INTRODUCTION TO PROFINITE GROUPS
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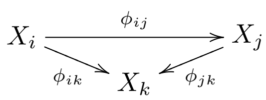
Dedicated to Peter Plaumann

Abstract. A *profinite space / group* is the *projective limit* of finite sets / groups. Galois theory offers a natural frame in order to describe Galois groups as profinite groups. Profinite groups have properties that correspond to some of finite groups: e.g., each profinite group does have p-Sylow subgroups for any prime p. In the same vein, every pro-solvable group (the projective limit of an inverse system of finite solvable groups) has Hall subgroups for any given set of primes. Any group can be equipped with the *profinite* topology turning it into a topological group. A basis of neighbourhoods of the identity-element consists of all normal subgroups of finite index. Any such group allows a completion w.r.t. this topology – the *profinite completion.* A *free profinite group* is the profinite completion of a free group. This can be considered an instance of the amalgamated free product and of the HNN extension (Higman-Neumann-Neumann). I do not include cohomological topics in this note.

Contents

1. Projective limits

Definition 1. *Let $(I, \leq)$ denote a set with a partial order order '$\leq$', directed upwards. So, '$\leq$' is a reflexive, antisymmetric and transitive relation on I and for all indices $i, j \in I$ there is k, so that $i \leq k$ and $j \leq k$. For every $i \in I$ let X_i be a set; we assume that for all $i \leq j$ there is a map ϕ_{ji} so that for all Indices $k \leq j \leq i$ the diagram*



commutes.

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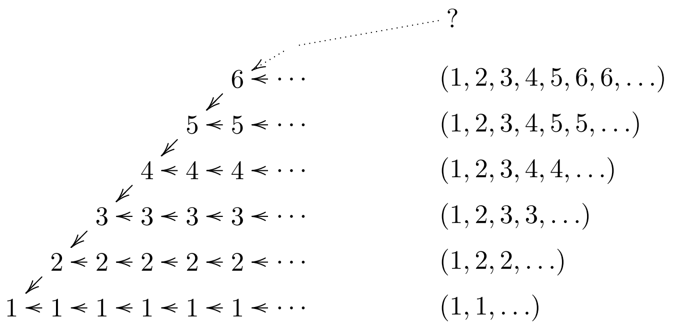
Here is an example:



In this example I is the set $\mathbb{N}$ of natural numbers and '$\leq$' is the natural ordering on $\mathbb{N}$. For $i \in \mathbb{N}$ set $X_i := \{1, 2, \ldots, i\}$ and let the arrows indicate the maps ϕ_{i+1i}. E.g., $\phi_{43}(4) = 3$, $\phi_{43}(i) = 3$ for $i \geq 3$ and $\phi_{43}(i) = i$ for $i < 4$.

Definition 2. In order to define the *projective limit* of the system $(I, \leq)$, proceed as follows:

(1) Form the Tychonov product $P := \prod_{i \in I} X_i$ (which consists of all the maps $f : I \to \bigcup_{i \in I} X_i$ with $f(i) \in X_i$).
(2) The projective limit $X := \varprojlim_{i \in I} X_i$ is the subset of all those $f \in P$ satisfying the *compatibility relations* $\phi_{ij}(f(i)) = f(j)$ for all $j \leq i$.
(3) The $\phi_i : P \to X_i$ satisfy $\phi_{ij}\phi_i = \phi_j$ for all indices $j \leq i$ and are nothing but the *canonical projections* of the Tychonov product restrichted to X.

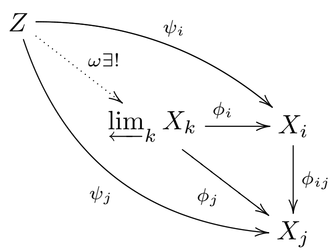
Theorem 3. *The following statements for a set X are equivalent:*

- *X is a compact topological space; moreover it is totally disconnected and Hausdorff.*
- *X is the projective limit of an inverse system of finite sets.*
- *X is a closed subset of the cartesian product $\prod_{j \in J} X_j$ of finite discrete topological spaces X_j.*

Such X is a *profinite space.*

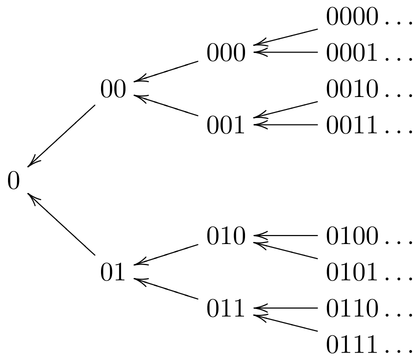
In the above example $X_i = \{1, 2, \ldots, i\}$. The elements of X are the infinite "rays" (f_i), for which $1 \leq f(i) \leq i$. In particular we find "rays" of the sort $(1, 2, 3, 4, \ldots, i, i, i, i, \ldots)$ for $i \in \mathbb{N}$ and the special one $\infty := (1, 2, 3, 4, 5, 6, \ldots)$. Thus $\varprojlim_{i \in \mathbb{N}} X_i$ coincides with the Aleksandrov-compactification of the natural numbers $\mathbb{N} \cup \{\infty\}$.

