Fundamental Groups of Solenoid Complements

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Definition

Let S^1 denoted the unit circle in the complex plane.

For an integer $n > 1$, define $f_n : S^1 \to S^1$ by $f_n(z) = z^n$.

Definition

Given a sequence $n_1, n_2, n_3, \ldots$, we get a *solenoid* as the inverse limit

$$\Sigma = \varprojlim (S^1, f_{n_i})$$

$$\cdots \to S^1 \xrightarrow{f_{n_3}} S^1 \xrightarrow{f_{n_2}} S^1 \xrightarrow{f_{n_1}} S^1$$

Dyadic Solenoid

If the sequence of integers is $2, 2, 2, \ldots$ $(n_i \equiv 2)$,
then the resulting solenoid is called the *dyadic solenoid*.

The dyadic solenoid is the most commonly discussed solenoid, and most of the specific examples in this talk will use the dyadic solenoid.

Properties of Solenoids

The sequence $\{n_i\}$ determines the solenoid, but not uniquely – different sequences can give rise to the same space.

The following do not change the solenoid:

- Removing finitely many numbers from the sequence.
- Infinite reordering of the sequence.
- Replacing a number by its factorization, e.g. $2, 6, 5, \ldots \sim 2, 2, 3, 5, \ldots$

Properties of Solenoids

A few facts about solenoids:

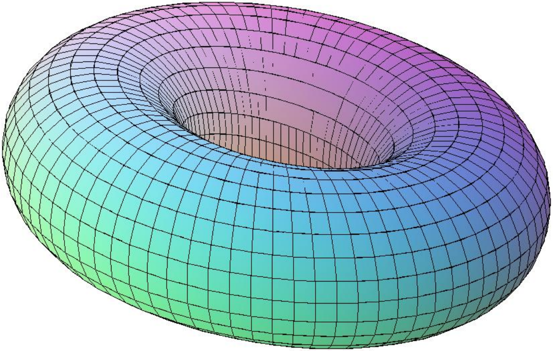
- A solenoid is a compact connected topological group.
- A solenoid has uncountably many path components.
- Each path component is dense in the solenoid.
- Each path component is “like unto” $\mathbb{R}$.
- A solenoid is not locally connected, nor is any of its path components.

Solenoids in $\mathbb{R}^3$

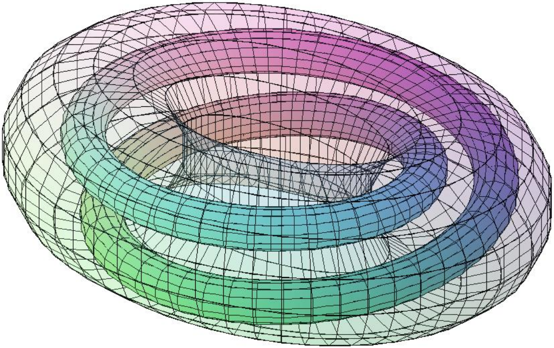
While it is not obvious from the definition as an inverse limit of circles, every solenoid can in fact be embedded in $\mathbb{R}^3$.

This can be achieved as a nested intersection of solid tori T_i, each of which loops around the previous torus n_i times.

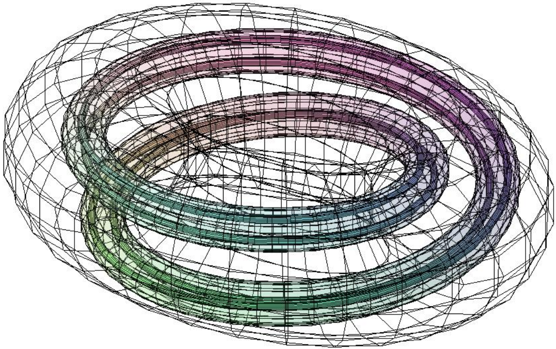
Solenoids in $\mathbb{R}^3$: T_0



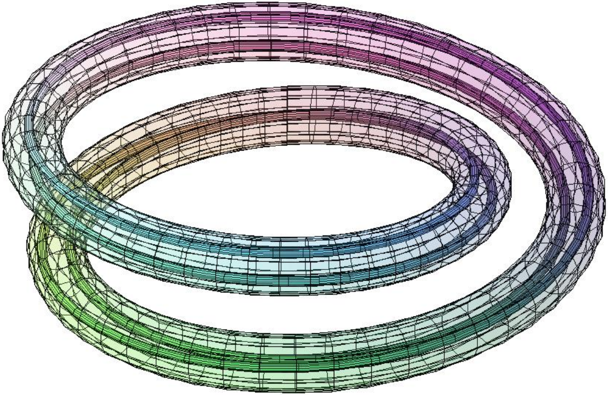
Solenoids in $\mathbb{R}^3$: $T_0 \supset T_1$



