Generalisations of Semigroups of Operators

by

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A thesis submitted in partial fulfillment
of the requirements for the
Degree of Bachelor of Science
in Technischer Mathematik

Vienna University of Technology
Vienna, Austria

November 3, 2009
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0.1 Preface

The theory of semigroups of operators introduced by Hille and Yoshida is the basis of this work. The main thought is to weaken the definition of a semigroup and therefore get a generalisation of the situation. The loss of information in relation to the "well-known" case is reflected in the injective operator $P(0)$. As in semigroup theory, one considers an Abstract Cauchy Problem for an operator $A : dom(A) \subset X \to X$,

$$u'(t) = Au(t), \quad u(0) = c,$$

for $t \in [0, \infty)$ where $u$ is a Banach space valued function.

Finally, the focus is on exponentially tamed pre-semigroups which can be identified with strongly continuous semigroups on a Banach subspace of the considered Banach space $X$. 

0.2 Notation, Definitions and Elementary Results

First we make some remarks and introduce some notation.

- Let $X$ always denote a Banach space with norm $\|\cdot\|$.
- An operator is always linear.
- $\mathcal{B}(X)$ is the set of all linear bounded operators from $X$ in $X$.
- A function $F : [0, \infty) \to \mathcal{B}(X)$ is called strongly continuous if
  \[
  \lim_{h \to 0} \|(F(t + h) - F(t))x\| = 0
  \]
  for all $t \in [0, \infty)$ and each fixed $x \in X$.

\(\forall t = 0\) we have the limit from the right side $h \to 0^+$).

- $C([a, b]; X) (a, b \in \mathbb{R})$ is the vector space of continuous functions $f : [a, b] : I \to X$ normed by $\|f\|_\infty := \sup_{t \in [a, b]} \|f(t)\|$. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $C([a, b]; X)$. By the definition of the $\|\cdot\|_\infty$-norm $(f_n(t))_{n \in \mathbb{N}}$ is Cauchy in $X$ and therefore converges to $f(t)$ for each fixed $t \in [a, b]$. Let $h$ be sufficiently small, then

\[
\|f(t + h) - f(t)\| \leq \|f(t + h) - f_n(t + h)\| + \|f_n(t + h) - f_n(t)\| + \|f_n(t) - f(t)\| < \epsilon,
\]

and hence $f$ is continuous using continuity of $f_n$. So $C([a, b]; X)$ is complete.

- $C_b([0, \infty); X)$ denotes the vector space of all bounded uniformly continuous functions $f : [0, \infty) \to X$ with norm $\|f\|_\infty := \sup_{t \geq 0} \|f(t)\|$. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $C_b([0, \infty); X)$. By definition of the norm, $(f_n(t))_{n \in \mathbb{N}}$ is Cauchy in $X$ and therefore converges to $f(t)$ for each fixed $t \geq 0$. Since $f_n$ is bounded, $f$ is bounded. Clearly, convergence in $\|\cdot\|_\infty$ is nothing else but uniform convergence, i.e.

\[
\forall \epsilon' > 0 \ \exists N_{\epsilon'} \in \mathbb{N} : \|f_n(t) - f(t)\| < \epsilon' \ \forall t \geq 0, n > N_{\epsilon'}.
\]

Using this and uniform continuity of $f_n$ (for a fixed $n$), i.e.

\[
\forall \epsilon > 0 \ \exists \delta_{\epsilon,n} > 0 : \|f_n(t + h) - f_n(t)\| < \epsilon \ \forall t \geq 0, |h| < \delta_{\epsilon,n},
\]

we get for $\epsilon > 0$

\[
\|f(t + h) - f(t)\| \leq \|f(t + h) - f_n(t + h)\| + \|f_n(t + h) - f_n(t)\| + \|f_n(t) - f(t)\| < 2\epsilon + \epsilon < \epsilon,
\]

for an arbitrarily fixed $n > N_{\epsilon'}$ and for all $|h| < \delta_{\epsilon,n}$. Obviously, $h$ and $n$ are independent of $t \in [0, \infty)$, hence $f$ is uniformly continuous. Therefore, $C_b([0, \infty); X)$ is a Banach space.
\( C_0(\mathbb{R}) \) is the space of all bounded, continuous functions \( f : \mathbb{R} \to \mathbb{R} \) with 
\[
\lim_{x \to \pm \infty} f(x) = 0, \text{ i.e. } \forall \epsilon > 0 \exists M_\epsilon > 0 : |f(x)| < \epsilon \quad \forall |x| > M_\epsilon \tag{1}
\]
With the norm \( \|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)| \) it is a Banach space. To see this, we consider the sequence \( f_n \in C_0(\mathbb{R}), \ n \in \mathbb{N} \) with \( f_n \to f \) for \( n \to \infty \) in \( \|f\|_\infty \). Convergence in the \( \|f\|_\infty \)-norm implies pointwise convergence. Because of (1) and continuity of \( f_n \) it follows 
\[
|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < 2\epsilon,
\]
for all \( |x| > M_{n,\epsilon} \) and an arbitrary \( n \geq N_\epsilon \in \mathbb{N} \). By the pointwise convergence, boundedness and continuity of \( f \) are clear (in analogue to \( C_b([0, \infty); X \)), see Notation, Definitions and Elementary Results), hence \( f \in C_0(\mathbb{R}) \).

\( C_{00}(\mathbb{R}) \) is the linear subspace of \( C_0(\mathbb{R}) \) of functions with compact support, where the support \( \text{supp}(f) \) of a function \( f \) is defined as
\[
\text{supp}(f) := \{x \in \mathbb{R} : f(x) \neq 0\}.
\]
It is easy to see that \( C_{00}(\mathbb{R}) \) lies dense in \( C_0(\mathbb{R}) \). For that, consider \( f \in C_0(\mathbb{R}) \) and define:
\[
f_n(x) := \begin{cases} f(x) & |x| \leq n \\ 0 & |x| > n \end{cases}
\]
It is obvious that the discontinuity of \( f_n \) at \( x = \pm n \) can be eliminated by a \( C^\infty \)-function that ”connects” \( f(\pm n) \) with 0 on an interval \([-n - \epsilon, -n] \) (\([n, n + \epsilon] \) respectively). Then, \( f_n \) clearly belongs to \( C_{00}(\mathbb{R}) \). Furthermore, 
\[
\|f_n - f\|_\infty = \sup_{|x| > n} |f(x)| \to 0,
\]
for \( n \to \infty \), hence \( C_{00}(\mathbb{R}) \) is dense in \( C_0(\mathbb{R}) \).
Note that with this definition, \( C_{00}(\mathbb{R}) \supset C_0(\mathbb{R}) \).

The strong derivative of a function \( f : [a, b] \to X \) at \( t \in (a, b) \) is defined as
\[
\frac{d}{dt}f(t) = f'(t) := \lim_{h \to 0} \frac{1}{h}(f(t + h) - f(t)),
\]
if the limit exists. The strong derivative at the boundary points is defined through the limit from the right hand side for \( t = a \) (strong right derivative) and through the limit from the left hand side for \( t = b \) respectively. As for \( \mathbb{R} \)-valued functions we have (see [Kal08b]): If \( f' = 0 \) on \([a, b] \), then \( f \) is constant on \([a, b] \).

For functions \( f : [a, b] \to X \) a Banach space valued Riemann integral \( \int_a^b f(s) \, ds \) can be defined in the same way as for \( \mathbb{R} \)-valued functions by Riemann-sums. See
[Kal08b] for details. Thus, many results and rules concerning the integral (e.g. linearity,..) are similar. We want to point out that for $T \in B(X)$:

$$\int T f(s) \, ds = T \int f(s) \, ds.$$  

This integral concept also includes improper Riemann-integrals. Such an improper integral is defined as

$$\int_a^\infty f(s) \, ds = \lim_{\beta \to \infty} \int_a^\beta f(s) \, ds,$$

where the limit is in the norm $\| \cdot \|$ of the Banach space. Since

$$\left\| \int_a^b f(s) \, ds \right\| \leq \int_a^b \| f(s) \| \, ds$$

(which follows easily by definition of Riemann sums and triangle inequality), a sufficient condition for the existence of this limit is the convergence of

$$\lim_{\beta \to \infty} \int_a^\beta \| f(s) \| \, ds$$

in $\mathbb{R}$. The case $\sim -\infty$ is completely analogue.

- $I : X \to X : x \mapsto x$ denotes the identity operator.

- The operator norm of a bounded operator $T : X \to X$ is

$$\| T \|_{B(X)} = \sup_{x \in X, x \neq 0} \frac{\| Tx \|}{\| x \|}.$$ 

We will write only $\| \cdot \|$, if it is clear that the object is an operator and if it is obviously on which space the map is defined.

- For an operator $A$ defined on a subset of $X$, $\text{dom}(A)$ denotes the domain. Furthermore $A(\text{dom}(A))$ denotes the image of $A$.

- For an operator $A : \text{dom}(A) \to X$ and a subspace $Y \subset X$, the part of $A$ in $Y$, $A_Y$, is defined as the operator with

$$\text{dom}(A_Y) = \{ x \in \text{dom}(A) : x \in Y \land Ax \in Y \}, \quad A_Y x = Ax.$$ 

- An operator $A : \text{dom}(A) \to X$ is called closed, if for all sequences $(x_n)_{n \in \mathbb{N}}$, $x_n \in \text{dom}(A)$ for all $n \in \mathbb{N}$, with

$$x_n \to x \in X \quad \text{and} \quad Ax_n \to y \in X,$$

it follows

$$x \in \text{dom}(A) \quad \text{and} \quad Ax = y.$$ 

- **Closed Graph Theorem**: Let $X, Y$ be Banach spaces and let $A : X \to Y$ be an operator ($\text{dom}(A) = X$). The following assertions are equivalent
1. $A$ is closed,
2. $A$ is continuous, i.e. $A \in \mathcal{B}(X)$.