The projective limit $\varprojlim_i X_i$ of an inverse system (X_i, ϕ_{ij}) enjoys a *universal property*: whenever Z is a profinite space and there are continuous maps $\psi_i : Z \to X_i$ so that for all indices $i \leq j$ the diagrams



commute then there is a *universal map* ω, so that $\phi_i\omega = \psi_i$ holds for all $i \in I$.

Example 4. Consider the diagram



This gives rise to defining an inverse system. How to define formally ϕ_{ij}?

Example 5. Let $X = \varprojlim_i X_i$ and no X_i be the empty set. Show the following assertions:

(1) X is not the empty set.
(2) For every point $x \in X$ and every neighbourhood U of x there is $i \in I$ so that $\phi_i^{-1}(\phi_i(x))$ is a closed and open (a *clopen*) neighbourhood of x contained in U.
(3) For a subset $A \subseteq X$ its topological closure $\bar{A}$ equals $\bigcap_{i\in I} \phi_i^{-1}\phi_i(A)$.
When ψ_{ij} denotes the restriction of ϕ_{ij} to $\phi_i(A)$ then $(\phi_i(A_i), \psi_{ij})$ is an inverse system. Show that $A = \varprojlim_i \phi_i(A_i)$.

Example 6. (Vietoris-topology) For a profinite space $X = \varprojlim_i X_i$ denote by $C(X)$ the set of all closed non empty subsets of X. For a continuous map $f : X \to Y$ define $C(f) : C(X) \to C(Y)$ by $C(f)(A) := \{f(a) \mid a \in A\}$. The map $C(-)$ is a functor. We obtain the inverse system $(C(X_i), C(\phi_{ij}))$. Show that $C(X) = \varprojlim_i C(X_i)$. Show that the subsets $W(U_1, \ldots, U_n)$ of $C(X)$, where U_i are clopen subsets of X, and

$$W(U_1, \ldots, U_n) := \{C \in C(X) \mid C \subseteq \bigcup_{j=1}^{n} U_j \ \wedge\ C \cap C_j \neq \emptyset,\ j = 1, \ldots n\},$$

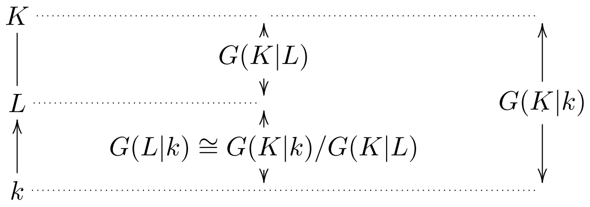
form a basis of the *Vietoris-topology.*

Example 7. Let k be a field and I the set of all finite Galois extensions of k. The relation $K \leq L$ if $k \subseteq K \subseteq L$ is a partial ordner. It is well-known that for all finite Galois extensions K and L there is a finite Galois extension M so that $K \leq M$ and $L \leq M$. Hence $\leq$ is a directed order.

Lemma 8. *If for all $i \in I$ the set X_i is a group and all ϕ_{ji} are homomorphisms then one can define a group multiplication on X so that X becomes a topological group and the canonical projections become continuous homomorphisms.*

Proof. The product $P := \prod_{i\in I} X_i$ is a group with group operation $(x_i)(y_i) := (x_i y_i)$. Observe the continuity of this operation w.r.t. the product topology. The projective limit $\varprojlim_i X_i$ is a closed subgroup of P. □

Example 9. As in Ex. 7 consider the finite Galois extensions of a field k. Let $G(K|L)$ be the Galois group of a Galois extension K of L. Galois' theorem says that $G(L|k)$ is isomorphic to $G(K|k)/G(K|L)$.



Hence there is $K \leq L$ ($L \subseteq K$) and a canonical homomorphism $\phi_{KL} : G(K|k) \to G(L|k)$. The inverse system $(G(K|k), \phi_{LK})$ gives rise to the projective limit $\varprojlim_K G(K|L)$. Let $\hat{k} := \bigcup_K K$. It is clear that $\hat{k}$ is a field and an algebraic and normal extension of k. The Galois group $G(\hat{k}|k)$ agrees with the projective limit $\varprojlim_K G(K|L)$.

Replacing in the diagram K by $\hat{k}$ one observes that $G(\hat{k}|L)$ is a normal subgroup of finite index in $G(\hat{k}|k)$. The *Krull-topology* on $G(\hat{k}|k)$ has a basis of open neighbourhoods of the identity consisting of all subgroups of the form $G(\hat{k}|L)$.