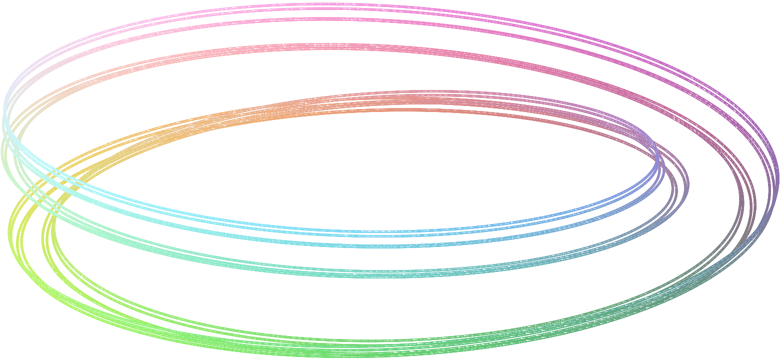
Solenoids in $\mathbb{R}^3$: $T_0 \supset T_1 \supset T_2$



Solenoids in $\mathbb{R}^3$: $T_1 \supset T_2$



Solenoids in $\mathbb{R}^3 : \Sigma = \bigcap T_i$



Fundamental Groups

When a solenoid Σ is embedded in S^3 (or $\mathbb{R}^3$), the complement $\Sigma^c = S^3 - \Sigma$ is an open 3-manifold.

We will discuss the fundamental groups of such manifolds, which depends on the embedding of Σ into S^3.

We use the Seifert Van Kampen Theorem to get a presentation for the fundamental group.

Solenoids

Recall that the solenoid is the nested intersection of solid tori:

$$T_0 \supset T_1 \supset T_2 \supset \dots$$

$$\Sigma = \bigcap T_i$$

Similarly, the complement is an increasing union:

$$S^3 - T_0 \subset S^3 - T_1 \subset S^3 - T_2 \subset \dots$$

$$\Sigma^c = \bigcup (S^3 - T_i)$$

Solenoids

Lemma

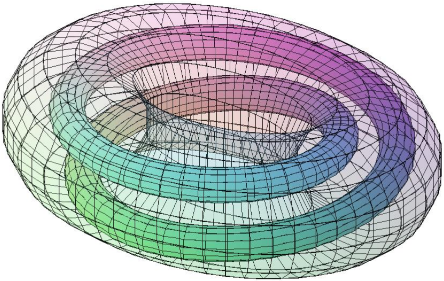
Let $\Sigma = \bigcap T_i$ *be a solenoid in* S^3 *realized as the nested intersection of solid tori.*

Then $\pi_1(S^3 - \Sigma) = \varinjlim_i \pi_1(S^3 - T_i) = \bigcup_i \pi_1(S^3 - T_i)$*.*

In particular, the maps $\pi_1(S^3 - T_i) \to \pi_1(S^3 - \Sigma)$ are injective.

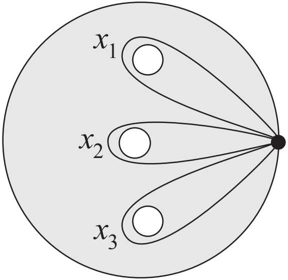
$\pi_1(T_{i-1} - T_i)$

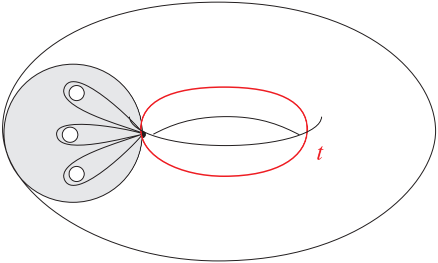
Our solenoid complement can be broken up into pieces $(T_{i-1} - T_i)$ that are each a solid torus minus a smaller solid torus that wraps around n_i times.



$\pi_1(T_{i-1} - T_i)$

The fundamental group $\pi_1(T_{i-1} - T_i)$ can be calculated by considering it as a mapping cylinder over an n_i-punctured disk.