- $C^1([a, b]; X)$ denotes the vector space of all functions $f : [a, b] \to X$ which are continuously (strong) differentiable, with norm $\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$, with $\|f\|_\infty = \sup_{t \in [a, b]} |f(t)|$. To show that this space is complete, we consider $C^1([a, b]; X)$ as 

\[ \{(f; g) \in C([a, b]; X) \times C([a, b]; X) : g = f' \}. \] (2)

Since this is a subset of the Banach space $C([a, b]; X) \times C([a, b]; X)$ with the norm $\|\cdot\|_\infty + \|\cdot\|_\infty$, it suffices to show that the set in (2) is closed. It is equivalent to show that the differentiation operator 

$$D : dom(D) \to C([a, b]; X) : f \mapsto f',$$

$$dom(D) := \{f \in C([a, b]; X) : f' \text{ exists and is continuous}\} \subset C([a, b]; X)$$

is closed. Let $f_n \in dom(D), n \in \mathbb{N}$, $f_n \to f \in C([a, b]; X)$ and $Df_n \to g \in C([a, b]; X)$ (limits in $C([a, b]; X)$). Since convergence in $C([a, b]; X)$ implies uniform convergence, $g = \lim_{n \to \infty} f_n'$ is continuous. Furthermore, uniform convergence gives us, $(t, t_0 \in [a, b] \text{ and } t > t_0)$

$$\int_{t_0}^t g(s) \, ds = \lim_{n \to \infty} \int_{t_0}^t f_n'(s) \, ds.$$ 

By the fundamental theorem of calculus we get

$$\int_{t_0}^t g(s) \, ds + \lim_{n \to \infty} f_n(t_0) = \lim_{n \to \infty} \left( \int_{t_0}^t f_n'(s) \, ds + f_n(t_0) \right)$$

$$= \lim_{n \to \infty} f_n(t)$$

$$= f(t).$$

The left hand side is differentiable at $t$ since $g$ is continuous, and therefore $f' = g$. Hence, $D$ is closed and $C^1([a, b]; X)$ is a Banach space.

- **Principle of uniform boundedness theorem**: Let $X, Y$ be Banach spaces and \{T_i : i \in I\} a family of bounded operators. If the family is bounded pointwisely, i.e. for all $x \in X$ there exists a $M_x > 0$ so that

$$\sup_{i \in I} \|T_i x\| \leq M_x,$$

then there exists a $M > 0$, so that

$$\sup_{i \in I} \|T_i\| < M < \infty.$$

- For a closed operator $A$, the **resolvent set** $\rho(A)$ is the set of all $\lambda \in \mathbb{C}$ for which the operator $(\lambda I - A) : dom(A) \to X$ is bijective. For $\lambda \in \rho(A)$, the **resolvent** $R_{\lambda, A}$ denotes $(\lambda I - A)^{-1}$ which is necessarily also closed and therefore in $\mathcal{B}(X)$ since
A is closed (by the Closed Graph Theorem). We point out that \((\lambda I - A)\) is not necessarily bounded for \(\lambda \in \rho(A)\). Apparently the following relations hold true:

\[ R_{\lambda, A}(\lambda I - A)x = x \quad x \in \text{dom}(A), \]
\[ (\lambda I - A)R_{\lambda, A}x = x \quad x \in X. \]

The existence of a map \(R_{\lambda, A} : X \to \text{dom}(A)\), which satisfies these two equations, is also obviously sufficient for \(\lambda \in \rho(A)\).

\[ R_{\lambda, A}(\lambda I - A)x = (\lambda I - A)R_{\lambda, A}x, \]

for all \(x \in \text{dom}(A)\).

**Lemma 0.1** For a closed operator \(A : \text{dom}(A) \subset X \to X\) which commutes with \(B : X \to X\), i.e. \(Bx \in \text{dom}(A)\) and

\[ BAx = ABx \quad \forall x \in \text{dom}(A), \]

it follows that the resolvent \(R_{\lambda, A}\) commutes with \(B\), i.e.

\[ BR_{\lambda, A}x = R_{\lambda, A}Bx \quad \forall x \in X, \lambda \in \rho(A). \]

**Proof:** By definition of \(R_{\lambda, A}\) and using the assumption we see

\[ BAy = ABy \quad \forall y \in \text{dom}(A) \]
\[ \Leftrightarrow \lambda By - BAy = \lambda By - ABy \quad \forall y \in \text{dom}(A) \]
\[ \Leftrightarrow B(\lambda I - A)y = (\lambda I - A)By \quad \forall y \in \text{dom}(A) \]
\[ \Leftrightarrow R_{\lambda, A}B(\lambda I - A)y = By \quad \forall y \in \text{dom}(A) \]
\[ \Leftrightarrow R_{\lambda, A}Bx = BR_{\lambda, A}x \quad \forall x \in X. \]

\[ \blacksquare \]

**Lemma 0.2** Let \(A : X \to X\), \(B : X \to X\) be operators. Let \(B\) be injective and \(AB=BA\). Then, \(A(BX) \subset BX\) and

\[ AB^{-1}x = B^{-1}Ax, \]

for all \(x \in BX\).

**Proof:** Commutativity gives \(A(BX) \subset BX\). Using this and \(x \in BX\), it follows

\[ Ax = Ax \]
\[ \Rightarrow ABB^{-1}x = BB^{-1}Ax \]
\[ \Rightarrow AB^{-1}x = B^{-1}Ax. \]

\[ \blacksquare \]
• **Lemma 0.3** For operators $A : \text{dom}(A) \subset X \to X$, $B : \text{dom}(B) \subset X \to X$ with surjective $A$, injective $B$ the relation $A \subset B$, i.e.

$$\text{dom}(A) \subset \text{dom}(B) \land Ax = Bx \quad \forall x \in \text{dom}(A),$$

implies

$$A = B.$$

**Proof:** It suffices to show that $\text{dom}(A) = \text{dom}(B)$. $\text{dom}(A) \subset \text{dom}(B)$ is fulfilled by assumption. Let $x$ be in $\text{dom}(B)$. Since $A$ is surjective, there exists a $y \in \text{dom}(A)$ so that $Ay = Bx$. By assumption $A \subset B$. Therefore, $y \in \text{dom}(B)$ and $Ay = By$. Thus $Bx = By$. The injectivity of $B$ leads to $x = y \in \text{dom}(A)$, that is $\text{dom}(B) \subset \text{dom}(A)$ and hence $\text{dom}(A) = \text{dom}(B)$. □
Chapter 1

Pre-Semigroups

DEFINITION 1.1 A family \( \{P(t)\}_{t \geq 0} \) of operators is called \textbf{pre-semigroup}, if

1. \( P : [0, \infty) \to B(X) \) is strongly continuous, i.e.
   \[ \lim_{h \to 0} \|P(t+h)x - P(t)x\| = 0 \quad \forall x \in X, \forall t \in [0, \infty) \]
2. \( P(0) : X \to X \) is injective
3. \( P(t-u)P(u) \) is independent of \( u \) for all \( 0 \leq u \leq t \)

This definition is a generalisation of strongly continuous semigroups of operators (\( C_0 \)-semigroups). Point (3.) in the given form is not really convenient for the following statements and their proofs. That is why we reformulate it in the next lemma.

LEMMMA 1.2 For a family \( \{P(t)\}_{t \geq 0} \) of operators the following points are equivalent:

- \( P(t-u)P(u) \) is independent of \( u \) for \( 0 \leq u \leq t \)
- \( P(t-u)P(u) = P(0)P(t) \) for \( 0 \leq u \leq t \)
- \( P(0)P(u+s) = P(s)P(u) \) for all \( u, s \geq 0 \)

PROOF: (1.) \( \iff \) (2.): One direction follows by setting \( u = t \). The other implication is trivial.

(2.) \( \iff \) (3.): Set \( t = s + u \).

The last point of this lemma,

\[ P(0)P(u+s) = P(s)P(u) \quad u, s \geq 0 \]  \hfill (ADD)

reflects some kind of additivity of the pre-semigroup and immediately implies the commutativity of the operators \( P(s) \),

\[ P(s)P(u) = P(u)P(s) \quad u, s \geq 0 \]  \hfill (COM)
REMARK 1.3 In the property $P(0)P(u+s) = P(s)P(u)$ we can see the connection and the difference to "normal" $C_0$-semigroups: It is the injective operator $P(0)$ which controls the additivity of $P(.)$. Now we have noticed that it is just $P(0)$ which generalises the situation of a strongly continuous semigroup. That is the reason why pre-semigroups are sometimes called "$C$-semigroups" where $C$ denotes the injective operator $P(0)$. Probably this definition is not really suitable since this can be easily confused with $C_0$-semigroups. That is why for example in [deL94] the term "$C$-regularized-semigroup" is introduced. The notation "pre-semigroups" has been adopted from [Kan95].

The next lemma shows a basic property of a pre-semigroup.