Example 10. For a profinite group G consider the set $\mathcal{N}$ of all open normal subgroup and the order '$\preceq$' given as $N \leq M$ whenever M is a subgroup of N. Show that

(1) every $N \in \mathcal{N}$ is closed.
(2) $(\mathcal{N}, \preceq)$ is a directed set.
(3) Let $\phi_N : G \to G/N$ denote the canonical homomorphism. There exist induced homomorphisms $\phi_{NM} : G/M \to G/N$ whenever M is contained in N. Obtain that $G = \varprojlim_N G/N$.

Example 11. Show that for a profinite group G the closed subgroups form a closed subset $S(G)$ of $C(G)$. For an open normal subgroup N and a finite subset Y of G set $W(N) := \{S \in S(G) | SN = \langle Y \rangle N\}$. Show that the sets $W(Y, N)$ form a basis of the induced Vietoris-topology on $S(G)$.

2. Profinite groups are "large" finite groups

A projective limit of an inverse system (G_i, ϕ_{ij}) of finite p-groups (p prime) is a *pro-p group*. A profinite group G is pro-p if and only if G/N is a finite p-group for every open normal subgroup N of G.

Example 12. Let us return to Ex. 4. Every natural number n has a binary expansion

$$n = a_0 2^0 + a_1 2^1 + a_2 2^2 + \ldots + a_k 2^k$$

with $a_j \in \{0, 1\}$. It is common to write $n = a_k \cdots a_2 a_1 a_0$ so that, e.g., $5 = 1 + 4 = 101$. How to present addition modulo 4 (or more generally modulo 2^k)? Can one interpret the diagram in Ex. 4 as an inverse system of finite 2-groups?

Example 13. Let $\mathbb{Z}_2$ be the set of all "formal" infinite binary expansions

$$\sum_{j=0}^{\infty} a_j 2^j$$

(we may think of a sequence $(a_0, a_1, a_2, \ldots)$.). Define a group operation for elements $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ using "binary addition": Set $r_{-1} := 0$. For $n = 0, 1, 2, \ldots$ define $c_n := a_n + b_n + r_{n-1} \pmod 2$ and $r_n := (a_n + b_n + r_{n-1} - c_n)/2$. Every natural number $n = a_0 + 2a_1 + 4a_2 + \ldots$ corresponds to a finite sequence in $\mathbb{Z}_2$

and addition in $\mathbb{N}$ amounts to the usual binary addition. The formal computation

$$-1 = \frac{1}{1-2} = \sum_{j=0}^{\infty} 2^j$$

yields a "geometric series" and indicates that $\mathbb{Z}$ may be considered a subgroup of $\mathbb{Z}_2$. This is the right moment to give a definion of $\mathbb{Z}_2$. Consider the inverse system (C_{2^i}, ϕ_{ij}), where ϕ_{i+1i} is the canonical epimorphism of $C_{2^{i+1}} \to C_{2^i}$. The above computation reads in C_{2^i} as

$$-1 \equiv 2^i - 1 = \sum_{j=0}^{i-1} 2^j.$$

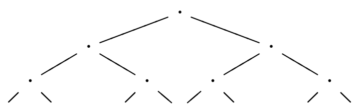
Therefore the infinite geometric series can be interpreted in $\varprojlim_i C_{2^i}$. So every number $z \in \mathbb{Z}$ has a formal presentation

$$z = \sum_{j=0}^{\infty} a_j 2^j$$

und $z \pmod{2^i} = \sum_{j=0}^{i-1} a_j 2^j$. It is easy to see that the sequence $(a_0, a_1, a_2, \ldots)$ is periodic for all $z \in \mathbb{Z}$. The "other" formal sequences do have an interpretation in $\varprojlim_i C_{2^i}$. This is the reason why $\mathbb{Z}_2 = \varprojlim_i C_{2^i}$.

Generally, when p is a prime then $\mathbb{Z}_p = \varprojlim_i C_{p^i}$ where ϕ_{i+1i} is the canonical epimorphism of $C_{p^{i+1}} \to C_{p^i}$. $\mathbb{Z}_p$ is an example of a *pro-p-ring*.

Example 14. (Wreath product and automorphisms) For a binary tree (in the diagram the points indicate *levels* 1,2, and 3.)



one can describe the automorphism group as follows: denoting by G_n the automorphism group of the tree of depths n then the subtrees in the next level become permuted and generate a group $V_n \cong C_2 \times C_2 \ldots \times C_2$ having 2^n factors. Inspection shows that $G_1 = \{1\}$, $G_2 = C_2$, $G_3 = (C_2 \times C_2) \rtimes C_2$, $G_4 = (((C_2 \times C_2) \rtimes C_2) \times (C_2 \times C_2) \rtimes C_2) \rtimes C_2$; for short $G_3 = C_2 \wr C_2$, $G_4 = (C_2 \wr C_2) \wr C_2$, etc.

