AN OPERATOR THEORETIC APPROACH
TO DEGENERATED NEVANLINNA-PICK
INTERPOLATION

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1 Introduction

In this paper we consider an interpolation problem of Nevanlinna-Pick type: If finitely many points $z_1, \ldots, z_n$ of the open upper half plane $\mathbb{C}^+$ are given, we study the existence of functions $f \in \mathbb{N}_\pi^\nu$ assuming prescribed values $w_1, \ldots, w_n \in \mathbb{C}$ in these points:

$$f(z_i) = w_i \text{ for } i = 1, \ldots, n.$$ (1)

Furthermore a description of all solutions of (1) through selfadjoint extensions of a certain symmetric operator is given.

We recall the definition of the classes $\mathbb{N}_\pi^\nu$. If $f$ is a complex function denote by $\rho(f)$ the domain of holomorphy of $f$.

Definition 1 Let $\pi$ and $\nu$ be nonnegative integers or $\infty$. Denote by $\mathbb{N}_\pi^\nu$ the set of all functions $f$ which are meromorphic in $\mathbb{C}^+$, such that the kernel

$$\frac{f(z) - f(w)}{z - w} \text{ for } z, w \in \rho(f)$$

has $\pi$ positive and $\nu$ negative squares, i.e., it satisfies

1. For each $k \in \mathbb{N}$ and points $\zeta_1, \ldots, \zeta_k \in \rho(f)$ the hermitian form

$$\sum_{i,j=1}^{n} \frac{f(\zeta_i) - f(\zeta_j)}{\zeta_i - \zeta_j} \zeta_i \zeta_j$$ (2)

has at most $\pi$ ($\nu$, respectively) positive (negative, respectively) squares.

2. If $\pi$ ($\nu$, respectively) is finite, then for some $k \in \mathbb{N}$ and points $\zeta_1, \ldots, \zeta_k \in \rho(f)$ the hermitian form (2) has exactly $\pi$ ($\nu$, respectively) positive (negative, respectively) squares. If $\pi$ ($\nu$, respectively) equals $\infty$, then it is not
possible to give a uniform bound for the number of positive (negative, respectively) squares of the hermitian forms (2).

We assign to the prescribed data the so-called Pick matrix

\[
P = \begin{pmatrix}
    w_{ij} - \overline{w}_i \\
    z_j - \overline{z}_i
\end{pmatrix}_{i,j=1}^n.
\]

Denote by \(\pi_d\) (\(\nu_d\), respectively) the number of positive (negative, respectively) eigenvalues of \(P\), and let \(\delta_d = n - (\pi_d + \nu_d)\) be its defect. Obviously it is impossible to find solutions \(f\) of (1) in some set \(N_\pi^\nu\) with \(\pi < \pi_d\) or \(\nu < \nu_d\). In [12] it is shown that there are also no solutions in sets \(N_{\pi_d}^\nu\) with \(\pi_d < \pi < \pi_d + \delta_d\) or \(\nu_d < \nu < \nu_d + \delta_d\). In case \(\delta_d > 0\) there exists a unique solution in the set \(N_{\pi_d}^\nu\) if and only if the data points do not belong to a certain "small" exceptional set.

In this note we prove that in each set \(N_\pi^\nu\) with \(\pi \geq \pi_d + \delta_d\) and \(\nu \geq \nu_d + \delta_d\) there exist infinitely many solutions, and establish a one-to-one correspondence between solutions of (1) in \(N_\pi^\nu\) and selfadjoint relations in certain Pontrjagin spaces. We always suppose, that at least one of the numbers \(\pi, \nu\) is finite.

These results are well known in the case \(P\) does not degenerate (see e.g. [6], [10]). Therefore we can suppose that \(\delta_d > 0\) holds, i.e. consider only the degenerated case \(|P| = 0\), although the developed method does not require any restrictions on the defect of \(P\).

In §2 we examine a connection between the sets \(N_\pi^\nu\), interpolation problems and inner product spaces. We assign to a complex function and especially to given interpolation data an inner product space and a symmetric operator. §3 proceeds with the definition of induced functions of an operator, and establishes the above mentioned correspondence (Theorem 1). In the remaining part of the paper we examine some more interpolation properties of the induced functions. §4 considers the case of semisimple eigenvalues in data points (Theorem 2), whereas §5 gives sufficient conditions for interpolation in the most general case (Theorem 3).

Besides the method used in this note there are many different approaches to interpolation problems of Nevanlimna-Pick type. Also there are various other generalizations of the classical Nevanlimna-Pick problem or of related questions. For an extensive bibliography on these subjects we refer to [2].

We use the notation and some results of the theory of Pontrjagin spaces and linear relations, which can be found in [1], [3], [4], [5] and [7].
2 Inner product spaces and interpolation

In this section we associate certain inner product spaces and symmetric operators with an interpolation problem. Denote in the following by $\mathbb{C}^0$ the set $\mathbb{C}^+ \cup \mathbb{C}^-$ of all nonreal complex numbers.

Definition 2 Let $f : D(f) \to \mathbb{C}$ with $D(f) \subseteq \mathbb{C}^0$ be a complex function. Here $D(f)$ is an arbitrary subset of $\mathbb{C}^0$. Denote by $H_f$ the linear space of all formal sums

$$H_f = \{ \sum_{z \in D(f)} x_z e_z | x_z \in \mathbb{C}, x_z = 0 \text{ for all but finitely many } z \in D(f) \},$$

equipped with the inner product defined by the relation

$$[e_z, e_w]_{H_f} = \frac{f(z) - f(w)}{z - \overline{w}} \text{ for } z, w \in D(f), z \neq \overline{w}.$$

If for some $z \in D(f)$ also $\overline{z} \in D(f)$, we assume that $f$ is holomorphic at $z$ and satisfies $f(\overline{z}) = \overline{f(z)}$ in a neighborhood of $z$. In this case we define

$$[e_z, e_{\overline{z}}] = f'(z).$$

If no confusion can occur the index $H_f$ at the inner product will be dropped. Note that by definition the elements $e_z$ for $z \in D(f)$ are linearly independent.

If $f$ is an extension of $g$, i.e. $D(g) \subseteq D(f)$ and $f|_{D(g)} = g$, then $H_g$ can be considered in a canonical way as subspace of $H_f$.