**LEMMA 1.4** For a pre-semigroup $\{P(t)\}_{t \geq 0}$ the family of operators $\{P(s) : s \in [a, b]\}$ is uniformly bounded for each compact interval $[a, b]$ in $[0, \infty)$, i.e. there exists a $M > 0$:

$$\|P(s)\| < M \quad \forall s \in [a, b].$$

**PROOF:** We have to show that $\{P(s) : s \in [a, b]\}$ is bounded pointwisely. Since the norm $\|.\| : X \to [0, \infty)$ is continuous, it follows from strong continuity of the pre-semigroup that $\|P(.)x\| : [0, \infty) \to [0, \infty)$ is continuous for all $x \in X$. Such a function clearly has a maximum on a compact interval. Hence for each $x \in X$ there exists a $M_x$, so that $\sup_{s \in [a,b]} \|P(s)x\| < M_x$. With the Principle of uniform boundedness (see Notation, Definitions and Elementary Results) the proof is completed. ■

**Example 1.5** Consider the Banachspace $X = C_0(\mathbb{R})$ and the the family of operators $\{P(t)\}_{t \geq 0}$, defined through

$$P(t)f(x) = e^{-x^2+tx}f(x),$$

for $x \in \mathbb{R}$. We will see that this is a pre-semigroup. For that, we have to check the conditions of DEFINITION 1.1.

1. $P : [0, \infty) \to \mathcal{B}(X)$ is strongly continuous.

Fix $t \geq 0$. First, we have to assure that $P(t)f$ is in $C_0(\mathbb{R})$. This is clear since $\lim_{x \to \pm \infty} e^{-x^2+tx} = 0$ and $f \in C_0(\mathbb{R})$. The parable $x \mapsto -x^2 + tx$ has its maximum $\frac{t^2}{4}$ at $x_m = \frac{t}{2}$. Therefore,

$$\|P(t)f\|_\infty = \sup_{x \in \mathbb{R}} e^{-x^2+tx}f(x)$$

$$\leq \sup_{x \in \mathbb{R}} e^{-x^2+tx} \left( \sup_{x \in \mathbb{R}} |f(x)| \right)$$

$$= e^{\frac{t^2}{4}} \|f\|_\infty,$$

hence $P(t) \in \mathcal{B}(X)$. Fix $f \in C_0(\mathbb{R})$. For strong continuity we have to show that for all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ so that

$$\sup_{x \in \mathbb{R}} |e^{-x^2+(t+h)x}f(x) - e^{-x^2+tx}f(x)| < \epsilon,$$

(1.2)
for all $|h| < \delta_e$. First we consider functions $f \in C_{00}(\mathbb{R})$. Since $f$ has a compact support $K$ the left hand side in line (1.2) reads

$$
\sup_{x \in K} \left| e^{-x^2 + (t+h)x} f(x) - e^{-x^2 + tx} f(x) \right| < \sup_{x \in K} \left| e^{-x^2 + tx} f(x) \right| \sup_{x \in K} \left| e^{hx} - 1 \right|
$$

$$
= S_K \sup_{x \in K} \left| e^{hx} - 1 \right|,
$$

where $S_K$ denotes the maximum of $e^{-x^2 + tx} f(x)$ on $K$. From monotony of $y \mapsto e^y$ we get, with $K_{\max} = \max \{ |x| : x \in K \}$,

$$
\sup_{x \in K} \left| e^{hx} - 1 \right| \leq \left| e^{|h||x|} - 1 \right|
$$

$$
= \left| e^{|h|K_{\max}} - 1 \right| \to 0,
$$

for $h \to 0$. Hence we have strong continuity for $f \in C_{00}(\mathbb{R})$. Let $f \in C_0(\mathbb{R})$. Since $C_{00}(\mathbb{R})$ lies dense in $C_0(\mathbb{R})$, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of functions in $C_{00}(\mathbb{R})$ such that,

$$
\forall \epsilon > 0 \ \exists N_\epsilon \in \mathbb{N} : \|f - f_n\|_\infty < \epsilon \ \ \forall n \geq N_\epsilon.
$$

Therefore we can write

$$
\|P(t + h)f - P(t)f\|_\infty = \|P(t + h)f - P(t + h)f_n + P(t + h)f_n - P(t)f_n + P(t)f_n - P(t)f\|_\infty
$$

$$
\leq \|P(t + h)(f - f_n)\|_\infty + \|P(t)(f - f_n)\|_\infty + \|P(t + h)f_n - P(t)f_n\|_\infty
$$

$$
\leq (\|P(t + h)\| + \|P(t)\|) \|f - f_n\|_\infty + \|P(t + h)f_n - P(t)f_n\|_\infty.
$$

Because of strong continuity for functions in $C_{00}(\mathbb{R})$ we have

$$
\|P(t + h)f_n - P(t)f_n\|_\infty < \epsilon,
$$

for $|h| < \delta_e$. Using LEMMA 1.4 we know that $\|P(t + h)\|$ and $\|P(t)\|$ are bounded (independent of $h$) by a constant $M > 0$. Therefore,

$$
(\|P(t + h)\| + \|P(t)\|) \|f - f_n\|_\infty + \|P(t + h)f_n - P(t)f_n\|_\infty < \epsilon
$$

for an arbitrary $n \geq N_{\epsilon/2(2M)}$ and $|h| < \delta_e/2$. Hence, the family of operators is strongly continuous.

2. $P(0)$ is injective because for $P(0)f = P(0)g$ with $f, g \in C_0(\mathbb{R})$ we have for all $x \in \mathbb{R}$

$$
P(0)f(x) = P(0)g(x)
$$

$$
e^{-x^2} e^{-x^2} f(x) = e^{-x^2} g(x)
$$

$$
\Leftrightarrow f(x) = g(x).
$$

3. Clearly, for all $f \in C_0(\mathbb{R})$ and all $x \in \mathbb{R}$ the following holds true

$$
P(0)P(s + t)f(x) = e^{-x^2} e^{-x^2 + s + t} f(x) = e^{-x^2 + s} [e^{-x^2 + tx} f(x)] = P(s)P(t)f(x),
$$

hence $P(0)P(s + t) = P(s)P(t)$.

This example will accompany us throughout this work. Actually it can be weakened. Instead of the assumption that the functions tend to zero for $x \to \pm \infty$, we can just require $\lim_{x \to \pm \infty} f(x) = b_f$ for a $b_f \in \mathbb{R}$. Note, that then $P(t)f$ is still in $C_0(\mathbb{R})$. 11
For the definition of the "generator" of a pre-semigroup we need the right derivative of a Banach space valued function (in analogue to $\mathbb{R}$, see Notation, Definitions and Elementary Results).

**DEFINITION 1.6** Let $\{P(t)\}_{t \geq 0}$ be a pre-semigroup and $x \in X$. The strong right derivative $P^+(t)x \in X$ of $P(\cdot)x$ at $t$ is defined as

$$\lim_{h \to 0^+} \frac{1}{h} [P(t + h)x - P(t)x], \quad (1.3)$$

if the limit exists.

Now we can define an operator for the pre-semigroup connected with the derivative at zero.

**DEFINITION 1.7** Let $\{P(t)\}_{t \geq 0}$ be a pre-semigroup. Define the operator $A : \text{dom}(A) \to X$ by:

- $\text{dom}(A) = \{ x \in X : P^+(0)x \text{ exists in } X \text{ and belongs to } P(0)X \}$
- $Ax = P(0)^{-1}P^+(0)x$

$A$ is called the **generator** of the pre-semigroup $\{P(t)\}_{t \geq 0}$. We also say "$A$ generates the pre-semigroup $\{P(t)\}_{t \geq 0}$".

Because of the injectivity of $P(0)$, the generator is well-defined. The linearity follows, clearly, from the linearity of $P(t)$ for all $t \in [0, \infty)$. For $P(0) = I$ this definition obviously equals the definition of the generator for semigroups.

**Example 1.8** Consider again the pre-semigroup from Example 1.1. For $f \in C_0(\mathbb{R})$ we regard the strong right derivative of $P(\cdot)f$ at zero.

$$P^+(0)f = \lim_{h \to 0^+} \frac{1}{h} (P(h)f - P(0)f)$$

Assume that $f \in \text{dom}(A)$. Since point evaluations are continuous on $C_0(\mathbb{R})$, we obtain for $x \in \mathbb{R}$ with de L’Hospital

$$(P^+(0)f)(x) = \lim_{h \to 0^+} \frac{1}{h} (P(h)f(x) - P(0)f(x)) = \lim_{h \to 0^+} \frac{1}{h} (e^{-x^2+hx}f(x) - e^{-x^2}f(x)) = e^{-x^2} f(x) \lim_{h \to 0^+} \frac{e^{hx} - 1}{h} = e^{-x^2} f(x)x.$$ 

In the definition of $\text{dom}(A)$ we demand $P^+(0)f$ to be in the image of $P(0) : g \mapsto (x \mapsto e^{-x^2}g(x))$, therefore our function $f$ in $\text{dom}(A)$ satisfies $(x \mapsto xf(x)) \in C_0(\mathbb{R})$.

