\sqrt{n}\nabla\sqrt{n} - p(1+n-p)\nabla\psi}{1+n+p} \cdot \nabla\xi dxdt = \int_{Q_T} \frac{1-np}{1+n+p}\xi dxdt, \tag{12}$$

$$\int_\Omega \nabla\psi \cdot \nabla\chi dx = \int_\Omega (n - p - C(x))\chi dx, \tag{13}$$

for all $\xi \in L^{\sigma'}(0,T; H^s_D(\Omega))$ and $\chi \in H^1(\Omega)$, where $\sigma' = \sigma/(\sigma-1)$.

Here, $\langle\cdot,\cdot\rangle$ is the dual product between $(H^s_D(\Omega))'$ and $H^s_D(\Omega)$, and the space $W^{1,p}_D(\Omega)$ (resp. $H^s_D(\Omega)$) with $1 \leq p \leq \infty$ consists of all functions $u \in W^{1,p}(\Omega)$ (resp. $H^s(\Omega)$) satisfying $u = 0$ on Γ_D. We have assumed time-independent boundary data for simplicity only (see [9, Lemma 5.2.5] for the treatment of time-dependent boundary functions). Notice that the condition "Γ_N open and closed in $\partial\Omega$" implies that the Dirichlet boundary is separated from the Neumann boundary. This assumption is needed in our approximation argument in order to obtain smooth solutions of the Laplace equation (see section 2). The parameter $s \geq d/2+1$ is chosen in such a way that $H^s(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ holds.

For the proof of Theorem 1 we discretize the equations both in time and space. More precisely, we employ an implicit Euler discretization in time and a Galerkin method in space. Since the diffusion matrix is of degenerate type, we approximate $B(n,p)$ by $\varepsilon I + B(n,p)$ in (10) for $\varepsilon > 0$, where I is the unit matrix of $\mathbb{R}^{2\times 2}$, and perform the limit $\varepsilon \to 0$. This approximation simplifies considerably the arguments of [14]. Moreover, it is of numerical interest. Indeed, since we use a Galerkin space for the variables ϕ_n and ϕ_p containing $L^\infty(\Omega)$, the particle densities, defined by $n = \exp(\phi_n + \psi)$ and $p = \exp(\phi_p - \psi)$, are strictly positive. Thus, our approximation provides a positivity-preserving numerical scheme. We refer to [7, 12] for related numerical schemes which preserve the positivity of the solutions.

In order to derive uniform a priori estimates, we employ the free energy (or "relative entropy") of the system,

$$E(t) = \int_\Omega \Big(n(\ln n - 1 - \ln n_D) + n_D + p(\ln p - 1 - \ln p_D) + p_D + \frac{1}{2}|\nabla(\psi - \psi_D)|^2 \Big) dx.$$

Formally, it holds (see Lemma 2 for a rigorous proof in the discrete setting):

$$E(t) + \int_0^t \int_\Omega \frac{n|\nabla\phi_n|^2 + p|\nabla\phi_p|^2 + |n\nabla\phi_n + p\nabla\phi_p|^2}{1+n+p} dxdt \le c(E(0)+1), \tag{14}$$

where the constant $c > 0$ depends on the boundary data and on t. This estimate provides uniform $W^{1,1}(\Omega)$ bounds for $\sqrt{n}$ and $\sqrt{p}$. Then the weak solution is defined for $\sqrt{n}$ and $\sqrt{p}$ instead for ϕ_n and ϕ_p by observing that $n\nabla\phi_n = 2\sqrt{n}\nabla\sqrt{n} - n\nabla\psi$ (and similarly for $p\nabla\phi_p$). Notice that we do not obtain any gradient regularity for the Fermi potentials ϕ_n and ϕ_p.

In order to obtain strong convergence (in the Lebesgue sense) of the approximating sequences for $\sqrt{n}$ and $\sqrt{p}$, called here $\sqrt{n_\varepsilon}$ and $\sqrt{p_\varepsilon}$, we also need uniform bounds for its time derivatives. Our new idea is to consider the function

$$y_\varepsilon = \sqrt{n_\varepsilon + \gamma p_\varepsilon + 1}, \quad \text{where } \gamma = 1, 2.$$

The $L^2(0,T;W^{1,1}(\Omega))$ bounds for $\sqrt{n_\varepsilon}$ and $\sqrt{p_\varepsilon}$ show that y_ε is bounded in the same space. Furthermore, the (discrete) time derivative $\partial_t^\varepsilon y_\varepsilon$ is uniformly bounded in $L^\sigma(0,T;(H^s(\Omega))')$ for $1 < \sigma < 2$. Thus, by Aubin's lemma, we infer strong convergence in $L^1(0,T;L^1(\Omega))$ for n_ε and p_ε.

Finally, the limits of the nonlinear terms can be identified by using the special structure of the equations. In particular, the boundedness of the energy production term in (14) is heavily employed.

The originality of the paper consists in the facts that

- the system (10) is of degenerate type,
- the nonnegativity of the solutions is obtained without use of a maximum principle, and
- the approximation provides a positivity-preserving numerical scheme.

This paper is organized as follows. In section 2 we show the existence of weak solutions to a fully discrete problem corresponding to (10). Two technical lemmas about weak and strong convergence are recalled in section 3. The existence proof of section 2 provides a priori estimates allowing to perform the limit $\varepsilon \to 0$ and $N \to \infty$, where N is the dimension of the Galerkin space. This is shown in section 4. The limit of the vanishing time parameter then is proved in section 5.

2. Existence of solutions to an approximated problem

We show Theorem 1 by first studying a fully discrete problem. For this, we approximate the problem (7)-(10) by employing an implicit Euler scheme and a Galerkin approximation.

Let (v_j) be a dense subset of $H^s(\Omega)$ for $s \geq d/2+1$ such that $v_j = 0$ on Γ_D ($j \in \mathbb{N}$). This subset can be obtained as follows. Since $H^s(\Omega)$ is separable and the operator $-\Delta$ with homogeneous Dirichlet boundary conditions on Γ_D and homogeneous Neumann boundary conditions on Γ_N is self-adjoint and compact, there exists a Hilbert base of eigenfunctions v_j in $H^s(\Omega)$. The operator is well defined due to our assumption on $\partial\Omega$ in Theorem 1 (see, e.g., [21, p. 196]). Notice that the condition $s \geq d/2+1$ implies that the embedding $H^s(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ is continuous.

Next we introduce the finite-dimensional space $V_N = \operatorname{span}\{v_1, \ldots, v_N\} \subset L^\infty(\Omega)$. Since we are looking for solutions $\phi_n - \phi_{n,D}$, $\phi_p - \phi_{p,D}$, $\psi - \psi_D \in V_N \subset L^\infty(\Omega)$, where $\phi_{n,D} = \ln n_D - \psi_D$ and $\phi_{p,D} = \ln p_D + \psi_D$, the definitions $n = \exp(\phi_n + \psi)$ and $p = \exp(\phi_p - \psi)$ are well defined. This provides the positivity of n and p.

We define now the approximated problem. Let $T > 0$, let $\varepsilon > 0$ be a regularization parameter, and let $\tau > 0$ be the time discretization parameter such that $T = \tau K$ with $K \in \mathbb{N}$. Let $\phi_{n,0}^{(N)} - \phi_{n,D}$, $\phi_{p,0}^{(N)} - \phi_{p,D} \in V_N$ be such that $\phi_{n,0}^{(N)} \to \ln n_0 - \psi(\cdot,0)$ and $\phi_{p,0}^{(N)} \to \ln p_0 + \psi(\cdot,0)$ in $L^2(\Omega)$, where $\psi(\cdot,0)$ is the unique solution of

$$\Delta\psi(\cdot,0) = n_0 - p_0 - C(x) \text{ in } \Omega, \quad \psi(\cdot,0) = \psi_D \text{ on } \Gamma_D, \quad \nabla\psi(\cdot,0)\cdot\nu = 0 \text{ on } \Gamma_N.$$