The index $\text{ind}(H)$ of an inner product space $H$ is the pair $(\pi, \nu)$, where $\pi$ (respectively) equals the maximal dimension of a positive (negative, respectively) subspace of $H$. We agree that $\pi$ (respectively) is either a nonnegative integer or the symbol $\infty$.

Let $f$ be a function which is meromorphic in $\mathbb{C}^+$. We extend $f$ to a function $\hat{f}$, meromorphic in $\mathbb{C}^0$, by $\hat{f}(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{C}^-$. Denote by $\rho(\hat{f})$ the domain of holomorphy of $\hat{f}$ in $\mathbb{C}^0$, then we will consider $\hat{f}$ as function with domain $D(\hat{f}) = \rho(\hat{f})$. A function $g$ with $D(g) = \overline{D(g)}$ which satisfies $g(\overline{z}) = g(z)$ is called real.

Lemma 1 Let $f$ be a real meromorphic function in $\mathbb{C}^0$. Then $f \in N^\pi_\nu$ if and only if $\text{ind}(H_f) = (\pi, \nu)$.

The proof of Lemma 1 can be found e.g. in [10]. Note that if $\hat{f}$ is considered as function on $\rho(\hat{f}) \cap \mathbb{C}^+$, the above lemma just restates the definition of $N^\pi_\nu$. 
Thus the nontrivial assertion is, that, if \( f \) is extended to \( \mathbb{C}^0 \) (in the canonical way) the number of positive (negative, respectively) squares does not change. In the following we will always extend meromorphic functions on \( \mathbb{C}^+ \) to real meromorphic functions on \( \mathbb{C}^0 \).

Using a well known procedure we can produce a Pontrjagin space from \( H_f \) (see e.g. [8]): Let \( f \in N^\pi_\nu \), denote by \( H_f^0 \) the isotropic part of \( H_f \) and consider the inner product space \( H_f/H_f^0 \). This space is nondegenerated and has index \((\pi, \nu)\). Thus completion gives a Pontrjagin space \( P_f \), also with index \((\pi, \nu)\).

**Remark 1** The above construction works in two steps:

\[
H_f \xrightarrow{p} H_f/H_f^0 \xrightarrow{i} P_f
\]

with the canonical projection \( p \) and embedding \( i \). Both mappings preserve the inner product and the notion of convergence (introduced as in [8]). Note that \( H_f/H_f^0 \) is topologically exactly the \( T_1 \)-space associated to \( H_f \).

Actually in the above construction either \( p \) or \( i \) is the identity mapping. This follows from the fact that \( H_f^0 \neq \{0\} \) if and only if \( f \) is rational, and in this case \( \pi + \nu = \deg(f) < \infty \) (these results can be found e.g. in [12]).

With the given data we associate the function \( d \) defined on \( \{z_1, \ldots, z_n\} \) by \( d(z_i) = w_i \) for \( i = 1, \ldots, n \), and consider the inner product space \( H_d \). For further reference we state another lemma; for the proof we refer to [12].

**Lemma 2** Let \( f \) be a solution of (1) in \( N^\pi_\nu \) with \( H_f^0 \cap H_d \neq \{0\} \). Then \( \pi = \pi_d \) and \( \nu = \nu_d \).

With a function \( f \) we associate not only the inner product space \( H_f \), but also a linear operator in this space.

**Definition 3** Let \( f : D(f) \to \mathbb{C} \) with \( D(f) \subseteq \mathbb{C}^0 \) be a complex function. Denote by \( D_f \) the hyperplane

\[
D_f = \left\{ \sum_{z \in D(f)} x_z e_z \in H_f \mid \sum_{z \in D(f)} x_z = 0 \right\},
\]

and define an operator \( S_f \) with domain \( D_f \) as

\[
S_f \left( \sum_{z \in D(f)} x_z e_z \right) = \sum_{z \in D(f)} z x_z e_z.
\]

**Proposition 1** The operator \( S_f \) is symmetric.
Proof : Let \( x, y \in D_f \), \( x = \sum_{z \in D(f)} x_z e_z \), \( y = \sum_{w \in D(f)} y_w e_w \). Then

\[
[S_f x, y] = \left[ \sum_{z \in D(f)} x_z e_z, \sum_{w \in D(f)} y_w e_w \right] = \sum_{z,w \in D(f)} x_z \overline{y_w} \overline{e_z e_w} = \sum_{z,w \in D(f)} x_z \overline{y_w} \frac{f(z) - f(w)}{z - \overline{w}}.
\]

In the same way we obtain

\[
[x, S_f y] = \sum_{z,w \in D(f)} \overline{w} x_z \overline{y_w} \frac{f(z) - f(w)}{z - \overline{w}}.
\]

Subtraction gives

\[
[S_f x, y] - [x, S_f y] = \sum_{z,w \in D(f)} (x_z \overline{y_w} - \overline{w} x_z \overline{y_w}) \frac{f(z) - f(w)}{z - \overline{w}} = \sum_{z \in D(f)} \left( \sum_{w \in D(f)} x_z \overline{y_w} f(z) \right) - \sum_{z \in D(f)} \left( \sum_{w \in D(f)} x_z \overline{y_w} f(w) \right) = \sum_{z \in D(f)} \left( \sum_{w \in D(f)} \overline{y_w} \right) x_z f(z) - \sum_{w \in D(f)} \left( \sum_{z \in D(f)} x_z \right) \overline{y_w} f(w) = 0.
\]

We can restate the interpolation problem (1) as follows: which functions of \( N^* \) extend \( d \)? We will show that in some sense these extensions correspond to extensions of the operator \( S_d \).

3 Interpolation properties of induced functions

Definition 4 Let \( P \) be a Pontrjagin space with \( H_d \subseteq P \). Further let \( S \) be a selfadjoint relation in \( P \) with nonempty resolvent set extending \( S_d \). Denote by \( f_S \) the complex function

\[
f_S(z) = \frac{\overline{w_1}}{w_1} + \frac{\overline{z_1}}{z_1} (z - \overline{z_1}) + (z - \overline{w_1})(z - z_1) \left[(S - z)^{-1} e_{z_1}, e_{w_1}\right].
\]

We refer to \( f_S \) as the induced function of \( S \) and consider \( f_S \) as function on its domain of holomorphy \( \rho(f_S) \) in \( \mathbb{C}^0 \).
By an elementary computation we obtain the representation

\[ f_S(z) = \Re w_1 + (z - \Re z_1) \frac{\Im w_1}{\Im z_1} + (z - z_1)(\Re S - z_1) \left[ (S - z)^{-1} e_{z_1}, e_{z_1} \right] \]

for \( f_S \). From this it follows immediately that \( f_S \) is a real function.