Conversely, let $f \in C_0(\mathbb{R})$ with $(x \mapsto xf(x)) \in C_0(\mathbb{R})$. We show that for such $f$, $P^+(0)f$ is $(x \mapsto e^{-x^2}xf(x))$. That is, for all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ so that

$$\sup_{x \in \mathbb{R}} \left| e^{-x^2} f(x) \frac{e^{hx} - 1}{h} - e^{-x^2} xf(x) \right| < \epsilon, \quad (1.4)$$
for $h < \delta$. We see that $|e^{-x^2}xf(x)|$ is sufficiently small, say $< \epsilon/2$, for $x \to \pm \infty$. Moreover for $h < 1$ we have

$$
\left|e^{-x^2}f(x)\frac{e^{hx} - 1}{h}\right| = \left|e^{-x^2}f(x)x \sum_{i=1}^{\infty} \frac{(xh)^i}{i!}\right|
\leq \left|e^{-x^2}xf(x)\right| \sum_{i=1}^{\infty} \frac{|x|^i}{i!}
\leq e^{-x^2}e|x| |xf(x)|.
$$

This expression is also $< \epsilon/2$ for $|x|$ sufficiently large. Hence we can define $S_\epsilon > 0$ so that (1.4) $< \epsilon$ for $|x| > S_\epsilon$ and $h < 1$. For the remaining $x$ we calculate

$$
\sup_{|x| \leq S_\epsilon} \left|e^{-x^2}f(x)\left(\frac{e^{hx} - 1}{h} - x\right)\right| \leq \|f\|_\infty \sup_{|v| \leq S_\epsilon} \left|\int_0^x (e^{hv} - 1) dv\right|
\leq \|f\|_\infty S_\epsilon \sup_{|v| \leq S_\epsilon} |e^{hv} - 1|
\leq \|f\|_\infty S_\epsilon \sup_{|v| \leq \Delta S_\epsilon} |e^v - 1| \to 0,
$$

for $h \to 0^+$. Hence, $P^+(0)f = (x \mapsto e^{-x^2}xf(x))$. Since $(x \mapsto xf(x)) \in C_0(\mathbb{R})$, $P^+(0)f$ is in the image of $P(0)$, and therefore, $f \in \text{dom}(A)$. Altogether we have (since $P(0)^{-1}(x \mapsto e^{-x^2}xf(x)) = (x \mapsto xf(x)))$,

$$
Af(x) = xf(x), \quad \text{dom}(A) = \{f \in C_0(\mathbb{R}) : (x \mapsto xf(x)) \in C_0(\mathbb{R})\}.
$$

After we have noticed that the operator $A$ is well defined, we want to know "if this map is reasonable in a certain sense". One question is about the domain of $A$: The pre-semigroup is per definitionem "only" (strongly) continuous. This does not really imply that there exists a (strong right) derivative. For example, we want to analyse how big the domain is.

The following theorem shows some basic results of the generator.

**THEOREM 1.9** For the generator $A$ of a given pre-semigroup $\{P(t)\}_{t \geq 0}$. The following assertions hold true.

1. $x \in \text{dom}(A) \Rightarrow P(t)x \in \text{dom}(A)$ for all $t \geq 0$
2. $AP(t)x = P(t)Ax$ for all $x \in \text{dom}(A)$
3. $P(.)x \in C^1([0, \infty) ; X)$ for $x \in \text{dom}(A)$,

$$
AP(t)x = \lim_{h \to 0} \frac{1}{h}(P(t + h)x - P(t)x) = \frac{d}{dt}P(t)x
$$

for all $x \in \text{dom}(A), t \in [0, \infty)$
4. For $x \in X$:

$$
\int_0^t P(s)x \, ds \in \text{dom}(A)
$$
5. A is closed and \( P(0)X \subseteq \text{dom}(A) \);

**PROOF:** Let be \( t \geq 0, h > 0 \) and \( x \in \text{dom}(A) \).

1. We use that the \( P(s), s \geq 0 \) commute to obtain

\[
\frac{1}{h} \left( P(h) \left[ P(t)x - P(0) \right] - P(0) \right) = \frac{1}{h} \left( P(t) \left[ P(h)x - P(0)x \right] \right).
\]

By the continuity of \( P(t) \) and because of \( x \in \text{dom}(A) \) the right hand side tends to the strong right derivative of \( P(t)P(.)x \) at 0 for \( h \to 0^+ \), hence

\[
\lim_{h \to 0^+} \frac{1}{h} \left( P(h) \left[ P(t)x - P(0) \right] - P(0) \right) = P(t) \left[ P^*(0)x \right].
\]

In particular, the limit for \( h \to 0^+ \) on the left hand side, i.e. the strong right derivative \( P^*(0) \left[ P(t)x \right] \), exists. With the definition of \( A \) and again with the commutativity of the operators \( P(s), s \geq 0 \) we get

\[
P^*(0) \left[ P(t)x \right] = P(t) \lim_{h \to 0^+} \frac{1}{h} \left( P(h)x - P(0)x \right)
\]

\[
= P(t)P(0)Ax
\]

\[
= P(0)P(t)Ax
\]

Therefore, \( P^*(0)P(t)x \in P(0)X \) and hence \( P(t)x \in \text{dom}(A) \).

2. Furthermore with the definition of the operator \( A \) and using (1.8) it follows

\[
A \left[ P(t)x \right] = P(0)^{-1} \left[ P^*(0)P(t)x \right] = P(0)^{-1}P(0)P(t)Ax = P(t)Ax.
\]

3. We use \( P(0)P(t+h) = P(t)P(h) \) (ADD) to obtain

\[
\frac{1}{h} \left( P(t) \left[ P(h)x - P(0)x \right] \right) = \frac{1}{h} \left( P(0) \left[ P(t+h)x - P(t)x \right] \right).
\]

Letting \( h \to 0^+ \), we observe that, with the same argument as in 1. \( x \in \text{dom}(A) \) and \( P(t) \) continuous), the strong right derivative of \( P(0)P(.)x \) at \( t \geq 0 \),

\[
\lim_{h \to 0^+} \frac{1}{h} P(0) \left[ P(t+h)x - P(t)x \right] = \left[ P(0)P(.) \right]^*(t)x,
\]

exists and equals \( P(t)P(0)Ax = P(0)P(t)Ax \). We show that this is also the strong left derivative of \( P(0)P(.)x \) for \( t > 0 \). With the triangle inequality we see

\[
\left\| \frac{1}{h} \left[ P(0)P(t) - P(0)P(t-h) \right] x - P(0)P(t)Ax \right\| \\
\leq \left\| \frac{1}{h} \left[ P(t) - P(t-h) \right] P(h)x - P(0)P(t)Ax \right\| + \left\| (P(t-h) - P(t)) \frac{1}{h} [P(h)x - P(0)x] \right\|.
\]
Using triangle inequality again, we get that this expression is less or equal to
\[
\left\| \frac{1}{h} [P(t) - P(t - h)] P(h)x - P(0)P(t)Ax \right\| + \\
+ \left\| (P(t - h) - P(t)) \left\{ \frac{1}{h} [P(h)x - P(0)x] - P(0)Ax \right\} \right\| + \\
+ \left\| P(t - h)P(0)Ax - P(t)P(0)Ax \right\|.
\]

Now we show that all three terms on the right hand side converge to zero for \( h \to 0^+ \).
Because of (ADD) the first term can be written as
\[
\left\| \frac{1}{h} (P(0)P(t + h)x - P(0)P(t)x) - P(0)P(t)Ax \right\|.
\]

For \( h \to 0^+ \) this converges to 0, since \( P(0)P(t)Ax \) is the strong right derivative of \( P(0)P(.)x \) at \( t \) as shown before. LEMMA 1.4 can be applied on the interval \( [t - h, t] \) and gives us a constant \( M > 0 \), so that for the second term we get
\[
\left\| (P(t - h) - P(t)) \left\{ \frac{1}{h} [P(h)x - P(0)x] - P(0)Ax \right\} \right\| \leq \\
\leq \left\| P(t - h) - P(t) \right\| \left\| \frac{1}{h} [P(h)x - P(0)x] - P(0)Ax \right\| \leq \\
\leq 2M \left\| \frac{1}{h} [P(h)x - P(0)x] - P(0)Ax \right\| \to 0,
\]
for \( h \to 0^+ \) by definition of the strong right derivative and \( A \). The third term converges to 0 since \( P(.)P(0)Ax \) is continuous.

Therefore, \( P(0)P(.)x \) is differentiable for all \( t \in (0, \infty), x \in \text{dom}(A) \) and its derivative equals \( P(0)P(t)Ax \). That is,
\[
P(0)P(t)Ax = \lim_{h \to 0^+} \frac{1}{h} (P(0)P(t + h)x - P(0)P(t)x) = [P(0)P(.)x]'(t).
\]

Obviously, the left side is continuous (as a function in \( t \) and fixed \( x \)), since \( P(0) \) is continuous and because of strong continuity of \( P(.) \). Therefore, \( P(0)P(.)x \) is continuously differentiable and hence we can use the fundamental theorem of calculus,
\[
\int_t^{t+h} [P(0)P(.)x]'(s) \, ds = \int_t^{t+h} P(0)P(s)Ax \, ds = P(0)P(t + h)x - P(0)P(t)x.
\]

The fact that \( P(0) \) bounded yields (see: Notation, Definitions and Elementary Results)
\[
P(0) \int_t^{t+h} P(s)Ax \, ds = P(0)[P(t + h)x - P(t)x],
\]
and by injectivity of \( P(0) \) we have
\[
\int_t^{t+h} P(s)Ax \, ds = P(t + h)x - P(t)x. \quad (1.10)
\]
The integrand is obviously continuous, so dividing by \( h \) and letting \( h \to 0 \) directly gives us (by the fundamental theorem of calculus)

\[
P(t)Ax = \lim_{h \to 0} \frac{1}{h} (P(t+h)x - P(t)x) = [P(.)x]'(t).
\]

Clearly, also for \( t = 0 \) the strong right derivative of \( P(.)x \) exists and equals \( P(0)Ax \).