Hence $G_{n+1} = V_n \rtimes G_n$ is a semidirect product. There is a natural inverse system consisting of (G_n, ϕ_{kn}) and $\phi_{k+1,k}$ the canonical epimorphism of $G_{k+1} \to G_k$ (with kernel V_k). The automorphism group of the tree is then the projective limit of this system. One can show that every pro-2 group having a dense countably generated subgroup embeds in G.

For an arbitrary prime p the construction is similar and the corresponding result is known to hold.

Theorem 15. *(Sylow) Let p be prime. Every profinite group G contains a maximal pro-p subgroup. If H and K are maximal pro-p subgroups of G then there is $g \in G$ so that $K = H^g$.*

Proof. When G is a pro-p group there is nothing to prove. So suppose that G is not a pro-p group. Ex. 11 shows that the set of all closed subgroups of G is a closed subset of $C(G)$. The subset $S_p(G)$ of all pro-p subgroups is a closed subset of $C(G)$: Fix any $S \notin S_p(G)$. Then there is an open normal subgroup N so that $\langle S \rangle N/N$ is not a finite p-group. Now, every $T \in W(S,N)$ satisfies $TN/N = SN/N$ and and so such T is not a finite p-group. Hence the complement of $S_p(G)$ is open and thus $S_p(G)$ is closed, as has been claimed.

For showing the existence of maximal elements in $S_p(G)$ we want to apply Zorn's Lemma using the order '$\subseteq$'. The set $S_p(G)$ is not empty as it contains $\langle 1\rangle$. Consider an ascending sequence of subgroups $\{S_i\}$ in $S_p(G)$ and a cluster point S in $S_p(G)$. Clearly $S_iN = SN$ for a cofinal subset of I and any fixed open normal subgroup N of G. Since SN/N is a p-group and $S = \varprojlim_N (\bigcup_i S_i)N/N$ a pro-p group containing every S_i the assumptions of Zorn's Lemma hold. Hence maximal elements in $S_p(G)$ – the *p-Sylow subgroups* – exist.

For proving that any maximal pro-p subgroups H and K are conjugate consider for an open normal subgroup N the following commutative diagram

$$\begin{array}{ccc} S_p(G) & \longrightarrow & S_p(G/N) \\ \big\downarrow & & \big\downarrow \\ S_p(G)/G & \longrightarrow & S_p(G/N)/G/N \end{array}$$

Here $S_p(G)/G$ and $S_p(G/N)/G/N$ are the respective quotient spaces for the action of G by conjugation on $S_p(G)$ and the induced action of G/N on $S_p(G/N)$. Now, by the Sylow theorem for finite groups, $S_p(G/N)/G/N$ consists of a single point only – the single conjugacy class of p-Sylow subgroups of G/N. Therefore, since $S_p(G)/G = \varprojlim_N S_p(G/N)/G/N$, there can only be a single conjugacy class of maximal pro-p subgroups of G. □

Example 16. (Schur-Zassenhaus) For a finite group G and a normal subgroup $N \lhd G$ so that $|G/N|$ and $|N|$ are coprime there exists a *complement* $H \leq G$ so that $G = N \rtimes H$ is a semidirect product.

(J. Thompson – O. H. Kegel – A. I. Kostrikin) Suppose $G = N \rtimes H$ and for all $h \in H \setminus \langle 1\rangle$ and $n \in N$ the equation $n^h = n$ holds then $n = 1$ (i.e., H acts *fixed point free* by conjugation on N) then there is bound c on the nilpotency class of N depending only upon H.

How to formulate and prove these assertions for profinite groups?

Definition 17. Let G be a profinite group and Π be a set of primes. A Π-*Hall system* is a collection $\{G_p \mid p \in \Pi\}$ of Sylow subgroups of G, so that $G_pG_q = G_qG_p$ for any $p, q \in \Pi$. Then $G_\Pi = \prod_{p\in\Pi} G_p$ is a maximal closed subgroup of G so that in every finite quotient the order is divisible only by primes in Π. It is called a Π-*Hall subgroup* of G.

For a solvable finite group G the following statements hold true:

- There is a Π-Hall system for $\Pi = \Pi(G)$.
- All Π-Hall subgroups are conjugate in G.
- There exists a *system normalizer*, i.e., a subgroup $K \neq \{1\}$ of G so that for every G_p in the Π-Hall system we have $G_p^k = G_p$ for all $k \in K$.

Example 18. Let G be a *pro-solvable* group, i.e., a projective limit of finite solvable groups. Show the corresponding statements for G.