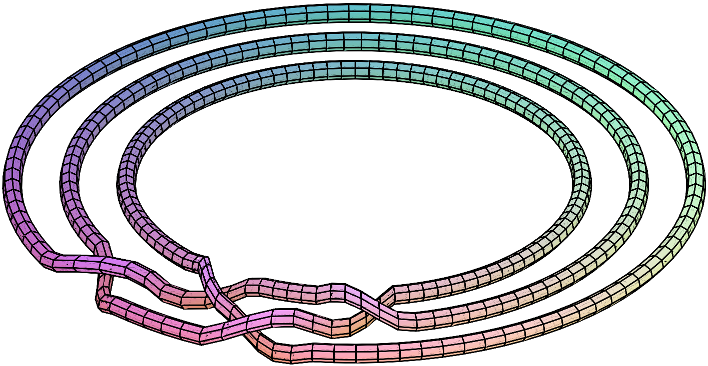
$\pi_1(T_{i-1} - T_i)$

The fundamental group $\pi_1(T_{i-1} - T_i)$ can be calculated by considering it as a mapping cylinder over an n_i-punctured disk.

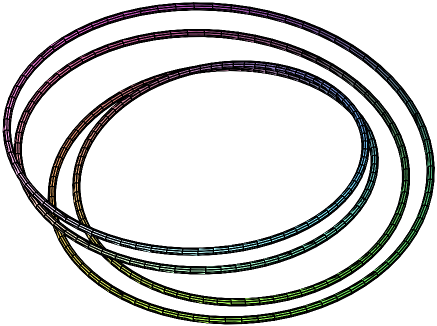
$$\pi_1(T_{i-1} - T_i) = \left\langle t, x_1, \ldots, x_{n_i} \;\middle|\; t^{-1} x_k t = w_k(x_1, \ldots, x_{n_i}) \right\rangle$$

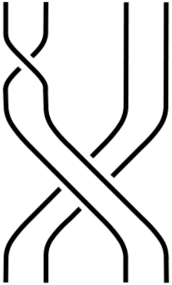
Here w_k is some word in the x_j's, depending on the embedding (braiding) of one solid torus inside the previous.

Figure 8 Knot

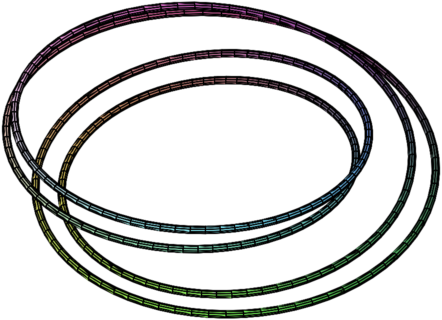


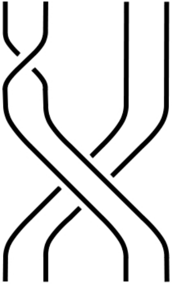
Dyadic Solenoid – Trefoil



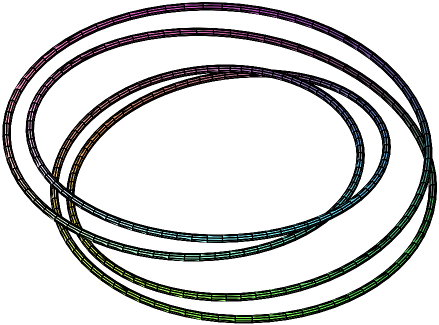


Dyadic Solenoid – Unknot



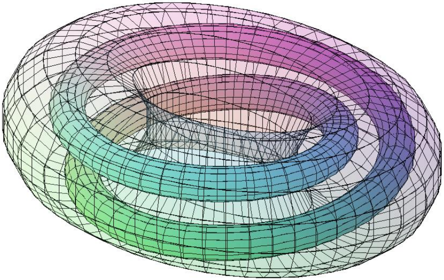


Dyadic Solenoid – Another Unknot



Seifert Van Kampen

Consecutive solid torus pieces share a common torus boundary, and we use this in the Seifert Van Kampen Theorem.