Since \( P(.)Ax \) is continuous on \([0, \infty)\) (because \( P(.) \) is strongly continuous), \( P(.)x \in C^1([0, \infty); X) \). With point 2. of this theorem we obtain

\[
AP(t)x = \lim_{h \to 0} \frac{1}{h} (P(t+h)x - P(t)x).
\]

4. For a fixed \( t > 0 \), we consider the strong right derivative of \( P(.) \left[ \int_0^t P(s) x ds \right] \) at 0:

\[
\lim_{h \to 0^+} \frac{1}{h} [P(h) - P(0)] \int_0^t P(s) x ds = \lim_{h \to 0^+} \frac{1}{h} \int_0^t [P(0)P(h+s) - P(0)P(s)] x ds
\]

\[
= \lim_{h \to 0^+} \frac{1}{h} P(0) \left[ \int_h^{t+h} P(u)x du - \int_0^t P(s)x ds \right]
\]

\[
= P(0) \lim_{h \to 0^+} \frac{1}{h} \left[ \int_t^{t+h} P(u)x du - \int_0^h P(s)x ds \right]
\]

\[
= P(0) \left[ P(t) - P(0) \right] x,
\]

where we used the fundamental theorem of calculus again. So the strong right derivative exists and belongs to \( P(0)X \).

5. With (1.10) from point 3. we have

\[
P(t)x - P(0)x = \int_0^t P(s)Ax ds \tag{1.11}
\]

First we show that \( A \) is closed. Let \( x_n \to x \) with \( x_n \in \text{dom}(A) \) and \( Ax_n \to y \). (1.11) and boundedness of \( P(t) \) for all \( t \in [0, \infty) \) yield to

\[
(P(h) - P(0))x = \lim_{n \to \infty} ((P(h) - P(0))x_n) = \lim_{n \to \infty} \int_0^h P(s)Ax_n ds.
\]

Because \( ||P(s)|| \) is bounded uniformly on the compact interval \([0, h]\) (LEMMA 1.4), the limit is uniformly, hence can be permuted with the integral (see [Kal08b]).

\[
(P(h) - P(0))x = \int_0^h \lim_{n \to \infty} P(s)Ax_n ds = \int_0^h P(s)y ds. \tag{1.12}
\]

Dividing by \( h \) and letting \( h \to 0^+ \) we get (with \( P(.)y \) being continuous and the fundamental theorem of calculus)

\[
P^{r+}(0)x = \lim_{h \to 0} \frac{1}{h} (P(h)x - P(0)x) = \lim_{h \to 0} \frac{1}{h} \int_0^h P(s)y ds = P(0)y.
\]
Hence \( P^+(0)x \in P(0)X \) and further \( x \in \text{dom}(A) \) by definition of \( A \) and \( Ax = y \).

Finally we show that \( P(0)X \subseteq \text{dom}(A) \). Let \( x \in X \). From 4., we see that \( t^{-1} \int_0^t P(s)x ds \in \text{dom}(A) \). Letting \( t \to 0^+ \) and using again the fundamental theorem of calculus we get

\[
P(0)x = \lim_{t \to 0^+} t^{-1} \int_0^t P(s)x ds.
\]

Thus \( P(0)x \in \text{dom}(A) \). ■

We see that there is a connection between the domain of \( A \) and the image of \( P(0) \). The bigger \( P(0)X \) is, the bigger will be \( \text{dom}(A) \). In the case that \( P(0) \) is bijective, it follows that \( \text{dom}(A) \) is dense in \( X \). Furthermore, the property \( P(0)P(u+s) = P(u)P(s) \) is responsible for the fact that \( \text{dom}(A) \) is invariant for \( P(t) \) and the commutativity of \( A \) and \( P(t) \). A main result is the differentiability of \( P(t)x \) for \( x \in \text{dom}(A) \). This will be used in chapter 2.

REMARK 1.10 An obvious question is "What happens if we have a pre-semigroup \( \{P(t)\}_{t \geq 0} \) and an injective, bounded operator \( G \) and we consider the family of operators \( \{W(t) := GP(t)\}_{t \geq 0} \)? Can we expect this family to be a pre-semigroup? If we look at the assumptions in DEFINITION 1.1, clearly, strong continuity is preserved by boundedness of \( G \) and injectivity of \( GP(0) \) is trivial. Concerning the additivity property, \( P(0)P(t+s) = P(t)P(s) \), we get

\[
W(0)W(s + t) = GP(0)GP(s + t),
\]

where we see that \( GP(t) = P(t)G \) for all \( t \geq 0 \) is a sufficient condition so that

\[
GP(0)GP(s + t) = GGP(0)P(s + t) = GGP(t)P(s) = GP(s)GP(t) = W(s)W(t)
\]

Therefore, additionally we have to require that the operator \( G \) commutes with \( P(t) \) for all \( t \geq 0 \). In this case, the domain of the generator of \( \{W(t) := GP(t)\}_{t \geq 0} \) includes the domain of \( A_P \), the generator of \( \{P(t)\}_{t \geq 0} \), since the boundedness of \( G \) gives us

\[
\left\| \frac{1}{h}(GP(h)x - GP(0)) - GP^+(0)x \right\| < \|G\| \left\| \frac{1}{h}(P(h)x - P(0)) - P^+(0)x \right\|.
\]

Finally, this thoughts inspire the idea to choose \( G = P(0)^{-1} \). Unfortunately, in general we can not expect continuity of the inverse of \( P(0) \). Although we will see in Chapter 3 that in some situations this is possible.

The situation is that we have a pre-semigroup which gives us the generator \( A \). Especially in connection with the ACP (see next chapter) and the uniqueness of its solution we are interested in a uniqueness of the generator. The following technical lemma will be useful for conclusions on the uniqueness of the pre-semigroup for a given generator. For that, we state the a product rule for Banach space-valued functions.

**LEMMA 1.11** Let \( W(.) : [a, b] \to B(X) \) be a strongly continuous function with \( W(.)x \in C^1([a, b]; X) \) for all \( x \) in a linear subspace \( U \subset X \). Furthermore, let \( v : [a, b] \to U \) be in \( C^1([a, b]; X) \). Then,

\[
(W(.)v(.))'(t) = W(t)v'(t) + W'(t)v(t), \tag{1.13}
\]

where \( W'(t)x := [W(.)x]'(t) \).
PROOF: Regard the function $g$ defined as follows:

$$g : [a, b] \rightarrow X : s \mapsto W(s)v(s)$$

We consider $g'(s)$, $s \in [a, b]$, with elementary rearrangements we get

$$g'(s) = \lim_{h \to 0} \frac{1}{h}[W(s + h)v(s + h) - W(s)v(s)]$$

$$= \lim_{h \to 0} \frac{1}{h}[W(s + h)v(s + h) - W(s + h)v(s) + W(s + h)v(s) - W(s)v(s)]$$

$$= \lim_{h \to 0} W(s + h)\frac{1}{h}[v(s + h) - v(s)] + \lim_{h \to 0} \frac{1}{h}[W(s + h) - W(s)]v(s)$$

$$= \lim_{h \to 0} W(s + h)(\frac{1}{h}[v(s + h) - v(s)] - v'(s)) + \lim_{h \to 0} W(s + h)v'(s) +$$

$$+ \lim_{h \to 0} \frac{1}{h}[W(s + h) - W(s)]v(s).$$

Due to the strong continuity of $W(.)$ and the principle of uniform boundedness theorem (compare: LEMMA 1.4), $|W(s + h)|$ is bounded (by a constant $S$) for $h$ in a compact interval. Therefore, we can write

$$\left\| W(s + h)(\frac{1}{h}[v(s + h) - v(s)] - v'(s)) \right\| \leq S \left\| \frac{1}{h}[v(s + h) - v(s)] - v'(s) \right\|,$$

where the right hand side clearly tends to 0 for $h \to 0$, since $v(.) \in C^1([a, b]; X)$. Hence, the first term in (1.17) is 0 $\in X$. The second term,

$$\lim_{h \to 0} W(s + h)v'(s) = W(s)v'(s),$$

since $W(.)$ is strongly continuous. Finally,

$$\lim_{h \to 0} \frac{1}{h}[W(s + h) - W(s)]v(s) = W'(s)v(s),$$

because $v(s) \in U$ and $W(.)x \in C^1([a, b]; X)$ for $x \in U$. Altogether,

$$g'(s) = W(s)v'(s) + W'(s)v(s),$$

which proves the lemma. (For $s = a$ or $s = b$ the limits above are to be considered for $h \to 0^+$ or $h \to 0^-$ )

We point out that $W'(t)v(t)$ is not the composition of the operators "$W'(t)$" and $v(t)$.