Example 19. Let G be a pro-solvable group and let $\Pi(G)$ denote the set of primes p so that a infinite p-Sylow subgroup $G_p \neq \langle 1\rangle$ exists. Show that a closed abelian subgroup A of G exists with $\Pi(A)$ infinite. Conclude that for a pro-solvable torsion group G the set $\Pi(G)$ must be finite.

The result holds for any profinite torsion group (W. Herfort). As a consequence every compact torsion group G has $\Pi(G)$ finite. E. Zelmanov has shown that every pro-p torsion group is *locally finite* and J. S. Wilson, using these results and CFSG, has shown that every profinite torsion group is locally finite.

3. Profinite topology

For a group G consider the set $\mathcal{N}$ of all normal subgroup N of G with finite index $|G : N|$. Then $\mathcal{N}$ may serve as a basis of the neighbourhoods of the identity of G turning G into a *topological group*. This is the *profinite topology*.

Example 20. Let us consider examples:

(1) Let $G = \mathbb{Z}$ be the additive group of integers. Every subgroup N of finite index has the form $n\mathbb{Z}$ where n is a positive number. If $x \neq y$ are elements in $\mathbb{Z}$ then there is a positive number $m \in \mathbb{N}$ so that $x - y$ is not divisible by m. Then x and y have open neighbourhoods $x + m\mathbb{Z} := \{x + km \mid k \in \mathbb{Z}\}$ and $y + m\mathbb{Z}$, and $(x + m\mathbb{Z}) \cap (y + m\mathbb{Z}) = \emptyset$. Hence $\mathbb{Z}$ is Hausdorff.

 There is an amusing application of the profinite topology due to H. Fürstenberg (1955), showing that the set of primes must be infinite:

 Let P denote the set of all primes and suppose that P were finite. Every set $p\mathbb{Z}$ is clopen (closed and open) and $\bigcup_{p \in P} pZ = \mathbb{Z} \setminus \{-1, 1\}$ is closed – a contradiction.

(2) Let G be a *residually finite* group, i.e., for all $1 \neq g \in G$ there exists a normal subgroup N of G of finite index so that $g \notin N$. Equivalently the intersection of all normal subgroup of G of finite index is trivial.

 So, a group G is *residually finite* if and only if it is Hausdorff w.r.t. to its profinite topology: if $x \neq y$ are elements in G and N is an open normal subgroup with $xy^{-1} \notin N$ then $xN \cap yN = \emptyset$.

 Simple examples of residually finite groups are $\mathbb{Z}$ and, more generally, free and pro-cyclic groups.

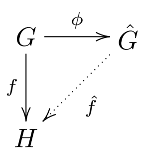
(3) Let G be a simple infinite group. Then $\mathcal{N} = \{G\}$.

Example 21. For a group G let $\mathcal{N}$ be the set of all normal subgroup of finite index in G. Order $\mathcal{N}$ by letting $M \preceq N$ if and only if $N \leq M$. Show that $(\mathcal{N}, \preceq)$ a directed set.

The preceding examples indicate that the set $\{G/N \mid N \in \mathcal{N}\}$ gives rise to an inverse system $(G/N, \phi_{NN'})$ where the canonical epimorphisms $\phi_{NN'} : G/N \to G/N'$ are induced from the canonical projections $G \to G/N$. The projective limit

$$\hat{G} := \varprojlim_{N} G/N$$

is the *profinite completion* of G and enjoys a universal property: there is a homomorphism $\phi : G \to \hat{G}$, so that $\phi(G)$ is dense in $\hat{G}$ and for every profinite group H and every continuous homomorphism $f : G \to H$ there is a unique continuous homomorphism $\hat{f} : \hat{G} \to H$ so that the diagram



commutes.

For an arbitrary group G there exists the *Bohr-compactification* βG (see the example p. 430 in [1]). It is easy to see that $\hat{G}$ is the factor group $\beta G / \beta G_0$ where βG_0 is the connected component of 1 in βG.

Example 22. Let us return briefly to Ex. 20.

(1) The profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$ is the set of *Hensel-numbers*.
(2) A group is residually finite if and only if ϕ is injective. The group $\widehat{F(x, y)}$ is the *free profinite group* – topologically freely generated by the set $\{x, y\}$.

(3) The third example has $\hat{G} = \langle 1 \rangle$.

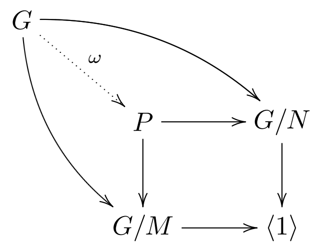
A category $\mathcal{C}$ of finite groups that admits pullbacks (=subdirect products) and contains H/K for all $H \in \mathcal{C}$ and K normal subgroup of H, is a *formation*.

An example is the class of all finite p-groups; another example is that of all finite solvable groups.