Note that the resolvent set \( \rho(S) \cap \mathbb{C}^0 \) is a dense subset of \( \rho(f_S) \). Furthermore we have the following lemma.

**Lemma 3** Consider the inner product space \( H_{f_S} \) and the subset

\[ D = \left\{ \sum_{z \in \rho(f_S)} x_z e_z \in H_{f_S} \mid x_z = 0 \text{ for } z \notin \rho(S) \right\}. \]

The linear manifold \( D \) is dense in \( H_{f_S} \).

**Proof**: Let \( z \in \rho(f_S) \setminus \rho(S) \). Choose a sequence \( (z_i)_{i \in \mathbb{N}} \), such that \( z_i \in \rho(S) \) and \( z_i \to z \). Then \( e_{z_i} \in D \),

\[ [e_{z_i}, e_w] = \frac{f(z_i) - f(w)}{z_i - w} \to \frac{f(z) - f(w)}{z - w} = [e_z, e_w] \text{ for } w \neq \overline{z}, \]

\[ [e_{z_i}, e_w] = \frac{f(z_i) - f(w)}{z_i - w} \to f'(z) = [e_z, e_w] \text{ for } w = \overline{z} \text{ and} \]

\[ [e_{z_i}, e_{z_i}] = \frac{f(z_i) - f(z_i)}{z_i - \overline{z_i}} \to \frac{f(z) - f(z)}{z - \overline{z}} = [e_z, e_z]. \]

Thus \( e_{z_i} \to e_z \) which shows that \( e_z \in \overline{D} \).

We define an operator \( U \) as

\[ U e_z = e_{z_1} + (z - z_1)(S - z)^{-1} e_{z_1} \text{ for } z \in \rho(S) \cap \mathbb{C}^0. \]

**Proposition 2** The operator \( U \) can be extended to an isometry from \( P_{f_S} \) into \( P \).

**Proof**: We are going to check the relation

\[ \frac{f_S(z) - f_S(w)}{z - \overline{w}} = [U e_z, U e_w]_P. \quad (4) \]

Once (4) is established the result follows, as \( U \) in first place is an isometric operator from \( D \) to \( P \). As \( D \) is dense in \( H_{f_S} \), we find \( U(\mathbb{H}_{f_S}^\circ) \subseteq \mathbb{H}_{f_S}^\circ \). Thus \( U \)
yields an isometry from $P_{fS}$ into $P$ with domain $D/H_{fS}$. Again as $D$ is dense in $H_{fS}$, $D/H_{fS}$ is dense in $H_{fS}/H_{fS}$ and thus also in $P_{fS}$. Now we can extend $U$ by continuity to the whole of $P_{fS}$ (see e.g. [11]).

It remains to verify (4). Note that we have 

$$[Ue_z, Ue_w] = \left[ \left( I + (z - z_1)(S - z)^{-1} \right) e_{z_1}, \left( I + (w - z_1)(S - w)^{-1} \right) e_{z_1} \right] =$$

$$= \left[ \left( I + (\overline{w} - z_1)(S - \overline{w})^{-1} \right) \left( I + (z - z_1)(S - z)^{-1} \right) e_{z_1}, e_{z_1} \right].$$

The following formulas can be checked by a straightforward computation:

$$\frac{f_{S}(z) - f_{S}(w)}{z - \overline{w}} = e_{z_1} + (z + \overline{w} - 2\Re z_1)(S - z)^{-1} e_{z_1} +$$

$$+(\overline{w} - \overline{z_1})(\overline{w} - z_1)(S - z)^{-1}(S - \overline{w})^{-1} e_{z_1}, e_{z_1}.$$ 

From this (4) follows.

In the following definition we recall the notion of generating elements.

**Definition 5** Let $A$ be a relation in a Pontrjagin space $P$, and let $x \in P$. We call $x$ a generating element for $A$ if the relation 

$$P = \langle x, (A - z)^{-1} x | z \in \rho(A) \rangle$$

holds ($\langle \ldots \rangle$ denotes the linear span of a set of vectors).

**Proposition 3** Let $S$ be a selfadjoint relation in $P$ extending $S_d$. Suppose that $z_1 \in \rho(S)$ and that $e_{z_1}$ is a generating element for $S$. Then $f_{S} \in N^{\pi}_{\nu}$, where $(\pi, \nu)$ is the index of $P$.

**Proof:** As $U : P_{fS} \to P$ is an isometry, the index $(\pi', \nu')$ of $P_{fS}$ is bounded by the index of $P$, i.e. $f_{S} \in N^{\pi'}_{\nu'}$ with $\pi' \leq \pi$ and $\nu' \leq \nu$.

From our assumptions we have 

$$e_{z_1} = Ue_{z_1} \in \mathcal{R}(U),$$

and for $z \in \rho(S) \cap \mathbb{C}^0$:

$$(S - z)^{-1} e_{z_1} = \frac{1}{z - z_1}(Ue_z - e_{z_1}) \in \mathcal{R}(U).$$
Thus
\[ P = \langle e_{z_1}, (S - z)^{-1} e_{z_1} | z \in \rho(S) \cap \mathbb{C}^0 \rangle \subseteq \mathcal{R}(U) \subseteq P \]
holds, which shows that \( \mathcal{R}(U) \) is dense in \( P \). Suppose e.g. \( \pi \) is finite. Then we find a \( \pi \)-dimensional positive subspace in \( \mathcal{R}(U) \), and therefore \( \pi' \geq \pi \). If \( \nu \) is finite a similar argument shows \( \nu' \geq \nu \). If, on the other hand, \( \nu \) is infinite, then \( \nu' \) must also be infinite, as no finite dimensional space can be dense in some infinite dimensional space.

\[ \square \]

**Proposition 4** Let \( S \) be a selfadjoint relation in \( P \) extending \( S_d \), and let \( i \in \{1, \ldots, n\} \). If \( z_i \in \rho(S) \), we have \( f_S(z_i) = w_i \).