**LEMMA 1.12** Let $\{P(t)\}_{t \geq 0}$ be a pre-semigroup generated by $A$. Let $v : [0, \infty) \to \text{dom}(A)$ be in $C^1([0, \infty); X)$ with $v' = Av$ and $v(0) = P(0)c$ for $c \in \text{dom}(A)$. Then,

$$P(.)c = v(.)$$
PROOF: We fix \( t > 0 \). Clearly the function \( h_{t,x} : [0,t] \to X : s \mapsto P(t - s)x \) is in \( C^1([0,t]; X) \) for \( x \in \text{dom}(A) \), since \( P(.)x \in C^1([0,t];X) \) for \( x \in \text{dom}(A) \) (see THEOREM 1.9). Therefore, the assumptions for LEMMA 1.11, where \( W(.) = P(t - .) \) and \( U = \text{dom}(A) \), are satisfied. Hence for \( f_t := P(t - .)v(.) : [0,t] \to X \) and \( s \in [0,t] \)

\[
f'_t(s) = (P(t - .)v(.))'(s) = P(t - s)v'(s) + P(t - s)'v(s).
\]

We know \( P(s)'x = AP(s)x \) for \( x \in \text{dom}(A) \) by THEOREM 1.9, which yields \( P(t - s)'x = -AP(t - s)x \) for \( x \in \text{dom}(A) \). Together with our assumption \( v' = Av \) we get

\[
f'_t(s) = P(t - s)Av(s) - AP(t - s)v(s) = 0 \in X,
\]

since \( A \) and \( P(r) \) commute for all \( r \geq 0 \) (see THEOREM 1.9). From \( f'_t = 0 \) and the theory of Riemann integrals of Banach space-valued functions (see: Notation, Definitions and Elementary Results) it follows that \( f_t \) is constant. Especially, \( f_t(0) = f_t(t) \) and with the definition of \( f_t \) we get

\[
P(t)v(0) = P(0)v(t).
\]

Due to the assumption \( v(0) = P(0)c \) and the commutativity of the operators \( P(s), s \geq 0 \) (COM) this leads to

\[
P(0)P(t)c = P(0)v(t).
\]

Because \( P(0) \) is injective, the claim is proven.

Now we can easily show a result on pre-semigroups with the same generator:

**THEOREM 1.13** Let \( \{P(t)\}_{t \geq 0}, \{W(t)\}_{t \geq 0} \) be pre-semigroups generated by \( A \). If in addition \( P(0) = W(0) \), then \( P(t)x = W(t)x \) for all \( t \geq 0 \) and all \( x \in \text{dom}(A) \).

**PROOF:** Let \( v(.) := W(.)c \) for \( c \in \text{dom}(A) \). By LEMMA 1.12, \( P(.)c = W(.)c \). Clearly, this is true for all \( c \) in \( \text{dom}(A) \).

We see that a generator characterizes the pre-semigroup at least on its domain. Again the image of \( P(0) \) plays an important role in the quality of the uniqueness. For a bijective \( P(0) \) (as in the semigroup situation) a generator has a unique pre-semigroup because then \( \text{dom}(A) \) is dense and due to the continuity of the \( P(t), W(t) \), we get \( P(t) = W(t) \).
Chapter 2

The Abstract Cauchy Problem

In this chapter we concentrate on a main application of semigroups and pre-semigroups. From THEOREM 1.9 we know that 
P′(t) = AP(t) for a pre-semigroup \( \{P(t)\}_{t \geq 0} \) generated by \( A \). This can be seen as a motivation for analysing the following type of differential equations.

**DEFINITION 2.1** Let \( A : \text{dom}(A) \to X \) be an operator and \( c \in \text{dom}(A) \). Then \( u \in C^1([0, \infty); X) \) with \( u(t) \in \text{dom}(A) \) for all \( t \geq 0 \) is a solution for the Abstract Cauchy Problem \( ACP \), if:

\[
\frac{d}{dt} u = Au \quad \text{and} \quad u(0) = c,
\]

where \( \frac{d}{dt} u \) denotes the strong derivative of \( u \). We denote \( c \) as the initial value.

The following examples are very special cases for \( X \) and the operator \( A \). Although, their solutions, which we get from ordinary theory of differential equations, have abilities of (pre-)semigroups.

**Example 2.2** Let be \( X = \mathbb{R}^n \).

\( n = 1 \): In this case we have the simple one dimensional differential equation (\( A \equiv a \in \mathbb{R} \))

\[
u' = au, \quad u(0) = C.
\]

With the solution \( u(t) = Ce^{at} \).

\( n > 2 \): Here we get a linear system of differential equations with the matrix \( A \)

\[
\frac{d}{dt} u = (\frac{d}{dt} u_i)_{i=1,\ldots,n} = (A_{ij}u_j)_{i=1,\ldots,n}
\]

The solution is given by the matrix exponential \( u(t) = e^{tA} \), where \( e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \).

From Theorem (1.9) we get solutions for an \( ACP \) through a pre-semigroup:

**COROLLARY 2.3** Let \( \{P(t)\}_{t \geq 0} \) be a pre-semigroup and let \( A \) be its generator. For \( c \in \text{dom}(A) \), \( u(.) = P(.)c \) is the unique solution of the \( ACP \),

\[
\frac{d}{dt} u = Au, \quad u(0) = P(0)c.
\]
PROOF: From THEOREM 1.9 point 2. we know that \( P(t)c \in \text{dom}(A) \) for all \( t \geq 0 \). Point three of this theorem gives us \( P(.)c \in C^1([0, \infty), X) \) and

\[
\frac{d}{dt} u = Au.
\]

Apparently, \( u(0) = P(0)c \). The uniqueness of the solution follows directly from LEMMA 1.12. Let \( v : [0, \infty) \rightarrow \text{dom}(A) \) be any further solution of (2.2). Then,

\[
P(.)c = v(.),
\]

by LEMMA 1.12. ■

**Example 2.4** For \( X = C_0(\mathbb{R}) \) and the operator \( A : \text{dom}(A) \rightarrow C_0(\mathbb{R}) \), \( \text{dom}(A) = \{ f \in C_0(\mathbb{R}) : (x \mapsto xf(x)) \in C_0(\mathbb{R}) \} \), \( Af(x) = xf(x) \), we have the following partial differential equation

\[
\frac{d}{dt} u(t,x) = x \cdot u(t,x), \quad u(0,x) = e^{-x^2}g(x), \quad (2.3)
\]

for all \( t \geq 0 \) and \( x \in \mathbb{R} \) and where \( g \in \text{dom}(A) \). Actually, in the sense of DEFINITION 2.1, \( \frac{d}{dt} u(t,x) \) has to be understood as \((\frac{d}{dt} u(t, .))(x)\) where we have the strong derivative in \( X = C_0(\mathbb{R}) \). Here, clearly, if the strong derivative exists, it equals the partial (pointwise) derivative \( \frac{d}{dt} u(t,x) \). Therefore a strong solution (in the sense of DEFINITION 2.1) is also a solution of (2.3). We know already from EXAMPLE 1.1 that \( A \) is the generator for the pre-semigroup \( \{ P(t) \}_{t \geq 0} \),

\[
P(t)f(x) = e^{-x^2+tx}f(x).
\]

Therefore, by COROLLARY 2.3 a solution for (2.3) is given by

\[
u(t,x) = e^{-x^2+tx}g(x),
\]

where \( u(t, .) \in C_0(\mathbb{R}) \) for all \( t \geq 0 \). The uniqueness is at least given for the situation of the strong solution.

From COROLLARY 2.3 we get a solution for the ACP implicated by a given pre-semigroup. The initial value is in \( P(0)\text{dom}(A) \). This solution is unique. In other words, we have a unique solution, if we know the pre-semigroup. Furthermore we are interested in the ”other direction”: If a function \( u = P(.)c \) is a solution of the ACP for an operator \( A \) and \( c \in \text{dom}(A) \), is \( P(.) \) a pre-semigroup? The following theorem answers this question for a situation with comparatively strong assumptions.

**THEOREM 2.5** For a a closed operator \( A \) consider following situation:

- \( \{ P(t) \}_{t \geq 0} \) is a family of bounded operators, which is strongly continuous;
- \( P(0) \) is injective;
- \( A \) commutes with \( P(s) \) for \( s \geq 0 \);
LEMMA 1.11 (with the functions $\lambda, \leq$)

Let $P$ be an element in $\mathbb{P}$.

Regard the derivative of $P(t - u)P(u)c$ with respect to $u$ for $c \in \text{dom}(A)$. We use LEMMA 1.11 (with the functions $W(\cdot) := P(t - \cdot), v(\cdot) := P(\cdot)c$ and $U := \text{dom}(A)$). As $(P(t - \cdot)x)'(u) = -AP(u)x$ for $x \in \text{dom}(A)$ we have

$$\frac{d}{du}P(t - u)P(u)c = P(t - u)AP(u)c - AP(t - u)P(u)c = 0,$$

(2.4)

for all $0 \leq u \leq t$ since $A$ and $P(t - u)$ commute by assumption. Hence $P(t - u)P(u)c = P(t)P(0)c$ for all $c \in \text{dom}(A)$ and for all $u \geq t$. Let $\text{dom}(A)$ be dense in $X$. Because $P(t - u)P(u)$ and $P(t)P(0)$ are continuous and coincide on the dense set $\text{dom}(A)$, they coincide on $X$. Now consider the situation where the resolvent set of $A$ is not empty. Let $\lambda$ be an element in $\rho(A)$. Regard the resolvent $R_{\lambda,A} = (\lambda I - A)^{-1} : X \to \text{dom}(A)$. Because of the injectivity of $R_{\lambda,A}$, it suffices to show that

$$R_{\lambda,A}P(t - u)P(u)x = R_{\lambda,A}P(0)P(t)x \quad \forall x \in X.$$

Because of the assumption $P(\cdot)Ax = AP(\cdot)x$ for all $x \in \text{dom}(A)$, $R_{\lambda,A}$ commutes with the operators $P(s)$, $s \in [0, \infty)$, i.e.,