Furthermore, let $\phi_{n,k-1}^{(N)} - \phi_{n,D}$, $\phi_{p,k-1}^{(N)} - \phi_{p,D}$, $\psi_{k-1}^{(N)} - \psi_D \in V_N$ for $k \geq 1$ be given. We set $n_{k-1}^{(N)} = \exp(\phi_{n,k-1}^{(N)} + \psi_{k-1}^{(N)}) > 0$ and $p_{k-1}^{(N)} = \exp(\phi_{p,k-1}^{(N)} - \psi_{k-1}^{(N)}) > 0$ and we wish to find $\psi_k^{(N)} - \psi_D \in V_N$ such that

$$\int_\Omega \nabla\psi_k^{(N)} \cdot \nabla\chi dx = \int_\Omega (n_k^{(N)} - p_k^{(N)} - C(x))\chi dx \tag{15}$$

for all $\chi \in V_N$ and $(\phi_{n,k}^{(N)} - \phi_{n,D}, \phi_{p,k}^{(N)} - \phi_{p,D}) \in V_N^2$ such that

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega \begin{pmatrix} n_k^{(N)} - n_{k-1}^{(N)} \\ p_k^{(N)} - p_{k-1}^{(N)} \end{pmatrix} \cdot \begin{pmatrix} \xi_n \\ \xi_p \end{pmatrix} dx + \int_\Omega \left[(\varepsilon I + B(n_k^{(N)}, p_k^{(N)})) \begin{pmatrix} \nabla\phi_{n,k}^{(N)} \\ \nabla\phi_{p,k}^{(N)} \end{pmatrix} \right] : \begin{pmatrix} \nabla\xi_n \\ \nabla\xi_p \end{pmatrix} dx \\ &\qquad = \int_\Omega \begin{pmatrix} R(n_k^{(N)}, p_k^{(N)}) \\ R(n_k^{(N)}, p_k^{(N)}) \end{pmatrix} \cdot \begin{pmatrix} \xi_n \\ \xi_p \end{pmatrix} dx. \end{aligned} \tag{16}$$

for all ξ_n, $\xi_p \in V_N$, where $n_k^{(N)} = \exp(\phi_{n,k}^{(N)} + \psi_k^{(N)})$ and $p_k^{(N)} = \exp(\phi_{p,k}^{(N)} - \psi_k^{(N)})$ in Ω. Here, the gradient is a row vector in $\mathbb{R}^{1\times d}$, the dot "$\cdot$" between vectors is the scalar product, and the product ":" between matrices is defined by

$$A : B = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij} \quad \text{for all } A = (a_{ij}),\ B = (b_{ij}) \in \mathbb{R}^{m\times n}.$$

The proof of existence of solutions of the above discrete system is based on a priori estimates for the discrete free energy (or entropy) of the system:

$$\begin{aligned} E_k^{(N)} &= \int_\Omega \Big(n_k^{(N)}(\ln n_k^{(N)} - 1 - \ln n_D) + n_D + p_k^{(N)}(\ln p_k^{(N)} - 1 - \ln p_D) + p_D \\ &\quad + \frac{1}{2}|\nabla(\psi_k^{(N)} - \psi_D)|^2\Big)dx. \end{aligned}$$

We show that the free energy is bounded uniformly in the discretization parameters N and τ and also in ε:

Lemma 2. *For given* $(\phi_{n,k-1}^{(N)} - \phi_{n,D}, \phi_{p,k-1}^{(N)} - \phi_{p,D}, \psi_{k-1}^{(N)} - \psi_D) \in V_N^3$*, there exists* $\tau_0 > 0$ *such that for all* $0 < \tau \le \tau_0$*, there is a solution* $(\phi_{n,k}^{(N)} - \phi_{n,D}, \phi_{p,k}^{(N)} - \phi_{p,D}, \psi_k^{(N)} - \psi_D) \in V_N^3$ *to* (15)-(16) *satisfying*

$$\begin{aligned} &E_k^{(N)} + \frac{\tau}{2}\sum_{j=0}^{k}\int_\Omega \frac{n_j^{(N)}|\nabla\phi_{n,j}^{(N)}|^2 + p_j^{(N)}|\nabla\phi_{p,j}^{(N)}|^2 + |n_j^{(N)}\nabla\phi_{n,j}^{(N)} + p_j^{(N)}\nabla\phi_{p,j}^{(N)}|^2}{1 + n_j^{(N)} + p_j^{(N)}}dx \\ &\quad + \frac{\varepsilon\tau}{2}\sum_{j=0}^{k}\int_\Omega (|\nabla\phi_{n,j}^{(N)}|^2 + |\nabla\phi_{p,j}^{(N)}|^2)dx \le c(E_0^{(N)} + 1), \end{aligned} \tag{17}$$

where the constant $c > 0$ *depends on the boundary data and* T *but is independent of* k*,* N*,* τ*, and* ε*.*

Proof. In order to simplify the presentation, we omit the indices k and N. The proof is based on the Leray-Schauder fixed-point theorem. For this, we define a fixed-point operator $S : V_N^3 \times [0,1] \to V_N^3$ in the following way. Let $\sigma \in [0,1]$ and $(\widetilde{\phi}_n - \phi_{n,D}, \widetilde{\phi}_p - \phi_{p,D}, \widetilde{\psi} - \psi_D) \in V_N^3$ be given and set $\widetilde{n} = \exp(\widetilde{\phi}_n + \widetilde{\psi}) > 0$, $\widetilde{p} = \exp(\widetilde{\phi}_p - \widetilde{\psi}) > 0$. First, let $\psi - \psi_D \in V_N$ be the unique solution of

$$\int_\Omega \nabla\phi \cdot \nabla\chi dx = \sigma \int_\Omega (\widetilde{n} - \widetilde{p} - C(x))\chi dx, \quad \chi \in V_N.$$

Next, consider the linear problem

$$a((\phi_{n,k} - \phi_{n,D}, \phi_{p,k} - \phi_{p,D}), (\xi_n, \xi_p)) = \sigma F((\xi_n, \xi_p)) \quad \text{for all } (\xi_n, \xi_p) \in V_N^2, \tag{18}$$

where

$$
\begin{aligned}
a((\eta_n,\eta_p),(\xi_n,\xi_p)) &= \int_\Omega \left[(\varepsilon I + B(\widetilde{n},\widetilde{p}))\begin{pmatrix}\nabla\eta_n\\ \nabla\eta_p\end{pmatrix}\right] : \begin{pmatrix}\nabla\xi_n\\ \nabla\xi_p\end{pmatrix} dx,\\
F((\xi_n,\xi_p)) &= -\frac{1}{\tau}\int_\Omega \begin{pmatrix}\widetilde{n}-n_{k-1}\\ \widetilde{p}-p_{k-1}\end{pmatrix}\cdot\begin{pmatrix}\xi_n\\ \xi_p\end{pmatrix} dx + \int_\Omega \begin{pmatrix}R(\widetilde{n},\widetilde{p})\\ R(\widetilde{n},\widetilde{p})\end{pmatrix}\cdot\begin{pmatrix}\xi_n\\ \xi_p\end{pmatrix} dx\\
&\quad - \int_\Omega \left[(\varepsilon I + B(\widetilde{n},\widetilde{p}))\begin{pmatrix}\nabla\phi_{n,D}\\ \nabla\phi_{p,D}\end{pmatrix}\right] : \begin{pmatrix}\nabla\xi_n\\ \nabla\xi_p\end{pmatrix} dx,
\end{aligned}
$$

for all (η_n,η_p), $(\xi_n,\xi_p)\in V_N^2$. Since the matrix $\varepsilon I + B(\widetilde{n},\widetilde{p})$ is positive definite, the Lax-Milgram lemma shows the existence of a (unique) solution of (18) in V_N^2. Finally, we set $S(\tilde\phi_n-\phi_{n,D},\tilde\phi_p-\phi_{p,D},\tilde\psi-\psi_D,\sigma) = (\phi_n-\phi_{n,D},\phi_p-\phi_{p,D},\psi-\psi_D)$ which defines the fixed-point operator. It holds $S(\tilde\phi_n-\phi_{n,D},\tilde\phi_p-\phi_{p,D},\tilde\psi-\psi_D,0)=(0,0,0)$. By standard arguments, $S(\cdot,\sigma)$ is continuous and also compact since $\dim(V_N)<\infty$.