**Proof**: First consider the case \( i = 1 \). Then
\[ f_S(z_1) = \overline{w_1} + \frac{\Im w_1}{\Im z_1} (z_1 - \overline{z_1}) = w_1. \]

Let \( i \neq 1 \), then \( e_{z_i} - e_{z_1} \in \mathcal{D}(S_d) \) and
\[ (S_d - z_i)(e_{z_i} - e_{z_1}) = (z_i e_{z_i} - z_1 e_{z_1}) - (z_i e_{z_i} - z_i e_{z_1}) = (z_i - z_1) e_{z_1}. \]

Thus
\[ (e_{z_i} - e_{z_1}, (z_i - z_1)e_{z_1}) \in S_d - z_i \subseteq S - z_i, \text{ and} \]
\[ ((z_i - z_1)e_{z_1}, e_{z_i} - e_{z_1}) \in (S - z_i)^{-1}. \]

We compute
\[ f_S(z_i) = \overline{w_i} + \frac{\Im w_1}{\Im z_1} (z_i - \overline{z_1}) + (z_i - \overline{z_1}) [e_{z_i} - e_{z_1}, e_{z_1}] = \]
\[ = \overline{w_i} + \frac{\Im w_1}{\Im z_1} (z_i - \overline{z_1}) + (z_i - \overline{z_1}) \left( \frac{w_i - \overline{w_1}}{z_i - \overline{z_1}} - \frac{\Im w_1}{\Im z_1} \right) = w_i. \]

\[ \square \]

In the following we consider the eigenvalues of the selfadjoint extensions of \( S_d \). Denote by \( \mathcal{K}(A) \) the kernel of the operator \( A \).
Lemma 4 Let $S$ be a selfadjoint relation in $P$ extending $S_d$, and let $\lambda \in \sigma_p(S), \lambda \neq 0$. Then for $x \in K(S - \lambda)$ and $i, j = 1, \ldots, n$ the relation

$$(\lambda - \overline{\alpha_j})[x, e_{z_i}] = (\lambda - \overline{\alpha_j})[x, e_{z_j}]$$  \hspace{1cm} (5)$$

holds. If in particular $\lambda = \overline{\alpha_{i_0}}$ for some $i_0 \in \{1, \ldots, n\}$, then

$$[x, e_{z_j}] = 0 \text{ for } j \neq i_0.$$

**Proof**: As $x \in K(S - \lambda)$ we have $(x, \lambda x) \in S$. Consider the element $e_{z_j} - e_{z_i}$ ($j \neq i$). Then

$$(e_{z_j} - e_{z_i}, z_je_{z_j} - z_ie_{z_i}) \in S_d \subseteq S.$$

As $S$ is selfadjoint we find

$$\lambda[x, e_{z_j}] - \lambda[x, e_{z_i}] = [\lambda x, e_{z_j} - e_{z_i}] =$$

$$= [x, z_je_{z_j} - z_ie_{z_i}] = \overline{\alpha_j}[x, e_{z_j}] - \overline{\alpha_i}[x, e_{z_i}],$$

which implies (5). The remaining assertion is obvious.

Lemma 5 Let $S$ be a selfadjoint relation in $P$ with nonempty resolvent set extending $S_d$, and suppose that $e_{z_1}$ is a generating element for $S$. Then $z_j \in \rho(S)$ for $j \neq 1$.

**Proof**: It suffices to show that $K(S - \overline{\alpha_j}) = \{0\}$ for $j = 2, \ldots, n$. If $x \in K(S - \overline{\alpha_j})$ and $z \in \rho(S)$, then $(S - \overline{\alpha_j})^{-1} x \in K(S - \overline{\alpha_j})$. The preceding lemma yields

$$[(S - \overline{\alpha_j})^{-1} x, e_{z_1}] = 0.$$

As $x$ is also orthogonal to $e_{z_1}$ and $e_{z_1}$ is a generating element for $S$, we find $x \perp P$, which implies $x = 0$.

Denote with $S_d(\pi, \nu)$ the class of all selfadjoint relations which operate in a Pontryagin space with index $(\pi, \nu)$, have $e_{z_1}$ as generating element, extend $S_d$ and satisfy $z_1 \in \rho(S)$. Combining the above results we obtain the following statement.

**Proposition 5** The assignment $S \rightarrow f_S$ defines a mapping $Q$ from $S_d(\pi, \nu)$ into the set of all solutions of (1) in $N^\pi_\nu$. 
Remark 2 Let \((\pi_d, \nu_d)\) be the signature of the Pick matrix \(P\) and let \(\delta_d\) be its defect. Then \(\text{ind} (H_d) = (\pi_d, \nu_d)\) and \(\dim (H_d^\circ) = \delta_d\). As \(P\) is a Pontrjagin space we conclude \(\pi \geq \pi_d + \delta_d\) and \(\nu \geq \nu_d + \delta_d\).

To show that every solution can be obtained in this way we proceed from a solution \(f\) of (1) in \(N_{\pi, \nu}\) with \(\pi \geq \pi_d + \delta_d\) and \(\nu \geq \nu_d + \delta_d\), and construct a relation \(S \in S_d(\pi, \nu)\) with \(f = f_S\). To do so consider the operator \(S f\), which extends \(S_d\), as \(f\) extends \(d\). As \(S f\) is symmetric, it yields a symmetric relation in the Pontrjagin space \(P_f\). Denote by \(S\) its closure.

Lemma 2 shows that \(H_d\) is embedded in \(P_f\) and therefore \(S\) is an extension of \(S_d\).

Lemma 6 The relation \(S\) is selfadjoint. Furthermore we have \(\rho(f) \subseteq \rho(S)\).