Seifert Van Kampen

Theorem (Seifert Van Kampen)

Let U, V be open sets in X such that $X = U \cup V$ and $U \cap V$ is path connected. Then

$$\pi_1(X) = \pi_1(U) *_C \pi_1(V) \quad \textit{where} \;\; C = \pi_1(U \cap V).$$

Seifert Van Kampen

With this we get relations such as

$$x_{(i-1)1} = \prod_{k=1}^{n_i} x_{(i)k}, \qquad t_{(i)} = t_{(i-1)}^{n_i} v_{(i)}(x_{(i)1}, \dots, x_{(i)n_i}).$$

- $x_{(i)k}$ is a meridian of T_i
- $t_{(i)}$ is a longitude of T_i
- $x_{(i)k}, t_{(i-1)}$ correspond to $\pi_1(T_{i-1} - T_i)$
- $v_{(i)}$ is a word determined by the embedding $T_i \subset T_{i-1}$

Presentation for π_1

In general, we get an infinite presentation for $\pi_1(\Sigma^c)$.

The generators are $t_{(i)}, x_{(i)k}$ from each level i, with $k = 1, \ldots, n_i$.

The relations come from each level and Van Kampen.

Also note that $t_{(0)} = e$, since the longitude of T_0 is trivial in S^3.

$$\pi_1(\Sigma^c) = \Big\langle t_{(i)}, x_{(i)k} \,\Big|\, t_{(i-1)}^{-1} x_{(i)k} t_{(i-1)} = w_{(i)k}(\{x_{(i)k}\}),\ t_{(0)} = e,$$

$$x_{(i-1)1} = \prod_{k=1}^{n_i} x_{(i)k},\ t_{(i)} = t_{(i-1)}^{n_i} v_{(i)}(\{x_{(i)k}\}) \Big\rangle$$

Dyadic Solenoid

In the case of the dyadic solenoid ($n_i \equiv 2$), our presentation for π_1 simplifies.

There are only two $x_{(i)k}$'s at each level i, and since $x_{(i-1)1} = x_{(i)1}x_{(i)2}$, we do not actually need the $x_{(i)2}$'s.

If we let $z_i = x_{(i)1}$ be the meridian of T_i, and $s_i = t_{(i)}$ the longitude of T_i, we then get a simplified presentation:

$$\pi_1 = \left\langle s_i, z_i \;\middle|\; [s_i, z_i] = e, R, s_0 = e \right\rangle$$

Dyadic Solenoid – Unknotted

If each solid torus T_i is unknotted in S^3, we get:

$$\left\langle s_i, z_i \;\middle|\; [s_i, z_i] = e, s_i^{-1} z_{i+1} s_i = z_{i+1}^{-1} z_i, s_{i+1} = s_i^2 z_i z_{i+1}^{-2}, s_0 = e \right\rangle$$

Dyadic Solenoid – Unknotted

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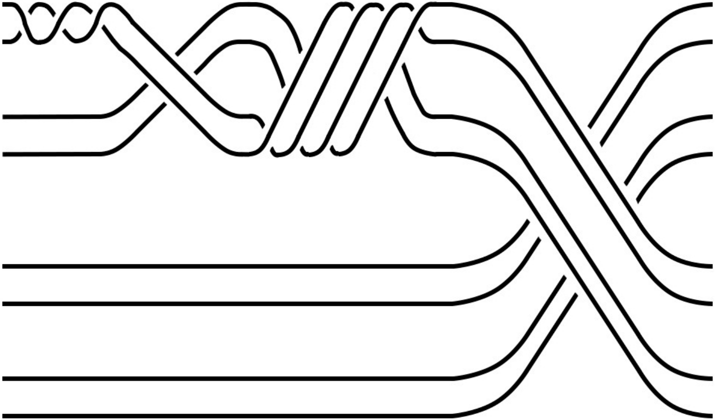
Note that if $s_i = e$, then $z_{i+1}^2 = z_i$, and then $s_{i+1} = s_i^2 = e$. Thus this group becomes

$$\left\langle z_i \;\middle|\; z_{i+1}^2 = z_i \right\rangle = \varinjlim(\mathbb{Z}, 2)$$

which is just the dyadic rationals $\left\{ \frac{a}{2^n} \in \mathbb{Q} \right\}$.

Unknotted Solenoids



Unknotted Solenoids

Theorem (Unknotted Embeddings)

- *For every solenoid Σ, there is an embedding into S^3 such that $\pi_1(\Sigma^c) \leq \mathbb{Q}$.*
- *For every non-trivial subgroup $G \leq \mathbb{Q}$, there is a solenoid Σ_G and an embedding such that $\pi_1(\Sigma_G^c) = G$.*

In particular, $\pi_1(\Sigma^c)$ is Abelian.

We call such an embedding of Σ *unknotted*.
This corresponds to each torus T_i being unknotted in S^3.

Unknotted Solenoids

Every solenoid can be embedded in an unknotted way.