Now let $(\phi_n-\phi_{n,D},\phi_p-\phi_{p,D},\psi-\psi_D)\in V_N^3$ be a fixed point of $S(\cdot,\sigma)$. In order to derive uniform estimates we use $(\phi_n-\phi_{n,D},\phi_p-\phi_{p,D})\in V_N^2$ as a test function in the weak formulation (18), where $\widetilde{n}=n=\exp(\phi_n+\psi)$, $\widetilde{p}=p=\exp(\phi_p-\psi)$. Since

$$
\begin{aligned}
\left[B(n,p)\begin{pmatrix}\nabla\phi_n\\ \nabla\phi_p\end{pmatrix}\right] : \begin{pmatrix}\nabla\phi_n\\ \nabla\phi_p\end{pmatrix} &= B_{11}|\nabla\phi_n|^2 + 2B_{12}\nabla\phi_n\cdot\nabla\phi_p + B_{22}|\nabla\phi_p|^2\\
&= \frac{n|\nabla\phi_n|^2}{1+n+p} + \frac{p|\nabla\phi_p|^2}{1+n+p} + \frac{|n\nabla\phi_n+p\nabla\phi_p|^2}{1+n+p},
\end{aligned}
$$

we obtain for $\sigma>0$

$$
\begin{aligned}
\frac{1}{\sigma}\int_\Omega &\left[\varepsilon(|\nabla\phi_n|^2+|\nabla\phi_p|^2) + \frac{1}{1+n+p}\big(n|\nabla\phi_n|^2+p|\nabla\phi_p|^2+|n\nabla\phi_n+p\nabla\phi_p|^2\big)\right] dx\\
&= -\frac{1}{\tau}\int_\Omega ((n-n_{k-1})(\phi_n-\phi_{n,D}) + (p-p_{k-1})(\phi_p-\phi_{p,D}))\,dx\\
&\quad + \int_\Omega (B_{11}\nabla\phi_n\cdot\nabla\phi_{n,D} + B_{12}\nabla\phi_p\cdot\nabla\phi_{n,D} + B_{12}\nabla\phi_n\cdot\nabla\phi_{p,D}\\
&\quad + B_{22}\nabla\phi_p\cdot\nabla\phi_{p,D})dx + \varepsilon\int_\Omega (\nabla\phi_n\cdot\nabla\phi_{n,D}+\nabla\phi_p\cdot\nabla\phi_{p,D})dx \qquad (19)\\
&\quad + \int_\Omega R(n,p)(\phi_n-\phi_{n,D}+\phi_p-\phi_{p,D})dx\\
&= I_1+\cdots+I_4.
\end{aligned}
$$

We first rewrite the integral I_1, using the definition of ϕ_n and ϕ_p:

$$
\begin{aligned}
I_1 &= -\frac{1}{\tau}\int_\Omega \big[n(\ln n - 1 - \ln n_D) - n_{k-1}(\ln n_{k-1} - 1 - \ln n_D) + n_{k-1}(\ln n_{k-1} - \ln n)\\
&\quad + n - n_{k-1} + p(\ln p - 1 - \ln p_D) - p_{k-1}(\ln p_{k-1} - 1 - \ln p_D)\\
&\quad + p_{k-1}(\ln p_{k-1} - \ln p) + p - p_{k-1}\big] dx \qquad (20)
\end{aligned}
$$

$$+\frac{1}{\tau}\int_\Omega \big((n-p)-(n_{k-1}-p_{k-1})\big)(\psi-\psi_D)dx.$$

For the last integral in the above equation we use the Poisson equation (15) and Young's inequality:

$$\begin{aligned}
&\int_\Omega \big((n-p-C)-(n_{k-1}-p_{k-1}-C)\big)(\psi-\psi_D)dx \\
&\quad= -\int_\Omega \nabla(\psi-\psi_{k-1})\cdot\nabla(\psi-\psi_D)dx \\
&\quad\le -\frac{1}{2}\int_\Omega |\nabla(\psi-\psi_D)|^2dx + \frac{1}{2}\int_\Omega |\nabla(\psi_{k-1}-\psi_D)|^2dx.
\end{aligned}$$

For the first integral in (20) we employ the elementary inequality $x(\ln x-\ln y)\ge x-y$ for $x,\, y>0$. This yields, after a short calculation,

$$I_1 \le -\frac{1}{\tau}(E_k-E_{k-1}).$$

We turn to the second integral I_2 which can be written, by Young's inequality, as

$$\begin{aligned}
I_2 &= \int_\Omega \frac{n\nabla\phi_n+p\nabla\phi_p}{1+n+p}\cdot(n\nabla\phi_{n,D}+p\nabla\phi_{p,D})dx + \int_\Omega \frac{n\nabla\phi_n}{1+n+p}\cdot\nabla\phi_{n,D}dx \\
&\quad + \int_\Omega \frac{p\nabla\phi_p}{1+n+p}\cdot\nabla\phi_{p,D}dx \\
&\le \int_\Omega \frac{1}{2(1+n+p)}(n|\nabla\phi_n|^2+p|\nabla\phi_p|^2+|n\nabla\phi_n+p\nabla\phi_p|^2)dx \\
&\quad + \int_\Omega \frac{1}{1+n+p}(n(1+n)|\nabla\phi_{n,D}|^2+p(1+p)|\nabla\phi_{p,D}|^2)dx \\
&\le \int_\Omega \frac{1}{2(1+n+p)}(n|\nabla\phi_n|^2+p|\nabla\phi_p|^2+|n\nabla\phi_n+p\nabla\phi_p|^2)dx + c\int_\Omega (n+p)dx \\
&\le \int_\Omega \frac{1}{2(1+n+p)}(n|\nabla\phi_n|^2+p|\nabla\phi_p|^2+|n\nabla\phi_n+p\nabla\phi_p|^2)dx + c(E_k+1),
\end{aligned}$$

where $c>0$ denotes here and in the following a constant independent of k, N, τ, and ε with values varying from occurence to occurence. In the above estimate we have used that $\phi_{\alpha,D}\in W^{1,\infty}(\Omega)$, $\alpha=n,p$.

For the fourth integral I_4 we use the monotonicity of $x\mapsto \ln x$:

$$\begin{aligned}
I_4 &= -\int_\Omega \frac{(\ln(np)-\ln(n_Dp_D))(np-n_Dp_D)}{1+n+p}dx \\
&\quad + \int_\Omega \frac{(\ln(np)-\ln(n_Dp_D))(1-n_Dp_D)}{1+n+p}dx \\
&\le \int_\Omega \left(\frac{\ln(np)}{1+n+p}(1-n_Dp_D)+|\ln(n_Dp_D)(1-n_Dp_D)|\right)dx \le c.
\end{aligned}$$

Finally, the third integral I_3 is estimated by Young's inequality, and the terms $(\varepsilon/2)\,|\nabla\phi_n|^2$ and $(\varepsilon/2)|\nabla\phi_p|^2$ are absorbed by the left-hand side of (19).

Putting together the above estimates gives

$$\frac{1}{\tau}(E_k - E_{k-1}) + \frac{1}{2}\int_\Omega \frac{n|\nabla\phi_n|^2 + p|\nabla\phi_p|^2 + |n\nabla\phi_n + p\nabla\phi_p|^2}{1+n+p}dx$$
$$+\frac{\varepsilon}{2}\int_\Omega (|\nabla\phi_n|^2 + |\nabla\phi_p|^2)dx \le c(E_k+1).$$

Then the discrete Gronwall lemma yields (17) if we choose τ sufficiently small. This also gives an estimate for $(\phi_n-\phi_{n,D},\phi_p-\phi_{p,D},\psi-\psi_D)$ in $(H^1(\Omega))^3$, and hence in V_N^3 since all norms in V_N are equivalent. Therefore, we can apply the Leray-Schauder fixed-point theorem to conclude the existence of a fixed point of $S(\cdot,1)$, i.e. a solution to (15)-(16). $\square$

3. Two technical lemmas

We recall two convergence results. The first one is well known.

Lemma 3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $(f_m) \subset L^\infty(\Omega)$, $(g_m) \subset L^p(\Omega)$ for some $1 \le p < \infty$ be sequences such that (f_m) is bounded in $L^\infty(\Omega)$, $f_m(x) \to f(x)$ a.e. in Ω, and $g_m \rightharpoonup g$ weakly in $L^p(\Omega)$ as $m \to \infty$. Then $f_m g_m \rightharpoonup fg$ weakly in $L^p(\Omega)$.*

Let $L_\Psi(\Omega)$ be the Orlicz space corresponding to $\Psi(x) = (1+x)\log(1+x) - x$, $x > 0$. We refer to [1] for its definition and properties (also see the appendix of [4]).