$$R_{\lambda,A}P(s)x = P(s)R_{\lambda,A}x \quad \forall x \in X,$$

(see LEMMA 0.1 in Notation, Definitions and Elementary Results). By (2.4) and $R_{\lambda,A}x \in \text{dom}(A)$, $P(t - u)P(u)R_{\lambda,A}x$ is constant with respect to $u$. This yields

$$R_{\lambda,A}P(t - u)P(u)x = P(t - u)P(u)R_{\lambda,A}x = P(0)P(t)R_{\lambda,A}x = R_{\lambda,A}xP(0)P(t),$$

which proves the present case. Denote the generator of the pre-semigroup $P(t)_{t \geq 0}$ by $A_P$. Since $P(\cdot)x$ is solution of the ACP for $x \in \text{dom}(A)$, $P^+(0)x$ exists and

$$P^+(0) = \frac{d}{dt}(P(\cdot)x)(0) = AP(0)x = P(0)Ax \in P(0)x,$$

where the last equality follows from the assumption that $A$ commutes with $P(\cdot)$. We obtain $\text{dom}(A_p) \supset \text{dom}(A)$. From definition of $A_P$ for $x \in \text{dom}(A)$ we get

$$P(0)A_P x = P^+(0)x = [P(\cdot)x]'(0) = AP(0)x = P(0)Ax,$$

which verifies that $A_P$ is an extension of $A$, because $P(0)$ is injective. $\blacksquare$

• $P(.)c$ solves ACP (2.2) for all $c \in \text{dom}(A)$;

If either $\text{dom}(A)$ is dense in $X$ or the resolvent set of $A$ is non-empty, then $P(.)$ is a pre-semigroup generated by an extension of $A$.

PROOF: It remains to show that $P(t - u)P(u)$ is independent of $u$ for all $0 \leq u \leq t$. We use LEMMA 1.11 (with the functions $W(\cdot) := P(t - \cdot), v(\cdot) := P(\cdot)c$ and $U := \text{dom}(A)$). As $(P(t - \cdot)x)'(u) = -AP(u)x$ for $x \in \text{dom}(A)$ we have

$$\frac{d}{du}P(t - u)P(u)c = P(t - u)AP(u)c - AP(t - u)P(u)c = 0,$$

(2.4)

for all $0 \leq u \leq t$ since $A$ and $P(t - u)$ commute by assumption. Hence $P(t - u)P(u)c = P(t)P(0)c$ for all $c \in \text{dom}(A)$ and for all $u \geq t$. Let $\text{dom}(A)$ be dense in $X$. Because $P(t - u)P(u)$ and $P(t)P(0)$ are continuous and coincide on the dense set $\text{dom}(A)$, they coincide on $X$. Now consider the situation where the resolvent set of $A$ is not empty. Let $\lambda$ be an element in $\rho(A)$. Regard the resolvent $R_{\lambda,A} = (\lambda I - A)^{-1} : X \to \text{dom}(A)$. Because of the injectivity of $R_{\lambda,A}$, it suffices to show that

$$R_{\lambda,A}P(t - u)P(u)x = R_{\lambda,A}P(0)P(t)x \quad \forall x \in X.$$

Because of the assumption $P(\cdot)Ax = AP(\cdot)x$ for all $x \in \text{dom}(A)$, $R_{\lambda,A}$ commutes with the operators $P(s)$, $s \in [0, \infty)$, i.e.,

$$R_{\lambda,A}P(s)x = P(s)R_{\lambda,A}x \quad \forall x \in X,$$

(see LEMMA 0.1 in Notation, Definitions and Elementary Results). By (2.4) and $R_{\lambda,A}x \in \text{dom}(A)$, $P(t - u)P(u)R_{\lambda,A}x$ is constant with respect to $u$. This yields

$$R_{\lambda,A}P(t - u)P(u)x = P(t - u)P(u)R_{\lambda,A}x = P(0)P(t)R_{\lambda,A}x = R_{\lambda,A}xP(0)P(t),$$

which proves the present case. Denote the generator of the pre-semigroup $P(t)_{t \geq 0}$ by $A_P$. Since $P(\cdot)x$ is solution of the ACP for $x \in \text{dom}(A)$, $P^+(0)x$ exists and

$$P^+(0) = \frac{d}{dt}(P(\cdot)x)(0) = AP(0)x = P(0)Ax \in P(0)x,$$

where the last equality follows from the assumption that $A$ commutes with $P(\cdot)$. We obtain $\text{dom}(A_p) \supset \text{dom}(A)$. From definition of $A_P$ for $x \in \text{dom}(A)$ we get

$$P(0)A_P x = P^+(0)x = [P(\cdot)x]'(0) = AP(0)x = P(0)Ax,$$

which verifies that $A_P$ is an extension of $A$, because $P(0)$ is injective. $\blacksquare$
Example 2.6 Let us again consider the family of operators from EXAMPLE 1.1. We know already that \( \{ P(t) \}_{t \geq 0} \) is a pre-semigroup, but, as an example, we want to use THEOREM 2.5 to prove that it is a pre-semigroup. Therefore, we show that the image of \( P(0) \) lies dense in \( C_0(\mathbb{R}) \).

Regard a function \( g \in C_00(\mathbb{R}) \) with compact support \( K \subset \mathbb{R} \). Since \((x \mapsto e^{x^2})\) is bounded on \( K \), \((x \mapsto e^{x^2}g(x))\) is also in \( C_00(\mathbb{R}) \), in particular in \( C_0(\mathbb{R}) \). Therefore, \( g \) has the form

\[
g(x) = e^{-x^2}(e^{x^2}g(x))
\]

Thus \( g \in P(0)C_0(\mathbb{R}) \). Hence, \( C_00(\mathbb{R}) \) lies in the image of \( P(0) \). Since \( C_00(\mathbb{R}) \) is dense in \( C_0(\mathbb{R}) \), \( P(0)C_0(\mathbb{R}) \) is dense in \( C_0(\mathbb{R}) \).

REMARK 2.7 In the last theorem, the condition ”resolvent set \( \rho(A) \) non-empty” can be weakened. Like in [deL94] one can define a special resolvent set for \( A \) by taking \( P(0) \) into consideration. \( \rho_{P(0)}(A) \) is defined as the set of all \( \lambda \in \mathbb{C} \) with \( (\lambda I - A) \) is injective and \( P(0)X \subseteq (\lambda I - A)[\text{dom}(A)] \). Clearly, \( \rho(A) \) is a subset of \( \rho_{P(0)}(A) \). This resolvent set is not only in this context the more natural one, since there is an explicit connection to the operator \( P(0) \). Note that for bijective \( P(0) \), \( \rho_{P(0)}(A) \) is the usual resolvent set.

The last theorems do not answer the question ”When do we get a pre-semigroup solution?” Basicaly we can not even expect to get such a solution since \( A \) can not be expected to be a generator of a pre-semigroup without further information. For that we introduce an operator \( B \) which is defined on a subset of \( \text{dom}(A) \). This \( B \) has some abilities that guarantee the existence of a pre-semigroup that is generated by an extension of \( A \).

THEOREM 2.8 Let \( A, B \) be closed operators, which are related as follows.

- \( \text{dom}(B) \subset \text{dom}(A) \);
- \( 0 \in \rho(B) \);
- \( \exists \lambda \in \rho(A), \lambda > 0 : R_{\lambda, A} B x = B R_{\lambda, A} x \, \forall x \in \text{dom}(B) \)

Then following assertions are equivalent:

1. The ACP for \( A \) has a unique solution for each initial value \( c \in \text{dom}(B) \).
2. There exists a pre-semigroup \( \{ P(t) \}_{t \geq 0} \) generated by an extension \( A_P \) of \( A \), such that \( P(0) = (\lambda I - A)B^{-1} \) and \( A \) commutes with \( P(s) \) for all \( s \geq 0 \).

PROOF: Notice that \( B^{-1} = R_{0,B} \) exists since \( 0 \in \rho(B) \) by assumption.

1. \( \Rightarrow \) 2.

Let \( u_c \in C^1([0,\infty) ; X) \) be the unique solution of the ACP for the initial value \( c \in \text{dom}(B) \). We have to construct a pre-semigroup \( P(.) \). For \( x \in X \), \( B^{-1}x \in \text{dom}(B) \) and hence the expression

\[
P(.)x := (\lambda I - A)u_{B^{-1}x}(.) = \lambda u_{B^{-1}x}(.) - u'_{B^{-1}x}(.), \quad (2.5)
\]

is well defined. We are going to show that \( P(.) \) is a pre-semigroup which is generated by an extension of \( A \). This includes following tasks:
• $P(.)$ is strongly continuous; This follows from the term on the right side in (2.5), since $u_{B^{-1}x} \in C^1([0, \infty))$.

• $P(0)$ is injective, because $P(0)x = (\lambda I - A)B^{-1}x$ and $\lambda \in \rho(A)$.