Proof : We consider the range of \((S_f - z)\) for \(z \in \rho(f)\). Since
\[
(S_f - z) \left( \frac{e_w - e_z}{w - z} \right) = e_w \text{ for } w \neq z
\]
we find that \(\mathcal{R}(S_f - z)\) is dense in \(H_f\), and therefore that \(\mathcal{R}(S - z)\) is dense in \(P_f\). Since \(S\) is a symmetric closed relation, the range of \(S - z\) is closed for all \(z \in \mathbb{C}^0\). Thus \(\mathcal{R}(S - z) = P_f\). From this we find
\[
K(S - z) \subseteq \mathcal{R}(S - z)^\perp = \{0\} \text{ for } z \in \mathbb{C}^0,
\]
Thus \(S\) is selfadjoint and \(\rho(S) \cap \mathbb{C}^0 = \mathbb{C}^0 \setminus \sigma_p(S)\). As \(K(S - z) = \{0\}\) for \(z \in \rho(f)\) we find \(\rho(f) \subseteq \rho(S)\).

\[\square\]

Proposition 6 Let \(f \in N_{\pi, \nu}^\circ\) be a solution of the interpolation problem (1). Then we have \(f = f_S\) for the selfadjoint relation \(S\) constructed above. Furthermore \(S \in S_d(\pi, \nu)\).

Proof : We first show that \(S \in S_d(\pi, \nu)\). As \(z_1 \in \rho(f)\) we find \(z_1 \in \rho(S)\). Let \(z \neq z_1\), then
\[
(S - z)^{-1} e_{z_1} = \frac{e_{z_1} - e_z}{z_1 - z},
\]
which implies
\[
ez \in \left\langle e_{z_1}, (S - z)^{-1} e_{z_1} \mid z \in \rho(S) \right\rangle.
\]
Thus
\[ P_f = \langle e_{z_1}, (S - z)^{-1} e_{z_1} | z \in \rho(S) \rangle, \]
which shows that \( e_{z_1} \) is an generating element for \( S \).

Let \( z \in \rho(f), z \neq z_1 \), then
\[
f_S(z) = \frac{\Im w_1}{\Im z_1} (z - \overline{z_1}) + (z - \overline{z_1}) \left( f(z) - f(z_1) \right) = f(z) = \frac{\Im w_1}{\Im z_1} (z - \overline{z_1}) + (z - \overline{z_1}) \left( \frac{\Im w_1}{\Im z_1} \right) = f(z).
\]

\[\square\]

**Remark 3** As \( \rho(f) \) is the domain of holomorphy of \( f \), \( f \) cannot be extended. Thus \( f_S(z) = f(z) \) for \( z \in \rho(f) \) implies \( \rho(S) \cap \mathbb{C}^0 \subseteq \rho(S) \). Together with Lemma 6 this shows \( \rho(S) \cap \mathbb{C}^0 = \rho(f) = \rho(f_S) \). Furthermore \( |\mathbb{C}^+ \setminus \rho(f)| \leq \min(\pi, \nu) \).

Due to Proposition 6 the mapping \( Q \) is surjective. We proceed determining the kernel of \( Q \).

**Definition 6** Let \( P_1 \) and \( P_2 \) be Pontryagin spaces extending \( H_d \), and let \( S_1 \) and \( S_2 \) be selfadjoint relations with nonempty resolvent set in \( P_1 \) and \( P_2 \) respectively. We call \( S_1 \) and \( S_2 \) unitary equivalent with respect to \( H_d \), if there exists a unitary operator \( U : P_1 \rightarrow P_2 \) which leaves \( H_d \) pointwise invariant and satisfies
\[
U \circ (S_1 - z)^{-1} = (S_2 - z)^{-1} \circ U \text{ for } z \in \rho(S_1) \cap \rho(S_2). \tag{6}
\]

**Lemma 7** Let \( S_1 \) and \( S_2 \) be relations in \( P_1 \) and \( P_2 \) respectively, which are unitary equivalent with respect to \( H_d \). Then the functions \( f_{S_1} \) and \( f_{S_2} \) coincide.

**Proof** : Let \( z \in \rho(S_1) \cap \rho(S_2) \), then we have (6), which shows
\[
\left[ (S_1 - z)^{-1} e_{z_1}, e_{z_1} \right] = [U^{-1} \circ (S_2 - z)^{-1} \circ U e_{z_1}, e_{z_1}] = \left[ (S_2 - z)^{-1} U e_{z_1}, U e_{z_1} \right] = \left[ (S_2 - z)^{-1} e_{z_1}, e_{z_1} \right],
\]
and therefore \( f_{S_1}(z) = f_{S_2}(z) \). Since \( \rho(S_1) \cap \rho(S_2) \) is dense in \( \rho(f_{S_1}) \) and \( \rho(f_{S_2}) \) we find \( f_{S_1} = f_{S_2} \).

\[\square\]

A certain converse to the above lemma is given by
Lemma 8 Let $S$ be a selfadjoint relation of class $S_d(\pi, \nu)$. Denote by $S_{fs}$ the selfadjoint relation in the Pontrjagin space $P_{fs}$, which has been constructed for the proof of Proposition 6, starting from the function $f_S$. Then $S$ and $S_{fs}$ are unitary equivalent with respect to $H_d$.

Proof : Consider the isometric operator $U$ defined in Proposition 2. We have shown in Proposition 3 that under the assumptions of the lemma $R(U)$ is dense in $P$. Thus $U$ is actually unitary. It remains to prove (6).

Consider the vector $e_z \in P_{fs}$ for some $z \in \rho(S) \cap \rho(S_{fs})$, then

$$Ue_z = e_{z_1} + (z - z_1)(S - z)^{-1}e_{z_1},$$

where the right hand side is an element of $P$. We compute for $w \in \rho(S) \cap \rho(S_{fs})$:

$$(S - w)^{-1} \circ Ue_z = (S - w)^{-1}e_{z_1} + (z - z_1)(S - w)^{-1}(S - z)^{-1}e_{z_1} =$$

$$= (S - w)^{-1}e_{z_1} + \frac{z - z_1}{z - w}[(S - z)^{-1}e_{z_1} - (S - w)^{-1}e_{z_1}] =$$

$$= \frac{z - z_1}{z - w}(S - z)^{-1}e_{z_1} - \frac{w - z_1}{z - w}(S - w)^{-1}e_{z_1}.$$

On the other hand we have

$$\left(\frac{e_z - e_w}{z - w}, e_z\right) \in S_{fs} - w,$$

which implies

$$(S_{fs} - w)^{-1}e_z = \frac{e_z - e_w}{z - w}.$$

Thus

$$U \circ (S_{fs} - w)^{-1}e_z = U\left(\frac{e_z - e_w}{z - w}\right) =$$

$$= \frac{1}{z - w}[(z - z_1)(S - z)^{-1}e_{z_1} - (w - z_1)(S - w)^{-1}e_{z_1}]$$

and (6) holds.