Lemma 4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set.*

(i) If (f_m) is bounded in $L^p(\Omega)$ with $p > 1$ and $f_m(x) \to f(x)$ a.e. in Ω, then $f_m \to f$ strongly in $L^q(\Omega)$ for all $1 \le q < p$.

(ii) If (f_m) is bounded in $L_\Psi(\Omega)$ and $f_m(x) \to f(x)$ a.e. in Ω, then $f_m \to f$ strongly in $L^1(\Omega)$.

The proof of (i) can be found in [15, Ch. 1.3 and p. 144]. The proof of (ii) is essentially contained in [1]. Indeed, Theorem 8.22 in [1] states that, given two Young functions Ψ_1 and Ψ_2 satisfying, for any $k > 0$,

$$\lim_{t\to\infty} \frac{\Psi_1(kt)}{\Psi_2(t)} = 0$$

and a sequence (f_m) which is bounded in $L_{\Psi_1}(\Omega)$ and convergent in measure, then (f_m) is convergent in $L_{\Psi_2}(\Omega)$. Notice that $\Psi_1(t) = \Psi(t)$ is a Young function, but $\Psi_2(t) = t$ is not. However, the proof of [1] can be extended also to this case. Since a.e. convergence implies convergence in measure, Lemma 4 (ii) follows.

4. The limit $\varepsilon \to 0$ and $N \to \infty$

In this section we perform the limit $\varepsilon \to 0$ and $N \to \infty$ for fixed $\tau > 0$. At this stage, the limit $\tau \to 0$ cannot be performed since we need some (discrete) regularity in time. The corresponding estimates, however, can only be achieved using a test function in a larger space than V_N. Therefore, the limiting procedure has to be divided into the two parts $\varepsilon \to 0$, $N \to \infty$ and $\tau \to 0$.

Lemma 5. *Let* $(n_k^{(N)}, p_k^{(N)}, \psi_k^{(N)})$ *be a solution to* (15)-(16). *Then there exists a constant* $c > 0$, *independent of* ε, N, *and* τ, *such that*

$$\|n_k^{(N)}\|_{L_\Psi(\Omega)} + \|p_k^{(N)}\|_{L_\Psi(\Omega)} + \|\psi_k^{(N)}\|_{H^1(\Omega)} \leq c, \tag{21}$$

$$\sum_{k=0}^{K} \tau \left(\int_\Omega \frac{|\nabla \sqrt{n_k^{(N)}}|^2}{1 + n_k^{(N)} + p_k^{(N)}} dx + \int_\Omega \frac{|\nabla \sqrt{p_k^{(N)}}|^2}{1 + n_k^{(N)} + p_k^{(N)}} dx \right) \leq c, \tag{22}$$

$$\sum_{k=0}^{K} \tau \left(\|\sqrt{n_k^{(N)}}\|^2_{W^{1,1}(\Omega)} + \|\sqrt{p_k^{(N)}}\|^2_{W^{1,1}(\Omega)} \right) \leq c, \tag{23}$$

$$\sum_{k=0}^{K} \tau \int_\Omega \frac{|2\sqrt{n_k^{(N)}}\nabla\sqrt{n_k^{(N)}} + 2\sqrt{n_k^{(N)}}\nabla\sqrt{n_k^{(N)}} - (n_k^{(N)} - p_k^{(N)})\nabla\psi_k^{(N)}|^2}{1 + n_k^{(N)} + p_k^{(N)}} dx \leq c, \tag{24}$$

where $L_\Psi(\Omega)$ *is the Orlicz space with* $\Psi(x) = (1+x)\ln(1+x) - x$, $x > 0$.

Proof. The convexity of the mapping $x \mapsto x \ln x$ implies that

$$\frac{1+x}{2} \ln \frac{1+x}{2} \leq \frac{1}{2}(1 \ln 1 + x \ln x) \quad \text{for all } x > 0,$$

which is equivalent to

$$\Psi(x) \leq x(\ln x - 1) + (1 - x)\ln 2.$$

Thus, the estimate (21) follows from (17) and the above inequality. The elementary inequality $(x-y)^2 \geq x^2/2 - y^2$ for all x, $y \in \mathbb{R}$ and (17) imply that

$$\begin{aligned} c &\geq \sum_{k=0}^{K} \tau \int_\Omega \frac{n_k^{(N)}|\nabla\phi_k^{(N)}|^2}{1 + n_k^{(N)} + p_k^{(N)}} dx = \sum_{k=0}^{K} \tau \int_\Omega \frac{|2\nabla\sqrt{n_k^{(N)}} - \sqrt{n_k^{(N)}}\nabla\psi_k^{(N)}|^2}{1 + n_k^{(N)} + p_k^{(N)}} dx \\ &\geq 2\sum_{k=0}^{K} \tau \frac{|\nabla\sqrt{n_k^{(N)}}|^2}{1 + n_k^{(N)} + p_k^{(N)}} dx - \sum_{k=0}^{K} \tau |\nabla\psi_k^{(N)}|^2 dx \end{aligned}$$

and a corresponding estimate for $p_k^{(N)}$. Hence, (22) follows from the uniform bound for $\nabla\psi_k^{(N)}$ in $L^2(\Omega)$. Young's inequality yields

$$\sum_{k=0}^{K} \tau \left(\int_\Omega |\nabla\sqrt{n_k^{(N)}}| dx \right)^2 \leq \sum_{k=0}^{K} \tau \int_\Omega \frac{|\nabla\sqrt{n_k^{(N)}}|^2}{1 + n_k^{(N)} + p_k^{(N)}} dx \int_\Omega (1 + n_k^{(N)} + p_k^{(N)}) dx \leq c,$$

using (21) and (22). This proves (23). Finally, the estimate (24) is a consequence from the formulation

$$n_k^{(N)}\nabla\phi_{n,k}^{(N)} = 2\sqrt{n_k^{(N)}}\nabla\sqrt{n_k^{(N)}} - n_k^{(N)}\nabla\psi_k^{(N)}.$$

and the uniform estimate (17). □

Lemma 6. *Let $\tau > 0$ be fixed and let $k \in \{1, \ldots, K\}$. Then, as $\varepsilon \to 0$, $N \to \infty$, $(n_k^{(N)}, p_k^{(N)}) \to (n_k, p_k)$ strongly in $L^1(\Omega)$, $\nabla\psi_k^{(N)} \rightharpoonup \nabla\psi_k$ weakly in $L^2(\Omega)$, and the limit satisfies*

$$\int_\Omega \nabla\psi_k \cdot \nabla\chi dx = \int_\Omega (n_k - p_k - C(x))\chi dx \tag{25}$$

for all $\chi \in H_D^s(\Omega)$ and

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})\xi_n dx + \int_\Omega \frac{2\sqrt{n_k}\nabla\sqrt{n_k} - n_k\nabla\psi_k}{1+n_k+p_k} \cdot \nabla\xi_n dx \\ &\qquad + \int_\Omega \frac{n_k(2\sqrt{n_k}\nabla\sqrt{n_k} - n_k\nabla\psi_k + 2\sqrt{p_k}\nabla\sqrt{p_k} + p_k\nabla\psi_k)}{1+n_k+p_k} \cdot \nabla\xi_n dx \\ &\quad = \int_\Omega R(n_k, p_k)\xi_n dx, \end{aligned} \tag{26}$$

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega (p_k - p_{k-1})\xi_p dx + \int_\Omega \frac{2\sqrt{p_k}\nabla\sqrt{p_k} + p_k\nabla\psi_k}{1+n_k+p_k} \cdot \nabla\xi_p dx \\ &\qquad + \int_\Omega \frac{p_k(2\sqrt{n_k}\nabla\sqrt{n_k} - n_k\nabla\psi_k + 2\sqrt{p_k}\nabla\sqrt{p_k} + p_k\nabla\psi_k)}{1+n_k+p_k} \cdot \nabla\xi_p dx \\ &\quad = \int_\Omega R(n_k, p_k)\xi_p dx \end{aligned} \tag{27}$$

for all ξ_n, $\xi_p \in H_D^s(\Omega)$. Furthermore, the following a priori estimates hold:

$$\|n_k\|_{L_\Psi(\Omega)} + \|p_k\|_{L_\Psi(\Omega)} + \|\psi_k\|_{H^1(\Omega)} \leq c, \tag{28}$$

$$\sum_{k=0}^K \tau\left(\left\|\frac{\nabla\sqrt{n_k}}{\sqrt{1+n_k+p_k}}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\nabla\sqrt{p_k}}{\sqrt{1+n_k+p_k}}\right\|_{L^2(\Omega)}^2\right) \leq c, \tag{29}$$

$$\sum_{k=0}^K \tau\left(\|\sqrt{n_k}\|_{W^{1,1}(\Omega)}^2 + \|\sqrt{p_k}\|_{W^{1,1}(\Omega)}^2\right) \leq c, \tag{30}$$

$$\sum_{k=0}^K \tau\left\|\frac{2\sqrt{n_k}\nabla\sqrt{n_k} + 2\sqrt{p_k}\nabla\sqrt{p_k} - (n_k - p_k)\nabla\psi_k}{\sqrt{1+n_k+p_k}}\right\|_{L^2(\Omega)}^2 \leq c, \tag{31}$$

where $L_\Psi(\Omega)$ is the Orlicz space with $\Psi(x) = (1+x)\ln(1+x) - x$, $x > 0$.