For a formation $\mathcal{C}$ of finite groups the preceding constructions apply and give rise to the notion of pro-$\mathcal{C}$ completion:

Let G be an arbitrary group and $\mathcal{N}_{\mathcal{C}}$ the set of all normal subgroup N of G, so that $G/N \in \mathcal{C}$.

Let M, N be normal subgroup of G in $\mathcal{N}_{\mathcal{C}}$. There is a pullback P



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namely, $P = G/M \cap N \in \mathcal{C}$ and $M \cap N \in \mathcal{N}$. Define an order *preceq* on $\mathcal{N}_{\mathcal{C}}$ by setting $MpredeqN$ if and only if $N \leq M$. Then ($\mathcal{N}$, *preceq*) is a directed set and therefore we can provide the *pro-$\mathcal{C}$ completion* $\hat{G}_{\mathcal{C}}$ of G.

Example 23. Some examples of formations:

- As has been said, if p is a prime then the class of all finite p-groups is a formation. One obtains the *pro-p completion*.
  The pro-2 completion of $\mathbb{Z}$ has been constructed in Ex. 13.
- For Π a set of primes we may form the class of finite groups G with $\Pi(G) \subseteq \Pi$.
- The class of all solvable groups form a formation. For G we obtain its *pro-solvable completion.*

Example 24. For G and $\mathcal{C}$ determine its pro-$\mathcal{C}$ completion:

(1) $G = \mathbb{Q}$ (the rational numbers) and $\mathcal{C}$ the class of all finite groups;
(2) $G = C_2$ and $\mathcal{C}$ the class of all finite 3-groups;
(3) $G = C_2 * C_4$ (free product) and $\mathcal{C}$ the class of all finite 5-groups;
(4) $G = F(x, y)$ (group freely generated by the set $\{x, y\}$) and $\mathcal{C}$ the class of all nilpotent finite groups;
(5) $G = \langle x, t \mid t^{-1}x^2t = x^4 \rangle$ and $\mathcal{C}$ the class of all finite 2-groups.

Example 25. *W. Burnside* (1902): If $n \in \mathbb{N}$ and G is a finitely generated group so that all elements $g \in G$ satisfy $g^n = 1$ (i.e., G has finite exponent), is then G necessarily a finite group? I. Adian has shown this to be false. A variation of this question is the restricted Burnside problem: does such G have a maximal finite quotient? G. Higman (for G solvable) and J. S. Wilson (general case) found that it is enough to study groups of prime power exponent p^k. The final solution for p^k is due to E. Zelmanov.

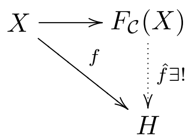
How can one formulate the restricted Burnside problem and its solution in terms of profinite topology?

The Krull-topology for the Galois group of a field is coarser than the profinite topology. When a profinite group G is topologically finitely generated – then, due to a result of N. Nikolov and D. Segal, every subgroup of finite index in G must be open. In this situation the topology of G agrees with the profinite topology.

4. Free constructions

Let $\mathcal{C}$ be a formation of finite groups. Every projective limit of groups in $\mathcal{C}$ is a *pro-$\mathcal{C}$ group*. This notion comprises the concept of pro-p group (2), for which $\mathcal{C}$ is the class of all finite p-groups.

Let X be any finite set and $F_0(X)$ the free group having basis X. The free pro-$\mathcal{C}$ group is denoted by $F_{\mathcal{C}}(X)$ and can be defined by a universal property: X is contained in $F_{\mathcal{C}}(X)$. Every map $f : X \to H$, where H is an arbitrary pro-$\mathcal{C}$ group, determines a unique continuous homomorphism $\hat{f}$, so that the following diagram commutes:



Example 26. Here are a few examples:

(1) If $|X| = 1$, then $F_{\mathcal{C}}(X)$ is *pro-cyclic*. The group $\mathbb{Z}_2$ in Ex. 13 is a free pro-2 group for $\mathcal{C}$ all finite 2-groups. It is not a free profinite group. Why?

In Ex. 12(1) the group $\hat{\mathbb{Z}}$ is the *free pro-cyclic group* and $\mathcal{C}$ consists of all finite (solvable) groups. Note that $\hat{\mathbb{Z}}_2$ is a closed subgroup of $\hat{\mathbb{Z}}$ – but it is not free profinite.

(2) In Ex. 12(2) we have $X = \{x, y\}$ and $\mathcal{C}$ consists of all groups.

(3) If X is finite then $F_{\mathcal{C}}(X)$ is the pro-$\mathcal{C}$ completion of the free group $F_0(X)$.