Let $\pi_d + \delta_d \leq \pi$ and $\nu_d + \delta_d \leq \nu$. Then we obtain by combining Proposition 5 and Proposition 6 with Lemma 7 and Lemma 8 the following result.

Theorem 1 The mapping $Q$ which assigns to each selfadjoint relation $S$ its induced function $f_S$ establishes a bijective correspondence between the set of all solutions of (1) in $N^\pi_\nu$ and the class $S_d(\pi, \nu)$ modulo unitary equivalence with respect to $H_d$. 

\[\square\]
Remark 4 Since unitary equivalent relations have the same spectrum and
\( \rho(S_f) \cap \mathbb{C}^0 = \rho(f_S) \), we have
\[
\rho(S) \cap \mathbb{C}^0 = \rho(f_S) \text{ for each relation } S \in \mathcal{S}_d(\pi, \nu).
\]

4 Semisimple eigenvalues in data points

We consider in this paragraph the case that for a selfadjoint extension \( S \) of \( \mathcal{S}_d \) (in an Pontrjagin space \( \mathcal{P} \)) with nonempty resolvent set the point \( z_j \) for some \( j \in \{1, \ldots, n\} \) is a semisimple eigenvalue, i.e., that
\[
\mathcal{K}(S - z_j)^2 = \mathcal{K}(S - z_j) \neq \{0\}.
\]
Denote the kernel \( \mathcal{K}(S - z_j) \) with \( \mathcal{K}^+ \) and \( \mathcal{K}(S - \overline{z_j}) \) with \( K^- \). The subspaces \( K^+ \) and \( K^- \) are neutral, skewly linked and finite dimensional. Thus \( \mathcal{P} \) decomposes as
\[
P = K^+ \oplus K^-
\]
where \( K^\perp \) denotes the orthogonal complement of \( K = K^+ \oplus K^- \). According to this decomposition we write vectors \( x \in \mathcal{P} \) as \( x = x^\perp + x^+ + x^- \).

The selfadjoint relation \( S \) also decomposes with respect to (7).

Proposition 7 Denote with \( S' \) the relation \( S' = S \cap (K^\perp)^2 = S|_{K^\perp} \) of \( K^\perp \). Then we have
\[
S = S' (\dot{+}) z_j I_{K^+} (\dot{+}) \overline{z_j} I_{K^-}.
\]
The relation \( S' \) is selfadjoint in the Pontrjagin space \( K^\perp \).

Proof : Obviously \( S' \), \( z_j I_{K^+} \), \( \overline{z_j} I_{K^-} \) \( \subseteq S \). Thus the inclusion "\( \supseteq \)" of (8) holds. To establish the converse inclusion let \( (x, y) \in S \), \( x = x^\perp + x^+ + x^- \). Then \( (x^+, z_j x^+) \) and \( (x^-, \overline{z_j} x^-) \) are elements of \( S \). Consider \( u = y - z_j x^+ - \overline{z_j} x^- \), then \( (x^\perp, u) \in S \). As \( S \) is selfadjoint we have for \( v \in \mathcal{K}(S - z_j) \)
\[
[u, v] = [x^\perp, z_j v] = 0,
\]
and for \( v \in \mathcal{K}(S - \overline{z_j}) \)
\[
[u, v] = [x^\perp, \overline{z_j} v] = 0.
\]
Thus \( u \in K^\perp \), and

\[
(x, y) = (x^\perp, u) + (x^+, z_j x^+) + (x^-, \overline{z}_j x^-).
\]

As the domains (ranges, respectively) of the relations on the right hand side of (8) are disjoint subspaces the sum in (8) is direct.

Obviously the restriction of \( \mathbf{S} \) to \( K^\perp \) is closed, and the inclusion \( \mathbf{S}' \subseteq \mathbf{S}'^* \) holds, as \( \mathbf{S} \) is selfadjoint. To prove the converse inclusion let \( (u, v) \in \mathbf{S}'^* \). Then we have \([u, y] = [v, x]\) for \((x, y) \in \mathbf{S}'\) by definition, and for \((x, z_j x) \in I|_{K^\perp}\) we find \([u, z_j x] = 0 = [v, x]\), as \(u, v \in K^\perp\). Similar \([u, z_j x] = 0 = [v, x]\) holds for \((x, z_j x) \in I|_{K^-}\). As (8) holds, we conclude \((u, v) \in \mathbf{S}' = \mathbf{S} \). Furthermore \(u, v \in K^\perp\) implies \((u, v) \in \mathbf{S}'\).

\[ \]

To prove the main result of this paragraph we have to do some computations, which are given in the following lemmata.

**Lemma 9** Using the above notation we have for \( i \in \{1, \ldots, n\} \):

\[
e^-_{z_i} = \frac{z_j - \overline{z}_j}{z_i - \overline{z}_j} e^-_{z_j},
\]

and for \( i \neq j \)

\[
e^+_{z_i} = 0.
\]

**Proof**: For \( i = j \) the assertion is trivial, so assume \( i \neq j \). Then

\[
(e_{z_i} - e_{z_j}, (z_i - \overline{z}_j)e_{z_i} - (z_j - \overline{z}_j)e_{z_j}) \in \mathbf{S}_d - \overline{z}_j \subseteq \mathbf{S} - \overline{z}_j.
\]

Comparing components with respect to (7) gives

\[
(e_{z_i}^\perp - e_{z_j}^\perp, (z_i - \overline{z}_j)e_{z_i}^\perp - (z_j - \overline{z}_j)e_{z_j}^\perp) \in \mathbf{S}' - \overline{z}_j,
\]

\[
(z_j - \overline{z}_j)e_{z_i}^+ - (z_j - \overline{z}_j)e_{z_j}^+ = (z_i - \overline{z}_j)e_{z_i}^+ - (z_j - \overline{z}_j)e_{z_j}^+,
\]

\[
0 = (z_i - \overline{z}_j)e_{z_i}^- - (z_j - \overline{z}_j)e_{z_j}^-.
\]

From (9) and (10) the assertion of the lemma follows.