Proof. We show first that the terms in the discrete problem (16) converge. Clearly, by the free energy estimate (17), as $\varepsilon \to 0$ and $N \to \infty$,

$$\varepsilon\nabla\phi_{n,k}^{(N)} \to 0, \quad \varepsilon\nabla\phi_{p,k}^{(N)} \to 0 \quad \text{strongly in } L^2(\Omega).$$

Since $W^{1,1}(\Omega)$ embeddes compactly into $L^q(\Omega)$ for $1 \leq q < d/(d-1)$, we obtain for a subsequence which is not relabeled,

$$\sqrt{n_k^{(N)}} \to \sqrt{n_k}, \quad \sqrt{p_k^{(N)}} \to \sqrt{p_k} \quad \text{strongly in } L^q(\Omega).$$

In particular, these sequences converge a.e. (up to a asubsequence). Thus, in view of the uniform bound (21) for $n_k^{(N)}$ and $p_k^{(N)}$, an application of Lemma 4 yields

$$n_k^{(N)} \to n_k, \quad p_k^{(N)} \to p_k \quad \text{strongly in } L^1(\Omega). \tag{32}$$

The $W^{1,1}(\Omega)$ bound for $\sqrt{n_k^{(N)}}$ does not imply weak compactness in $W^{1,1}(\Omega)$. However, this compactness follows from the uniform bounds for

$$\frac{\nabla\sqrt{n_k^{(N)}}}{\sqrt{1+n_k^{(N)}+p_k^{(N)}}} \quad \text{and} \quad \sqrt{1+n_k^{(N)}+p_k^{(N)}}$$

in $L^2(\Omega)$ (see (21) and (22)) since the product of both sequences converges (up to a subsequence) weakly in $L^1(\Omega)$,

$$\nabla\sqrt{n_k^{(N)}} = \frac{\nabla\sqrt{n_k^{(N)}}}{1+n_k^{(N)}+p_k^{(N)}}\sqrt{1+n_k^{(N)}+p_k^{(N)}} \rightharpoonup u \quad \text{weakly in } L^1(\Omega). \tag{33}$$

Clearly, $u = \nabla\sqrt{n_k}$. In a similar way, $\nabla\sqrt{p_k^{(N)}}$ converges to $\nabla\sqrt{p_k}$ weakly in $L^1(\Omega)$.

The sequence $(\sqrt{n_k^{(N)}}/(1+n_k^{(N)}+p_k^{(N)}))$ is bounded in $L^\infty(\Omega)$ and converges a.e. Taking into account (33), we can apply Lemma 3 to conclude that

$$\frac{\sqrt{n_k^{(N)}}\nabla\sqrt{n_k^{(N)}}}{1+n_k^{(N)}+p_k^{(N)}} \rightharpoonup \frac{\sqrt{n_k}\nabla\sqrt{n_k}}{1+n_k+p_k} \quad \text{weakly in } L^1(\Omega). \tag{34}$$

By (21), we infer that, up to the extraction of a subsequence, $\nabla\psi_k^{(N)} \rightharpoonup \nabla\psi_k$ in $L^2(\Omega)$. Furthermore, the sequence $(n_k^{(N)}/(1+n_k^{(N)}+p_k^{(N)}))$ is bounded in $L^\infty(\Omega)$ and converges a.e. Thus, again applying Lemma 3,

$$\frac{n_k^{(N)}\nabla\psi_k^{(N)}}{1+n_k^{(N)}+p_k^{(N)}} \rightharpoonup \frac{n_k\nabla\psi_k}{1+n_k+p_k} \quad \text{weakly in } L^2(\Omega). \tag{35}$$

The convergence results (34)-(35) and the bound in (17) imply that

$$\frac{n_k^{(N)}\nabla\phi_{n,k}^{(N)}}{1+n_k^{(N)}+p_k^{(N)}} \rightharpoonup \frac{2\sqrt{n_k}\nabla\sqrt{n_k}-n_k\nabla\psi_k}{1+n_k+p_k} \quad \text{weakly in } L^2(\Omega). \tag{36}$$

Analogously, we obtain

$$\frac{p_k^{(N)}\nabla\phi_{p,k}^{(N)}}{1+n_k^{(N)}+p_k^{(N)}} \rightharpoonup \frac{2\sqrt{p_k}\nabla\sqrt{p_k}+p_k\nabla\psi_k}{1+n_k+p_k} \quad \text{weakly in } L^2(\Omega). \tag{37}$$

The strong convergence of $(\sqrt{n_k^{(N)}})$ in $L^2(\Omega)$ and (36)-(37) show that

$$\frac{\sqrt{n_k^{(N)}}(n_k^{(N)}\nabla\phi_{n,k}^{(N)} + p_k^{(N)}\nabla\phi_{p,k}^{(N)})}{1 + n_k^{(N)} + p_k^{(N)}} \rightharpoonup \frac{2\sqrt{n_k}(\sqrt{n_k}\nabla\sqrt{n_k} + \sqrt{p_k}\nabla\sqrt{p_k}) - \sqrt{n_k}(n_k - p_k)\nabla\psi_k}{1 + n_k + p_k} \quad \text{weakly in } L^1(\Omega).$$

Actually, by (31), this convergence holds true in $L^2(\Omega)$. Using again (31) and the strong convergence of $(\sqrt{n_k^{(N)}})$ in $L^2(\Omega)$, we arrive at

$$\frac{n_k^{(N)}(n_k^{(N)}\nabla\phi_{n,k}^{(N)} + p_k^{(N)}\nabla\phi_{p,k}^{(N)})}{1 + n_k^{(N)} + p_k^{(N)}} \rightharpoonup \frac{2n_k(\sqrt{n_k}\nabla\sqrt{n_k} + \sqrt{p_k}\nabla\sqrt{p_k}) - n_k(n_k - p_k)\nabla\psi_k}{1 + n_k + p_k} \quad \text{weakly in } L^1(\Omega).$$

Finally, the source terms converge,

$$\frac{1 - n_k^{(N)}p_k^{(N)}}{1 + n_k^{(N)} + p_k^{(N)}} \to \frac{1 - n_k p_k}{1 + n_k + p_k} \quad \text{strongly in } L^1(\Omega).$$

Now, let ξ_n, ξ_p, $\chi \in H^s_D(\Omega)$ and $\xi_n^{(M)}$, $\xi_p^{(M)}$, $\chi^{(M)} \in V_M$ such that $\xi_n^{(M)} \to \xi_n$, $\xi_p^{(M)} \to \xi_p$, and $\chi^{(M)} \to \chi$ strongly in $H^s(\Omega)$ and hence also in $W^{1,\infty}(\Omega)$, as $M \to \infty$. (Here we need that $H^s(\Omega)$ is embedded continuously into $W^{1,\infty}(\Omega)$.) We choose the test functions $(\xi_n^{(M)}, \xi_p^{(M)}, \chi^{(M)})$ in (15)-(16). The above convergence results allow to perform the limit $\varepsilon \to 0$ and $N \to \infty$. This shows that the limit (n_k, p_k, ψ_k) solves the equations (25)-(27) for $\chi = \chi^{(M)}$, $\xi_n = \xi_n^{(M)}$, and $\xi_p = \xi_p^{(M)}$. Then $M \to \infty$ gives the conclusion of the lemma. □

5. The limit $\tau \to 0$

Let $(n_k, p_k, \psi_k) \in (H^s(\Omega))^3$ be a solution of (25)-(27). We introduce the piecewise constant functions

$$n^{(\tau)}(x,t) = n_k(x), \quad p^{(\tau)}(x,t) = p_k(x), \quad \psi^{(\tau)}(x,t) = \psi_k(x)$$

for $x \in \Omega$, $t \in (t_{k-1}, t_k]$, $k = 1, \ldots, K$. The following uniform bounds are a direct consequence of (28)-(31).