(4) If X is infinite then provide the free group $F_0(X)$ with the *restricted* pro-$\mathcal{C}$ topology: let $X \cup \{1\}$ be the Alexandrov-compactification of X and $\mathcal{N}$ the set of normal subgroups N in $F_0(X)$ containing almost all elements of X and so that $F_0(X)/N$ belongs to $\mathcal{C}$. Then $F_0(X)$ beomes a topological group and its completion is a profinite group – the *free pro-$\mathcal{C}$ group* $F_{\mathcal{C}}(X)$.

Theorem 27. *Let $\mathcal{C}$ be a formation of finite groups, so that with $G \in \mathcal{C}$ every subgroup $H \leq G$ belongs to $\mathcal{C}$. Then every open subgroup of a free pro-$\mathcal{C}$ group $F_{\mathcal{C}}(X)$ is itself free.*

Proof. (for finite X) Let H be any open subgroup of $F(X) := F_{\mathcal{C}}(X)$. The free group $F_0(X)$ is a dense subgroup of $F_{\mathcal{C}}(X)$ and $H \cap F_0(X)$ is a subgroup of finite index in $F_0(X)$. By the Nielsen-Schreier theorem for free (discrete) groups $F_0(X) \cap H$ is free. If we can show $F_0 \cap H$ must contain a normal subgroup K of $F_0(X)$ with $F_0(X)/K \in \mathcal{C}$ then we proved that the profinite topology on $F_0(X)$ induces the profinite topology on $F_0(X) \cap H$ and the theorem follows.

Note that the complement H' of H in $F(X)$ is closed and that $H' = \bigcup_M (HM)'$ where M runs through all open normal subgroups of $F(X)$ with $F(X)/M \in \mathcal{C}$. By the compactness of H' we can restrict ourselves to a finite subcollection of subgroups M and their intersection provides an open normal subgroup N of $F(X)$ with $N \leq H$. Therefore $K := N \cap F_0(X)$ has the required properties. □

Definition 28. In a similar manner one can define the free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups A and B: Let $A * B$ be their free product and consider the set $\mathcal{N}$ of all normal subgroup N with $A * B/N \in \mathcal{C}$ and so that $A \cap N$ and $B \cap N$ are closed subgroups of A and B respectively. $\mathcal{N}$ is then a basis of the open neighbourhoods of the identity of $A * B$. The completion is denoted by $A \amalg B$ and we refer to it as the *free pro-$\mathcal{C}$ product of A and B*.

Example 29. Examples:

- For $A = B = \hat{\mathbb{Z}}$ obtain $A \amalg B = F(x,y)$ with generators x and y of A and B respectively.
- For $A = C_2 = B$ we obtain the dihedral group $C_2 * C_2$ and the pro-$\mathcal{C}$ dihedral group $C_2 \amalg C_2$. Then $C_2 \amalg C_2$ is as well the pro-$\mathcal{C}$ completion of $C_2 * C_2$.
- Generally, when A and B are groups in $\mathcal{C}$ then the free pro-$\mathcal{C}$ product is the pro-$\mathcal{C}$ completion of $A * B$.

Theorem 30. *(Kurosh for two factors) Let $\mathcal{C}$ be a formation so that with every group in $\mathcal{C}$ each of its subgroups belongs to $\mathcal{C}$. Let $G = A \amalg B$ be the free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups A and B. Every open subgroup H of G is then a free pro-$\mathcal{C}$ product*

$$H = \coprod_{i=1}^{m} A^{g_i} \cap H \amalg \coprod_{j=1}^{n} B^{k_j} \cap H \amalg U$$

and U is a free pro-$\mathcal{C}$ group.

(Kurosh for n factors) If $G = A_1 \amalg A_2 \amalg \ldots \amalg A_n$ and H is an open group then

$$H = \coprod_{i=1}^{n} \coprod_{r \in A_i \backslash G/H} A_i^{g_{ir}} \cap H \amalg U,$$

where $A_i \backslash G/H$ is a double coset system and $g_{ir} \in G$ and $g_{i1} = 1$.

For finite A and B the proof is similar to the one of Theorem 27.

Here is an application:

Corollary 31. *Let $G = A_1 \amalg A_2 \amalg \ldots \amalg A_n$ be the free profinite product of finite groups $A_i \neq \langle 1 \rangle$. Then $A_i \cap A_j^x \neq \langle 1 \rangle$ if and only if $i = j$ and $x \in A_i$. For $1 \neq a \in A_i$ the centralizer $C_G(a)$ agrees with $C_{A_i}(a)$.*

Proof. (P. A. Zalesskii) Similarly as in the discrete case the canonical epimorphism $\psi : \coprod_i A_i \to \prod_i A_i$ implies that $A_i \cap A_j^g \leq \ker \psi$ and, as $A_i \cap \ker \psi = \langle 1 \rangle$, we conclude that $i = j$

Suppose next that $x \notin A_i$ and $A_i \cap A_i^x \neq \langle 1 \rangle$. Then there exists an open normal subgroup N with $x \notin A_i N$. Then $H := A_i N$ is an open subgroup of G and so we can apply the Kurosh subgroup theorem:

$$A_i N = \coprod_{i=1}^{n} \coprod_{r \in A_k \backslash G/A_i N} A_k^{g_{kr}} \cap (A_i N) \amalg U.$$

In this free decomposition $A_i \leq A_i N$ has "exponent" $g_{ir} = 1$. Hence there must exist $a_i, a_i' \in A_i$, g_{ir} and $n \in N$ so that $x = a_i g_{ir} a_i' n$. But $g_{ir} = 1$; since $x \notin A_i N$ this is a contradiction.