**Lemma 10** Let \( z \in \rho(\mathbf{S}) \). Then

\[
[(\mathbf{S} - z)^{-1} e_{z_1}, e_{z_2}] = [(\mathbf{S}' - z)^{-1} e_{z_1}^\perp, e_{z_2}^\perp] + \frac{1}{z_j - z}[e^+_{z_1}, e^-_{z_1}] + \frac{1}{\overline{z}_j - z}[e^-_{z_1}, e^+_{z_1}].
\]

(11)
Proof: As $K^+$ and $K^-$ are invariant under $(S - z)^{-1}$, so is $K^\perp$. Thus $(S' - z)^{-1}$ is an operator defined on the whole of $K^\perp$. Furthermore

$$(S - z)^{-1} e_{z_1} = (S' - z)^{-1} e_{z_1}^\perp.$$  

We compute

$$[(S - z)^{-1} (e_{z_1} + e_{z_1}^+), (e_{z_1} + e_{z_1}^+ + e_{z_1}^-)] =$$

$$= [(S - z)^{-1} e_{z_1} + e_{z_1}^+] + [(S - z)^{-1} e_{z_1}^+ + e_{z_1}^-] + [(S - z)^{-1} e_{z_1}^+, e_{z_1}^-].$$

As $(e_{z_1}^+, (z_j - z)e_{z_1}^+) \in S - z$ and $(e_{z_1}^-, (z_j - z)e_{z_1}^-) \in S - z$, we find

$$(S - z)^{-1} e_{z_1}^+ = \frac{1}{z_j - z} e_{z_1}^+ \text{ and } (S - z)^{-1} e_{z_1}^- = \frac{1}{z_j - z} e_{z_1}^-,$$

which proves (11).

\qed

Theorem 2 Let $j \in \{1, \ldots, n\}$ and let $S$ be a selfadjoint relation extending $S_d$ and satisfying

$$\mathcal{K}(S - z_j)^2 = \mathcal{K}(S - z_j) \neq \{0\}.$$  

Then $f_S(z_j) = w_j$ if and only if $[e_{z_j}^+, e_{z_j}^-]$ vanishes.

Proof: Consider first the case $j \neq 1$. Then

$$(e_{z_j} - e_{z_1}, (z_j - z_1)e_{z_1}) \in S - z_j,$$

which implies

$$\left(\frac{e_{z_j}^\perp - e_{z_1}^\perp}{z_j - z_1}, e_{z_1}^\perp\right) \in S' - z_j.$$  

As $\mathcal{K}(S' - z_j) = \{0\}$ we have $z_j \in \rho(S')$. Thus

$$(S' - z)^{-1} e_{z_1}^\perp = \frac{e_{z_j} - e_{z_1}}{z_j - z_1},$$

and, with regard to $e_{z_1}^+ = 0$ and (11):

$$\lim_{z \to z_j} (z - z_1) [(S - z)^{-1} e_{z_1}, e_{z_1}] = \lim_{z \to z_j} (z - z_1) [(S' - z)^{-1} e_{z_1}^\perp, e_{z_1}^\perp] =$$

$$= (z_j - z_1) [(S' - z_j)^{-1} e_{z_1}^\perp, e_{z_1}^\perp] = [e_{z_j}^\perp - e_{z_1}^\perp, e_{z_1}^\perp] =$$
\[ = [e_{z_j} - e_{z_1}, e_{z_1}] - [e_{z_j}^+, e_{z_1}^-]. \]  
(12)

In the proof of Proposition 4 we have seen that \( f_S(z_j) = w_j \) is equivalent to

\[ \lim_{z \to z_j} (z - z_1)[(S - z)^{-1} e_{z_1}, e_{z_1}] = [e_{z_j} - e_{z_1}, e_{z_1}]. \]

Due to (12) this is the case if and only if \([e_{z_j}^+, e_{z_1}^-] = 0\), which is equivalent to

\([e_{z_1}^+, e_{z_1}^-] = 0\), as \(e_{z_1}^-\) and \(e_{z_1}^-\) are linearly dependent.

In the case \(j = 1\) the interpolation condition \( f_S(z_1) = w_1 \) is satisfied if and only if the expression \([S - z_1)^{-1} e_{z_1}, e_{z_1}]\) is bounded at \(z_1\). As \(z_1 \in \rho(S')\) in this case this is, in view of (11), equivalent to \([e_{z_1}^+, e_{z_1}^-] = 0\).

\[ \square \]

5 Interpolation through arbitrary selfadjoint extensions

Let \( S \) be a selfadjoint extension of \( S_d \) in a Pontrjagin space with index \((\pi, \nu)\). Then the first part of the proof of Proposition 3 implies that \( f_S \in N_{\pi'}^{\nu'} \) for some \( \pi' \leq \pi \) and \( \nu' \leq \nu \). From Proposition 4 it follows that, if a data point \( z_j \) is in the resolvent set of \( S \), the function \( f_S \) satisfies \( f_S(z_j) = w_j \).

In this section we consider an arbitrary selfadjoint extension \( S \) of \( S_d \) and give a weaker condition, which is also sufficient for \( f_S(z_j) = w_j \).

We start with two lemmata about restrictions and factorizations of relations.

Lemma 11 Let \( A \) be a relation in a Pontrjagin space \( P \), let \( L \) be a (closed) subspace of \( P \) and consider the relation \( A' = A|_L \). If \( z \in \rho(A) \) and \( L \) is invariant under \((A - z)^{-1}\), then \( z \in \rho(A') \) and

\[ (A - z)^{-1}|_L = (A' - z)^{-1}. \]  
(13)

Proof: Obviously \( K(A' - z) = \{0\} \). Let \( x \in L \) and consider \( y = (A - z)^{-1} x \), then \( y \in L \). Thus \((y, x) \in (A - z) \cap L^2 = (A' - z) \) which implies \((A' - z)^{-1} x = y \). This proves (13) and shows that \((A' - z)^{-1}\) is defined on the whole of \( L \). As \((A - z)^{-1}\) is bounded, so is \((A' - z)^{-1}\) and thus \( z \in \rho(A') \). 

\[ \square \]

Note that the subspace \( L \) may be degenerated. We define a relation \( A'' \) in \( L/L^0 \) as

\[ (x + L^0, y + L^0) \in A'' \iff (x, y) \in A'. \]
Lemma 12 In the situation of Lemma 11 suppose that $L^o$ is invariant under $(A' - z)^{-1}$. Then $z \in \rho(A'')$ and
\[(A'' - z)^{-1} (x + L^o) = (A' - z)^{-1} x + L^o.\]

Proof: Let $x + L^o \in \mathcal{K}(A'' - z)$, i.e. $(x + L^o, L^o) \in (A'' - z)$. Then there exist $x', y' \in L$ such that $x' - x \in L^o, y' \in L^o$ and $(x', y') \in (A' - z)$. Thus $x' = (A' - z)^{-1} y'$ which implies $x' \in L^o$ and consequently $x + L^o = L^o$, hence $\mathcal{K}(A'' - z) = \{0\}$.