In this case, if Σ is defined by the sequence $\{n_i\}$ then

$$\pi_1(\Sigma^c) = H_1(\Sigma^c) = \left\{ \frac{p}{q} \in \mathbb{Q} \;\middle|\; q = \prod_{i=1}^{k} n_i \text{ for some } k \right\}.$$

Lemma

Every non-trivial subgroup of $\mathbb{Q}$ is isomorphic to one of these.

Dyadic Solenoid – Knotted

If each solid torus is a trefoil knot inside the previous:

$$\left\langle s_i, z_i \;\middle|\; [s_i, z_i] = e, s_i^{-1} z_{i+1} s_i = z_i^{-1} z_{i+1}^{-1} z_i^2, s_{i+1} = s_i^2 z_i^3 z_{i+1}^{-6}, s_0 = e \right\rangle$$

Dyadic Solenoid – Knotted

If each solid torus is a trefoil knot inside the previous:

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Note that if we Abelianize, then $z_{i+1}^2 = z_i$, and then $s_{i+1} = s_i^2 = e$.

Thus $H_1 = (\pi_1)_{\mathsf{Ab}}$ is the dyadic rationals as before.

Dyadic Solenoid – Knotted

However, this group is not Abelian, as it maps onto the infinite alternating group A_∞.

$$\left\langle s_i, z_i \;\middle|\; [s_i, z_i] = e, s_i^{-1} z_{i+1} s_i = z_i^{-1} z_{i+1}^{-1} z_i^2, s_{i+1} = s_i^2 z_i^3 z_{i+1}^{-6}, s_0 = e \right\rangle$$

Map the elements z_i to the generators $\big(i\ (i+1)\ (i+2)\big)$, and send each s_i to the identity.

We can easily check that the relations are satisfied.

(Note that consecutive 3-cycles satisfy $z_{i+1} = z_i^{-1} z_{i+1}^{-1} z_i^2$.)

Alexander Duality

For a fixed solenoid Σ, the first homology of the complement is always the same, independent of the embedding in S^3.

This result follows from Alexander Duality, but can also be seen directly from our presentations for π_1.

Theorem (Alexander Duality)

For compact $K \subset S^n$, $H_i(S^n - K) \cong \check{H}^{n-i-1}(K)$

$$(\pi_1(\Sigma^c))_{\text{Ab}} = H_1(\Sigma^c) \equiv \check{H}^1(\Sigma) \leq \mathbb{Q}$$

Knotted Solenoids

Theorem (Knotted Embeddings)

For every solenoid Σ, there is an embedding into S^3 such that $\pi_1(\Sigma^c)$ is non-Abelian.

Conjecture

Every solenoid has infinitely many knotted embeddings with distinct non-Abelian fundamental groups.

Distinct Knotted Embeddings

Theorem

Every solenoid has infinitely many knotted embeddings with non-homeomorphic complements.

The proof actually gives embeddings corresponding to almost all defining sequences $\{n_i\}$, with the only exception being for the dyadic solenoid in the case that eventually $n_i \equiv 2$.

Extended JSJ-Decomposition

Theorem

Let $\Sigma = \bigcap T_i$ be embedded as the intersection of nested solid tori T_i in S^3, such that infinitely many of the pieces $(T_{i-1} - T_i)$ are hyperbolic.

Then the complement $S^3 - \Sigma$ has a canonical extended JSJ-decomposition by incompressible tori into pieces that are either hyperbolic or Seifert fibered.

Hyperbolic n-Braids

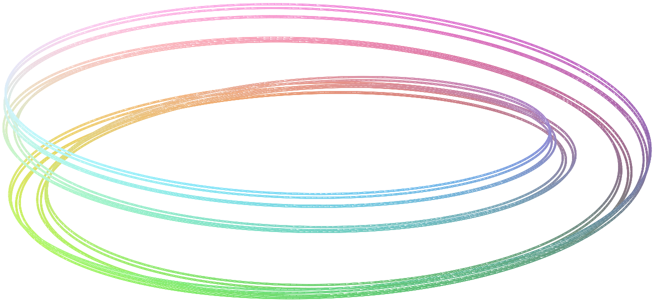
Proposition

Given $n \geq 3$, there are multiple n-braids in a solid torus with distinct hyperbolic structures.

Theorem (Mostow-Prasad Rigidity)

If a 3-manifold admits a complete hyperbolic structure with finite volume, then that structure is unique up to isometry.

The End



THE END.