Lemma 7. *There exists a positive constant c independent of τ such that*

$$\|n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} + \|p^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} + \|\psi^{(\tau)}\|_{L^\infty(0,T;H^1(\Omega))} \leq c, \tag{38}$$

$$\left\|\frac{\nabla\sqrt{n^{(\tau)}}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\nabla\sqrt{p^{(\tau)}}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}}\right\|_{L^2(0,T;L^2(\Omega))} \leq c, \tag{39}$$

$$\|\sqrt{n^{(\tau)}}\|_{L^2(0,T;W^{1,1}(\Omega))} + \|\sqrt{p^{(\tau)}}\|_{L^2(0,T;W^{1,1}(\Omega))} \leq c, \tag{40}$$

$$\left\|\frac{2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} - n^{(\tau)}\nabla\psi^{(\tau)} + 2\sqrt{p^{(\tau)}}\nabla\sqrt{p^{(\tau)}} + p^{(\tau)}\nabla\psi^{(\tau)}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}}\right\|_{L^2(0,T;L^2(\Omega))} \leq c. \tag{41}$$

Furthermore, we introduce the shift operator σ_K, defined by $(\sigma_K n^{(\tau)})(x,t) = n_{k-1}(x)$ for $x \in \Omega$, $t \in (t_{k-1}, t_k]$, $k = 1, \ldots, K$ (and similarly for $\sigma_K p^{(\tau)}$).

For the compactness argument, we need estimates for the functions

$$y_\gamma^{(\tau)} = \sqrt{n^{(\tau)} + \gamma p^{(\tau)} + 1} \quad \text{for } \gamma = 1, 2.$$

Lemma 8. *There exists a positive constant c independent of τ such that*

$$\|y_\gamma^{(\tau)}\|_{L^2(0,T;W^{1,1}(\Omega))} \leq c, \tag{42}$$

$$\|y_\gamma^{(\tau)} - \sigma_K y_\gamma^{(\tau)}\|_{L^1(0,T;(H_D^s(\Omega))')} \leq c. \tag{43}$$

Proof. The bound (42) follows directly from (40). In order to prove (43), we introduce $y_{\gamma,k} = \sqrt{n_k + \gamma p_k + 1}$ and $y_{\gamma,k}^{(N)} = \sqrt{n_k^{(N)} + \gamma p_k^{(N)} + 1}$. Let $\xi \in H_D^s(\Omega)$, $\xi \neq 0$. We employ the test function $\xi/(y_{\gamma,k}^{(N)} + y_{\gamma,k-1}^{(N)}) \in H_D^s(\Omega)$ in (26) to obtain

$$\begin{aligned}
\sum_{k=0}^K \tau \left\|\frac{y_{\gamma,k} - y_{\gamma,k-1}}{\tau}\right\|_{(H_D^s(\Omega))'} &= \sum_{k=0}^K \tau \sup_{\xi\neq 0}\left|\int_\Omega \frac{(n_k - n_{k-1}) + \gamma(p_k - p_{k-1})}{\tau}\frac{\xi}{y_{\gamma,k} + y_{\gamma,k-1}}dx\right| \\
&\leq \sum_{k=0}^K \tau \sup_{\xi\neq 0} \lim_{N\to\infty,\,\varepsilon\to 0}\left|\int_\Omega \frac{(n_k - n_{k-1}) + \gamma(p_k - p_{k-1})}{\tau}\frac{\xi}{y_{\gamma,k}^{(N)} + y_{\gamma,k-1}^{(N)}}dx\right| \\
&= \sum_{k=0}^K \tau \sup_{\xi\neq 0} \lim_{N\to\infty,\,\varepsilon\to 0}\int_\Omega \Bigg| \frac{2\sqrt{n_k}\nabla\sqrt{n_k} + 2\gamma\sqrt{p_k}\nabla\sqrt{p_k} - (n_k - \gamma p_k)\nabla\psi_k}{1+n_k+p_k}\cdot\nabla\frac{\xi}{y_{\gamma,k}^{(N)} + y_{\gamma,k-1}^{(N)}} \\
&\quad + \frac{(n_k+\gamma p_k)(2\sqrt{n_k}\nabla\sqrt{n_k} + 2\sqrt{p_k}\nabla\sqrt{p_k} - (n_k - p_k)\nabla\psi_k)}{1+n_k+p_k}\cdot\nabla\frac{\xi}{y_{\gamma,k}^{(N)} + y_{\gamma,k-1}^{(N)}} \\
&\quad + (1+\gamma)R(n_k,p_k)\frac{\xi}{y_{\gamma,k}^{(N)} + y_{\gamma,k-1}^{(N)}}\Bigg|dx.
\end{aligned} \tag{44}$$

Notice that the above test function is not an element of V_N (even if $\xi \in V_N$) and therefore, it cannot be used in (16). We need to perform the limit $\varepsilon \to 0$, $N \to \infty$ in (16) first in order to employ this test function in (26).

We claim that the right-hand side of (44) can be uniformly bounded. For this we first estimate the gradient of the test function $\xi/(y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1})$. Since

$$\sum_{k=1}^K \tau \left\| \frac{\nabla y^{(N)}_{\gamma,k}}{y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1}} \right\|^2_{L^2(\Omega)} \le \sum_{k=0}^K \tau \left\| \frac{\nabla y^{(N)}_{\gamma,k}}{y^{(N)}_{k}} \right\|^2_{L^2(\Omega)} = \sum_{k=0}^K \tau \left\| \frac{\nabla (n^{(N)}_k + \gamma p^{(N)}_k)}{2(n^{(N)}_k + \gamma p^{(N)}_k + 1)} \right\|^2_{L^2(\Omega)}$$

$$\le \sum_{k=0}^K \tau \left(\left\| \frac{\nabla \sqrt{n^{(N)}_k}}{\sqrt{n^{(N)}_k + \gamma p^{(N)}_k + 1}} \right\|^2_{L^2(\Omega)} + \left\| \frac{\gamma \nabla \sqrt{p^{(N)}_k}}{\sqrt{n^{(N)}_k + \gamma p^{(N)}_k + 1}} \right\|^2_{L^2(\Omega)} \right) \le c,$$

by (22), and similarly,

$$\sum_{k=1}^K \tau \left\| \frac{\nabla y^{(N)}_{\gamma,k-1}}{y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1}} \right\|^2_{L^2(\Omega)} \le c,$$

we infer that

$$\sum_{k=0}^K \tau \left\| \nabla \frac{\xi}{y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1}} \right\|^2_{L^2(\Omega)} = \sum_{k=0}^K \tau \left\| \frac{\nabla \xi}{y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1}} - \frac{\nabla (y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1})}{(y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1})^2} \xi \right\|^2_{L^2(\Omega)} \le c \|\xi\|^2_{H^s(\Omega)}.$$

Thus, employing (29), the first term on the right-hand side of (44) including n_k can be estimated as follows:

$$\sum_{k=0}^K \tau \int_\Omega \frac{2\sqrt{n_k}\nabla\sqrt{n_k} - n_k \nabla \psi_k}{1 + n_k + p_k} \cdot \nabla \frac{\xi}{y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1}} dx$$

$$\le \sum_{k=0}^K \tau \left(\left\| \frac{\nabla \sqrt{n_k}}{\sqrt{1 + n_k + p_k}} \right\|_{L^2(\Omega)} + \|\nabla \psi_k\|_{L^2(\Omega)} \right) \|\xi\|_{H^s(\Omega)} \le c \|\xi\|_{H^s(\Omega)}.$$

Clearly, the corresponding integral including p_k can be treated similarly. For the second term on the right-hand side of (44) we use the fact that the strong convergence of $n^{(N)}_k$ and $p^{(N)}_k$ in $L^1(\Omega)$ implies that

$$\left\| \frac{\sqrt{n_k + \gamma p_k}}{\sqrt{n^{(N)}_k + \gamma p^{(N)}_k + 1}} \right\|_{L^\infty(\Omega)} \le c. \tag{45}$$