Let $x \in L$ and consider $y \in (A' - z)^{-1} x$. Then $(y, x) \in (A' - z)$ and therefore $(y + L^o, x + L^o) \in (A'' - z)$. This implies $y + L^o = (A'' - z)^{-1} (x + L^o)$ and shows that $(A'' - z)^{-1}$ is everywhere defined.

It remains to show that $(A'' - z)^{-1}$ is continuous. Consider the decomposition
\[P = L_1[\mathcal{L}^o + \mathcal{L}^2][\mathcal{L}^2]L^o\]
with $L = L_1[\mathcal{L}^o]L^o$. Let $\hat{x}_n \in L/L^o$ be a sequence which converges to $\hat{x}$. We may assume that the representors $x_n$ and $x$ of $\hat{x}_n$ and $\hat{x}$ are elements of $L_1$. As $L_1$ is orthocomplemented the sequence $x_n$ converges to $x$ in the topology of $P$. Thus $(A - z)^{-1} x_n \to (A - z)^{-1} x$, which shows that also $(A'' - z)^{-1} \hat{x}_n \to (A'' - z)^{-1} \hat{x}$.

\begin{flushright} \Box \end{flushright}

We now apply the above lemmata to the following situation. Let $S$ be a selfadjoint extension of $S_d$ in a Pontrjagin space $P$ with nonempty resolvent set. Choose the subspace $L$ as
\[L = \langle e_{z_i}, (S - w)^{-1} e_{z_i} | w \in \rho(S), i = 1, \ldots, n \rangle. \tag{14}\]

Lemma 13 The subspaces $L$ and $L^o$ are invariant under the resolvents $(S - z)^{-1}, z \in \rho(S)$.

Proof: Let $i \in \{1, \ldots, n\}$. Then for $z, w \in \rho(S), z \neq w$ we have $(S - z)^{-1} e_{z_i} \in L$ and
\[(S - z)^{-1} (S - w)^{-1} e_{z_i} = (z - w) \left( (S - z)^{-1} e_{z_i} - (S - w)^{-1} e_{z_i} \right) \in L.\]
As $(S - z)^{-1}$ depends continuously on $z$ also
\[(S - w)^{-1} (S - w)^{-1} e_{z_i} \in L.\]
As $S$ is selfadjoint this also implies that $L^o$ is invariant under each resolvent.

\begin{flushright} \Box \end{flushright}

In the sequel we use the notation developed above. The next lemma provides the main tool for proving the below theorem.
Lemma 14 Let \( e_{z_j} \in \mathcal{R}(S' - z_j) \). Then \( \mathcal{R}(S' - z_j) = L \).

**Proof:** As \( S' \) is closed it suffices to show that \( \mathcal{R}(S' - z_j) \) is dense in \( L \). As \( \mathcal{R}(S' - z_j) \) is invariant under each resolvent \( (S' - z)^{-1} \) it suffices to show \( e_{z_i} \in \mathcal{R}(S' - z_j) \) for \( i = 1, \ldots, n \). Due to the assumption of the lemma we may assume \( i \neq j \). Then \( e_{z_i} - e_{z_j} \in L \) and

\[
(e_{z_i} - e_{z_j}, (z_i - z_j)e_{z_i}) \in (S' - z_j).
\]

\[ \square \]

**Theorem 3** Let \( S \) be a selfadjoint relation with nonempty resolvent set extending \( S_d \) and operating in a Pontrjagin space \( P \) with index \((\pi, \nu)\). Suppose

\[
e_{z_j} \in \mathcal{R}(S - z_j)|_L \text{ for } j = 1, \ldots, n,\]

where the subspace \( L \) is defined as in (14). Then the induced function \( f_S \) is a solution of the interpolation problem (1) in some set \( N^\pi_{\nu'} \) with \( \pi' \leq \pi \) and \( \nu' \leq \nu \).

**Proof:** The relation \( S'' \) is symmetric and closed. Furthermore \( \rho(S) \subseteq \rho(S'') \) which implies \( \rho(S'') \cap \mathbb{C}^\perp \neq \emptyset \). Thus \( \dim (\mathcal{K}(S'' - z_j)) - \dim (\mathcal{R}(S'' - z_j)^\perp) = 0 \) on \( \mathbb{C}^0 \). As \( \mathcal{R}(S'' - z_j) = L/L^0 \) we must also have \( \mathcal{K}(S'' - z_j) = \{0\} \) and therefore \( z_j \in \rho(S'') \).

Let \( j \neq 1 \), then we have

\[
(e_{z_j} - e_{z_1} + L^0, (z_j - z_1)e_{z_1} + L^0) \in (S'' - z_j),
\]

and thus

\[
\lim_{z \to z_j} (z - z_1)[(S' - z)^{-1}e_{z_1}, e_{z_1}] = \lim_{z \to z_j} (z - z_1)[(S'' - z)^{-1}e_{z_1} + L^0, e_{z_1} + L^0] = (z_j - z_1)[(S'' - z_j)^{-1}e_{z_1} + L^0, e_{z_1} + L^0]
\]

\[
= (z_j - z_1)[e_{z_j} - e_{z_1} + L^0, e_{z_1} + L^0],
\]

which implies \( \lim_{z \to z_j} f_S(z) = w_j \).

Consider now the case \( j = 1 \). Then \( z_1 \in \rho(S'') \) implies that \((S'' - z)^{-1}e_{z_1} + L^0, e_{z_1} + L^0] \) is bounded at \( z_1 \). Thus \( \lim_{z \to z_1} (z - z_1)[(S' - z)^{-1}e_{z_1}, e_{z_1}] = 0 \) and \( f_S(z_1) = w_1 \).

\[ \square \]
References


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