The second assertion is a consequence of the first one. $\square$

Example 32. (1) Show that Theorem 27 can be deduced from the Kurosh subgroup theorem.

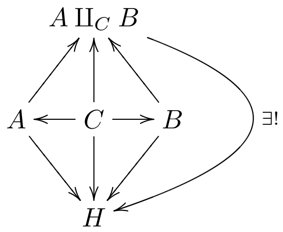
(2) Let $G = C_2 \amalg C_2 = \langle a, b \rangle$. Apply the Kurosh subgroup theorem to $H = \langle ab \rangle$.
(3) Let $G = C_2 \amalg C_2 \amalg C_2 = \langle a, b, c \rangle$. Apply the Kurosh subgroup theorem to $H = \langle ab, bc, ca \rangle$.
(4) Let A, B, C, D be groups in $\mathcal{C}$ and $G = A \amalg B = C \amalg D$. Show the existence of $g \in G$ so that either $C = A^g$ or $D = A^g$.

Definition 33. Let A, B, C be pro-$\mathcal{C}$ groups and $G := A *_C B$ be the free amalgamated product. Let $\mathcal{N}$ be the system of normal subgroups of G so that $G/N \in \mathcal{C}$ and $A \cap N$, $B \cap N$ and $C \cap N$ are all open subgroups of respectively A, B and C. We let $\mathcal{N}$ be basis of the filter of open neighbourhoods of the identity. The completion w.r.t. to this topology of G is the *free amalgamated pro-$\mathcal{C}$ product* and is denoted by $A \amalg_C B$.

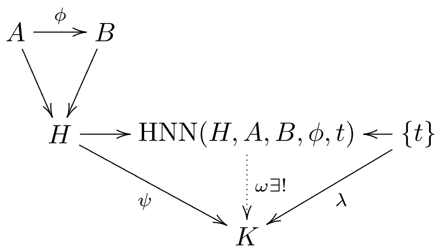
Let $\phi : A \to B$ be a continuous isomorphism between closed subgroups of a pro-$\mathcal{C}$ group H and let $G = \mathrm{HNN}(H, A, B, \phi, t)$ be the HNN extension. In G form $\mathcal{N}$ as just explained. The completion of G is the *pro-$\mathcal{C}$ HNN extension* with G the *basic subgroup*, A and B the *associated subgroups* and t the *stable letter*.

Recall that $\mathrm{HNN}(H, A, B, t)$ is the factor group of the free product $H * \langle t \rangle$ modulo the normal closure of the set $\{\phi(a)tat^{-1} \mid a \in A\}$.

As in the discrete case $A \amalg_C B$ and $\mathrm{HNN}(A, B, \phi, t)$ enjoy universal properties. Let A, B be (finite) groups and $C \to A$, $C \to B$ monomorphisms. The upwards arrows are the natural maps into the free amalgamated product which turns out to be the pushout of the diagram in the category of pro-$\mathcal{C}$ groups.



The next diagram displays the universal property of $\mathrm{HNN}(H, A, B, \phi, t)$:



Given a group K and a homomorphism $\psi : H \to K$ and an element in K (to be considered the image under a map $\lambda : \{t\} \to K$) so that $\psi\phi(a)) = \psi(a)^{\lambda(t)}$ holds for all $a \in A$. Then there exists a unique homomorphism $\omega : \mathrm{HNN}(H, A, B, \phi, t) \to K$ such that the diagram commutes.

Example 34. (HNN-extensions)

(1) Let $\phi : C_4 \to C_4$ be automorphism $\phi(x) := x^{-1}$ and $A = B = C_4$. Explain $\mathrm{HNN}(A, A, B, \phi, t)$.
(2) How can one present the profinite group in Ex. 24(5) in the form of an HNN-extension?

These constructions have remarkable applications (similar to the discrete case): Every profinite topologically countably generated group can be embedded in topologically 2-generated group (Z. Chatzidakis, P.A. Zalesskii).

Another embedding result, using the HNN-extensions (W. Herfort and P.A. Zalesskii), says that every profinite group possessing an open torsion free subgroup can be embedded in a profinite group so that all elements of the same finite order are conjugate.

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