Thus, by Young's inequality and (31),

$$\sum_{k=0}^K \tau \int_\Omega \frac{(n_k + \gamma p_k)(2\sqrt{n_k}\nabla\sqrt{n_k} + 2\sqrt{p_k}\nabla\sqrt{p_k} - (n_k - p_k)\nabla\psi_k)}{1 + n_k + p_k} \cdot \nabla \frac{\xi}{y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1}} dx$$

$$\le c \sum_{k=0}^K \tau \left\| \frac{n_k + \gamma p_k}{\sqrt{1 + n_k + p_k}} \nabla \frac{\xi}{y^{(N)}_{\gamma,k} + y^{(N)}_{\gamma,k-1}} \right\|_{L^2(\Omega)}$$

$$
\begin{aligned}
&\le c\sum_{k=0}^K \tau\left(\left\|\sqrt{n_k+\gamma p_k}\frac{\nabla\xi}{y_{\gamma,k}^{(N)}+y_{\gamma,k-1}^{(N)}}\right\|_{L^2(\Omega)} + \left\|\sqrt{n_k+\gamma p_k}\frac{\nabla(y_{\gamma,k}^{(N)}+y_{\gamma,k-1}^{(N)})}{(y_{\gamma,k}^{(N)}+y_{\gamma,k-1}^{(N)})^2}\xi\right\|_{L^2(\Omega)}\right) \\
&\le c\sum_{k=0}^K \tau\left\|\frac{\sqrt{n_k+\gamma p_k}}{y_{\gamma,k}^{(N)}}\right\|_{L^\infty(\Omega)}\left(\|\nabla\xi\|_{L^2(\Omega)} + \left\|\frac{\nabla(y_{\gamma,k}^{(N)}+y_{\gamma,k-1}^{(N)})}{y_{\gamma,k}^{(N)}+y_{\gamma,k-1}^{(N)}}\right\|_{L^2(\Omega)}\|\xi\|_{L^\infty(\Omega)}\right) \\
&\le c\|\xi\|_{H^s(\Omega)},
\end{aligned}
$$

employing (45).

Finally, the last term on the right-hand side of (44) can be bounded by

$$
\sum_{k=0}^K \tau(1+\gamma)\int_\Omega R(n_k,p_k)\frac{\xi}{y_{\gamma,k}^{(N)}+y_{\gamma,k-1}^{(N)}}dx \le (\|n_k\|_{L^1(\Omega)}+\|p_k\|_{L^1(\Omega)})\|\xi\|_{L^\infty(\Omega)} \le c\|\xi\|_{H^s(\Omega)}.
$$

The above inequalities show that (44) can be uniformly bounded. This proves (43). ☐

Lemma 8 allows to apply Aubin's lemma and hence to conclude strong convergence. More precisely, the following result holds.

Lemma 9. *There exist subsequences of* $(n^{(\tau)})$ *and* $(p^{(\tau)})$ *(not relabelled) such that for all* $r<\infty$, *as* $\tau\to 0$,

$$
n^{(\tau)} \to n, \quad p^{(\tau)} \to p \qquad \text{strongly in } L^r(0,T;L^1(\Omega)), \tag{46}
$$

$$
\nabla\sqrt{n^{(\tau)}} \rightharpoonup \nabla\sqrt{n}, \quad \nabla\sqrt{p^{(\tau)}} \rightharpoonup \nabla\sqrt{p} \qquad \text{weakly in } L^1(Q_T), \tag{47}
$$

where we recall that $Q_T=\Omega\times(0,T)$.

Proof. Thanks to the estimates (42) and (43) and the compact embedding $W^{1,1}(\Omega)\hookrightarrow L^q(\Omega)$ for $1\le q<d/(d-1)$, we can apply Aubin's lemma [20] to obtain the existence of a subsequence (not relabeled) such that, as $\tau\to 0$,

$$
y_\gamma^{(\tau)} \to y_\gamma \quad \text{strongly in } L^2(0,T;L^q(\Omega)).
$$

In particular, up to the extraction of a subsequence, the sequence is converging a.e. in Q_T. Therefore,

$$
n^{(\tau)}+\gamma p^{(\tau)} = (y_\gamma^{(\tau)})^2-1 \to y_\gamma^2-1 \quad \text{a.e. in } Q_T.
$$

Therefore, for $\gamma=1,2$, we have

$$
n^{(\tau)} \to n := 2y_1-y_2, \quad p^{(\tau)} \to p := y_2-y_1 \quad \text{a.e. in } Q_T.
$$

With the help of the a.e. convergence and the uniform $L^\infty(0,T;L_\Psi(\Omega))$ bounds for $n^{(k)}$ and $p^{(k)}$, Lemma 4 gives (46).

In order to prove the second result, we argue as in the proof of Lemma 6. The estimates (38) and (39) show that (a subsequence of) the sequence $(\nabla\sqrt{n^{(\tau)}}/\sqrt{n^{(\tau)}+p^{(\tau)}+1})$ converges weakly in $L^2(Q_T)$ and $(\sqrt{n^{(\tau)}+p^{(\tau)}+1})$ converges strongly in $L^2(Q_T)$; therefore, the product converges weakly in $L^1(Q_T)$. ☐

Lemma 10. *There exist subsequences (not relabeled) such that, as $\tau \to 0$,*

$$\frac{2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} - n^{(\tau)}\nabla\psi^{(\tau)}}{1+n^{(\tau)}+p^{(\tau)}} \rightharpoonup \frac{2\sqrt{n}\nabla\sqrt{n} - n\nabla\psi}{1+n+p}, \tag{48}$$

$$\frac{2\sqrt{p^{(\tau)}}\nabla\sqrt{p^{(\tau)}} + p^{(\tau)}\nabla\psi^{(\tau)}}{1+n^{(\tau)}+p^{(\tau)}} \rightharpoonup \frac{2\sqrt{p}\nabla\sqrt{p} + p\nabla\psi}{1+n+p} \tag{49}$$

weakly in $L^2(Q_T)$ and

$$\begin{aligned} &\frac{2n^{(\tau)}(\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} + \sqrt{p^{(\tau)}}\nabla\sqrt{p^{(\tau)}}) - n^{(\tau)}(n^{(\tau)}-p^{(\tau)})\nabla\psi^{(\tau)}}{1+n^{(\tau)}+p^{(\tau)}} \\ &\quad \rightharpoonup \frac{2n(\sqrt{n}\nabla\sqrt{n} + \sqrt{p}\nabla\sqrt{p}) - n(n-p)\nabla\psi}{1+n+p}, \end{aligned} \tag{50}$$

$$\begin{aligned} &\frac{2p^{(\tau)}(\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} + \sqrt{p^{(\tau)}}\nabla\sqrt{p^{(\tau)}}) - p^{(\tau)}(n^{(\tau)}-p^{(\tau)})\nabla\psi^{(\tau)}}{1+n^{(\tau)}+p^{(\tau)}} \\ &\quad \rightharpoonup \frac{2p(\sqrt{p}\nabla\sqrt{p} + \sqrt{n}\nabla\sqrt{n}) - p(n-p)\nabla\psi}{1+n+p} \end{aligned} \tag{51}$$

weakly in $L^\sigma(0,T;L^1(\Omega))$ for all $1 \le \sigma < 2$.

Proof. The sequence $(\sqrt{n^{(\tau)}}/(1+n^{(\tau)}+p^{(\tau)}))$ is bounded in $L^\infty(Q_T)$ and converges (up to a subsequence) a.e. to $\sqrt{n}/(1+n+p)$, by (46). In view of (47), we can apply Lemma 3 to obtain, as $\tau \to 0$,

$$\frac{2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}}}{1+n^{(\tau)}+p^{(\tau)}} \rightharpoonup \frac{2\sqrt{n}\nabla\sqrt{n}}{1+n+p} \quad \text{weakly in } L^1(Q_T).$$

Now we argue similarly as in the proof of Lemma 6. The estimate (38) provides the weak convergence of (a subsequence of) $(\nabla\psi^{(\tau)})$ to $\nabla\psi$ in $L^2(Q_T)$, and thus, again by Lemma 3,

$$\frac{n^{(\tau)}\nabla\psi^{(\tau)}}{1+n^{(\tau)}+p^{(\tau)}} \rightharpoonup \frac{n\nabla\psi}{1+n+p} \quad \text{weakly in } L^2(Q_T).$$

This shows that

$$\frac{2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} - n^{(\tau)}\nabla\psi^{(\tau)}}{1+n^{(\tau)}+p^{(\tau)}} \tag{52}$$

converges to $(2\sqrt{n}\nabla\sqrt{n}-n\nabla\psi)/(1+n+p)$ weakly in $L^1(Q_T)$. In fact, the weak convergence holds true in $L^2(Q_T)$ since (52) is uniformly bounded in $L^2(Q_T)$ by (38)-(39). This shows (48). The proof of (49) is similar.

Next, we show (50) and (51). By (46) and (17), up to the extraction of subsequences, $\sqrt{n^{(\tau)}} \to \sqrt{n}$ strongly in $L^2(Q_T)$ and $\nabla\psi^{(\tau)} \rightharpoonup \nabla\psi$ weakly in $L^2(Q_T)$ and therefore, $\sqrt{n^{(\tau)}}\nabla\psi^{(\tau)} \rightharpoonup \sqrt{n}\nabla\psi$ weakly in $L^1(Q_T)$. Then Lemma 3 implies

$$\frac{n^{(\tau)}\nabla\psi^{(\tau)}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}} = \frac{\sqrt{n^{(\tau)}}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}} \cdot \sqrt{n^{(\tau)}}\nabla\psi^{(\tau)} \to \frac{n\nabla\psi}{\sqrt{1+n+p}} \quad \text{weakly in } L^1(Q_T).$$

Lemma 3 implies further

$$\frac{2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}} \rightharpoonup \frac{2\sqrt{n}\nabla\sqrt{n}}{\sqrt{1+n+p}} \quad \text{weakly in } L^1(Q_T). \tag{53}$$

This shows that

$$\frac{2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} - n^{(\tau)}\nabla\psi^{(\tau)}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}} \rightharpoonup \frac{2\sqrt{n}\nabla\sqrt{n} - n\nabla\psi}{\sqrt{1+n+p}} \quad \text{weakly in } L^1(Q_T).$$

In a similar way, we deduce that

$$\frac{2\sqrt{p^{(\tau)}}\nabla\sqrt{p^{(\tau)}} + p^{(\tau)}\nabla\psi^{(\tau)}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}} \rightharpoonup \frac{2\sqrt{p}\nabla\sqrt{p} + p\nabla\psi}{\sqrt{1+n+p}} \quad \text{weakly in } L^1(Q_T).$$

The sum of both sequences also converges weakly in $L^1(Q_T)$. In fact, in view of (41), the sum is uniformly bounded in $L^2(Q_T)$ and hence,

$$\begin{aligned} &\frac{2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} - n^{(\tau)}\nabla\psi^{(\tau)} + 2\sqrt{p^{(\tau)}}\nabla\sqrt{p^{(\tau)}} + p^{(\tau)}\nabla\psi^{(\tau)}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}} \\ &\quad \rightharpoonup \frac{2\sqrt{n}\nabla\sqrt{n} - n\nabla\psi + 2\sqrt{p}\nabla\sqrt{p} + p\nabla\psi}{\sqrt{1+n+p}} \quad \text{weakly in } L^2(Q_T). \end{aligned}$$

By Lemma 3, we deduce that

$$\begin{aligned} &\frac{\sqrt{n^{(\tau)}}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}} \frac{2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} - n^{(\tau)}\nabla\psi^{(\tau)} + 2\sqrt{p^{(\tau)}}\nabla\sqrt{p^{(\tau)}} + p^{(\tau)}\nabla\psi^{(\tau)}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}} \\ &\quad \rightharpoonup \frac{\sqrt{n}(2\sqrt{n}\nabla\sqrt{n} - n\nabla\psi + 2\sqrt{p}\nabla\sqrt{p} + p\nabla\psi)}{1+n+p} \quad \text{weakly in } L^2(Q_T). \end{aligned} \tag{54}$$

Since $\sqrt{n^{(\tau)}} \to \sqrt{n}$ strongly in $L^r(0,T;L^2(\Omega))$ for all $r<\infty$, the product of this sequence and (54) converges weakly in $L^\sigma(0,T;L^1(\Omega))$ for all $\sigma<2$:

$$\begin{aligned} &\frac{n^{(\tau)}(2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} - n^{(\tau)}\nabla\psi^{(\tau)} + 2\sqrt{p^{(\tau)}}\nabla\sqrt{p^{(\tau)}} + p^{(\tau)}\nabla\psi^{(\tau)})}{1+n^{(\tau)}+p^{(\tau)}} \\ &\quad \rightharpoonup \frac{n(2\sqrt{n}\nabla\sqrt{n} - n\nabla\psi + 2\sqrt{p}\nabla\sqrt{p} + p\nabla\psi)}{1+n+p} \quad \text{weakly in } L^\sigma(0,T;L^1(\Omega)). \end{aligned}$$

This proves (50). The proof of (51) is analogous. □

Lemma 11. *There exist subsequences (not relabeled) such that, as $\tau\to 0$,*

$$\frac{1-n^{(\tau)}p^{(\tau)}}{1+n^{(\tau)}+p^{(\tau)}} \to \frac{1-np}{1+n+p} \qquad \text{strongly in } L^r(0,T;L^1(\Omega)) \text{ for all } r<\infty, \tag{55}$$

$$\frac{1}{\tau}(n^{(\tau)} - \sigma_N n^{(\tau)}) \rightharpoonup \partial_t n \qquad \text{weakly in } L^\sigma(0,T;(H^s_D(\Omega))'), \tag{56}$$

$$\frac{1}{\tau}(p^{(\tau)} - \sigma_N p^{(\tau)}) \rightharpoonup \partial_t p \qquad \text{weakly in } L^\sigma(0,T;(H^s_D(\Omega))'), \tag{57}$$

where $1 < \sigma < 2$.

Proof. For the result (55) we observe that the sequences

$$\frac{n^{(\tau)}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}} \quad \text{and} \quad \frac{p^{(\tau)}}{\sqrt{1+n^{(\tau)}+p^{(\tau)}}}$$

converge strongly in $L^{2r}(0,T;L^2(\Omega))$, by (46), and therefore, the product converges strongly in $L^r(0,T;L^1(\Omega))$ for all $r<\infty$, which implies (55).

Lemma 10 shows that

$$\begin{aligned} A^{(\tau)} &= \frac{2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} - n^{(\tau)}\nabla\psi^{(\tau)}}{1+n^{(\tau)}+p^{(\tau)}} \\ &\quad + \frac{n^{(\tau)}(2\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}} - n^{(\tau)}\nabla\psi^{(\tau)} + 2\sqrt{p^{(\tau)}}\nabla\sqrt{p^{(\tau)}} + p^{(\tau)}\nabla\psi^{(\tau)})}{1+n^{(\tau)}+p^{(\tau)}} \end{aligned}$$

is uniformly bounded in $L^\sigma(0,T;L^1(\Omega))$ for all $\sigma<2$ and moreover,

$$A^{(\tau)} \rightharpoonup \frac{2\sqrt{n}\nabla\sqrt{n} - n\nabla\psi}{1+n+p} + \frac{n(2\sqrt{n}\nabla\sqrt{n} - n\nabla\psi + 2\sqrt{p}\nabla\sqrt{p} + p\nabla\psi)}{1+n+p}.$$

weakly in $L^\sigma(0,T;L^1(\Omega))$. By the weak formulation of the approximation problem, we conclude the estimate

$$\left\| \frac{1}{\tau}(n^{(\tau)} - \sigma_K n^{(\tau)}) \right\|_{L^\sigma(0,T;(H^s_D(\Omega))')} \le c,$$

which shows (56) and, similarly, (57). □

The proof of Theorem 1 follows from Lemmas 10 and 11. It only remains to show that $\sqrt{n}$ and $\sqrt{p}$ satisfy the boundary data. In fact, $\sqrt{n_k^{(N)}} - \sqrt{n_D}$, $\sqrt{p_k^{(N)}} - \sqrt{p_D} \in H^1_D(\Omega)$ and thus, in view of (33), $\sqrt{n^{(\tau)}} - \sqrt{n_D}$, $\sqrt{p^{(\tau)}} - \sqrt{p_D} \in L^2(0,T;W^{1,1}_D(\Omega))$. Then, the limit (47) implies that $\sqrt{n} - \sqrt{n_D}$, $\sqrt{p} - \sqrt{p_D} \in L^2(0,T;W^{1,1}_D(\Omega))$.

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