• $P(t) \in \mathcal{B}(X)$ for all $t \in [0, \infty)$; $P(t) : X \to X$ is linear, due to $B^{-1} \in \mathcal{B}(X)$ and the uniqueness of the solution of the ACP for a given initial value: Let $x, y \in X$ and $k \in \mathbb{C}$ (it is clear that $u_{B^{-1}x} + u_{B^{-1}ky}$ solves the ACP for $u(0) = B^{-1}x + kB^{-1}y$)

$$P(t)(x + ky) = (\lambda I - A)u_{B^{-1}x + B^{-1}ky}(t)$$
$$= (\lambda I - A)(u_{B^{-1}x}(t) + ku_{B^{-1}y}(t))$$
$$= P(t)x + kP(t)y.$$ 

For the boundedness of $P(t)$ we regard the operator

$$W(.) : X \to C^1([0, a] ; X) : x \mapsto W(.)x = u_{B^{-1}x}(.) ,$$

and $a \geq 0$ fixed. Here, $C^1([0, a] ; X)$ is equipped with the norm $\|u\|_{C^1} = \|u(t)\|_{\infty} + \|u'(t)\|_{\infty}$ (see: Notation, Definitions & Elementary Results).

We show that $W(.)$ is closed. Let $x_n \to x$ in $X$, and $W(.)x_n \to y$ in $C^1([0, a] ; X)$. From the definition of $W(.)$ and the convergence of $W(.)x_n$ in the $\|.\|_{C^1}$-Norm (this implies pointwise convergence) it follows for fixed $t \leq a$ that

$$AW(t)x_n = Au_{B^{-1}x_n}(t) = u'_{B^{-1}x_n}(t) = [W(.)x_n]'(t) \to y'(t).$$

We now regard the sequences $W(t)x_n \to y(t)$ and $AW(t)x_n \to y'(t)$. Using the fact that $A$ is closed, we get $y(t) \in dom(A)$ and $y'(t) = Ay(t)$. Furthermore,

$$y(0) = \lim_{n \to \infty} W(0)x_n = \lim_{n \to \infty} u_{B^{-1}x_n}(0) = \lim_{n \to \infty} B^{-1}x_n = B^{-1}x.$$ 

The uniqueness of the solution of the ACP with the initial value $B^{-1}x \in dom(B)$ yields $y = u_{B^{-1}x} = W(.)x$ on $[0, a]$ (the second equality holds per definitionem). So $W(.)$ is closed. As $C^1([0, a] ; X)$ is a Banach space, by the Closed Graph Theorem $W(.)$ is even bounded, i.e. $\exists M > 0$: $\|W(.)x\|_{C^1} \leq M \|x\|$ $\forall x \in X$. Since $\lambda > 0$ it follows from (2.5) for a fixed $t \leq a$ that

$$\|P(t)x\| = \|(\lambda I - A)W(t)x\| \leq \|\lambda W(t)x\| + \|[W(.)x]'(t)\|$$
$$\leq (\lambda + 1)(\|W(t)x\| + \|[W(.)x]'(t)\|)$$
$$\leq (\lambda + 1)\|W(.)x\|_{C^1}$$
$$\leq (\lambda + 1)M \|x\| .$$

Since $a$ can be chosen big enough for each $t$, so that $t \leq a$, $P(t)$ is in $\mathcal{B}(X)$ for all $t \in [0, \infty)$. 

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\begin{itemize}
  \item \( AP(t)x = P(t)Ax \) for all \( x \in \text{dom}(A) \) and \( t \in [0, \infty) \);
  \[
P(t)Ax = AP(t)x \quad \forall x \in \text{dom}(A) \tag{2.6}
\]
  \[
\Leftrightarrow P(t)(\lambda I - A)x = (\lambda I - A)P(t)x \quad \forall x \in \text{dom}(A) \tag{2.7}
\]
  \[
\Leftrightarrow R_{\lambda,A}P(t)c = P(t)R_{\lambda,A}c, \quad \forall c \in X \tag{2.8}
\]
\end{itemize}

it suffices to show (2.8). For that we consider \( y(\cdot) := R_{\lambda,A}u_{B^{-1}c}(\cdot) \) for \( c \in X \). Then (using that an operator commutes with its resolvent, see LEMMA 0.1)

\[
\frac{d}{dt}y = R_{\lambda,A} \frac{d}{dt}u_{B^{-1}c} = R_{\lambda,A}Au_{B^{-1}c} = Ay.
\]

Because \( BR_{\lambda,A}d = R_{\lambda,A}Bd \) for all \( d \in \text{dom}(B) \) by assumption, it follows

\[
BR_{\lambda,A}d = R_{\lambda,A}Bd \quad \forall d \in \text{dom}(B) \tag{2.9}
\]
\[
\Leftrightarrow R_{\lambda,A}d = B^{-1}R_{\lambda,A}Bd \quad \forall d \in \text{dom}(B) \tag{2.10}
\]
\[
\Leftrightarrow R_{\lambda,A}B^{-1}c = B^{-1}R_{\lambda,A}c \quad \forall c \in X. \tag{2.11}
\]

Hence,

\[
y(0) = R_{\lambda,A}B^{-1}c = B^{-1}R_{\lambda,A}c \in \text{dom}(B).
\]

So \( y \) solves ACP with initial value \( R_{\lambda,A}B^{-1}c \). By uniqueness of the solution, it follows

\[
R_{\lambda,A}u_{B^{-1}c}(\cdot) = u_{R_{\lambda,A}B^{-1}c}(\cdot). \tag{2.12}
\]

Now we prove (2.8). Using the definition of \( P(t)x \) in (2.5), (2.12) and the commutativity of the resolvents (2.11), we get

\[
R_{\lambda,A}P(t)x = u_{B^{-1}x}(t) = (\lambda I - A)R_{\lambda,A}u_{B^{-1}x}(t)
= (\lambda I - A)u_{R_{\lambda,A}B^{-1}x}(t)
= (\lambda I - A)u_{B^{-1}R_{\lambda,A}x}(t)
= P(t)R_{\lambda,A}x.
\]

\begin{itemize}
  \item \( P(t - u)P(u) \) is indepent of \( u \) for \( 0 \leq u \leq t \) and \( P(\cdot) \) is generated by an extension of \( A \); Let \( c \in \text{dom}(A) \). Clearly, we can write \( c = R_{\lambda,A}d \) for a certain \( d \in X \). With (2.11) and (2.12) it follows that

  \[
P(\cdot)c = (\lambda I - A)u_{B^{-1}c}(\cdot)
= (\lambda I - A)u_{B^{-1}R_{\lambda,A}d}(\cdot)
= (\lambda I - A)u_{R_{\lambda,A}B^{-1}d}(\cdot)
= u_{B^{-1}d}(\cdot).
\]

  Hence, \( P(\cdot)c \) solves the ACP since the initial value \( P(0)c = B^{-1}d \in \text{dom}(B) \) for each \( c \in \text{dom}(A) \). So together with the points above, the conditions of theorem (2.5) are fulfilled (\( \rho(A) \) is non-empty by assumption). This theorem completes the proof of this direction.
\end{itemize}
Let $P(.)$ be a pre-semigroup generated by an extension $A_P$ of $A$, with $P(0) = (\lambda I - A)B^{-1}$ and Furthermore, let $A$ commute with $P(s)$ for all $s \geq 0$. We want to show that the ACP
\[ u' = Au; \quad u(0) = c, \]  
has a unique solution for each $c \in \text{dom}(B)$. From COROLLARY 2.3 it is clear that for $d \in \text{dom}(A) \subseteq \text{dom}(A_P)$ the function $P(.)d$ is the unique solution of the ACP
\[ u' = A_Pu; \quad u(0) = P(0)d. \]
As $d \in \text{dom}(A)$ we have $Ad = A_Pd$. $A_P$ commutes with $P(s)$ for all $s \geq 0$ by THEOREM 1.9. Because $A$ commutes with $P(s)$ for all $s \geq 0$ by assumption, it follows that $A_P(.)d = A_PP(.)d$. Hence, $P(.)d$ solves ACP (2.13) with initial value $c := P(0)d$ uniquely. Since $\text{dom}(A) = R_{\lambda,A}X$ and $R_{\lambda,A}$ commutes with the operators $P(s), s \geq 0$ (see LEMMA 0.1 in Notation, Definitions and Elementary Results), it follows from $P(0) = (\lambda I - A)B^{-1}$ that
\[ P(0)\text{dom}(A) = P(0)R_{\lambda,A}X = R_{\lambda,A}P(0)X = R_{\lambda,A}(\lambda I - A)B^{-1}X = \text{dom}(B), \]
which shows $c \in \text{dom}(B)$. Hence, for a given $c \in \text{dom}(B)$, $P(0)^{-1}c \in \text{dom}(A)$ and $u = P(.)P(0)^{-1}c$ is the unique solution of (2.13).
Chapter 3

Exponentially tamed pre-semigroups

In the following we will analyse pre-semigroups with an additional property. This reduction will give us more power in creating a similar situation as there is in the theory of strongly continuous semigroups.

**DEFINITION 3.1** A pre-semigroup \( \{P(t)\}_{t \geq 0} \) is **exponentially tamed**, if there exists \( \omega > 0 \) so that
\[
fx : [0, \infty) \to X : t \mapsto e^{-\omega t}P(t)x,
\]
is bounded and uniformly continuous for all \( x \in X \).

In the theory of common semigroups such a relation emerges as a property of strongly continuous semigroups. There we have constants \( M, a > 0 \) so that
\[
\|P(t)\| \leq Me^{at} \quad \text{for all} \quad t \in [0, \infty),
\]
Using this (and properties of a semigroup) we see
\[
\|e^{-a(t+h)}P(t+h)x - e^{-at}P(t)x\| \leq e^{-at}\|P(t)\|\|e^{-ah}P(h)x - x\|
\leq M\|e^{-ah}P(h)x - x\| \quad \forall t \in [0, \infty), x \in X,
\]
which implies the uniform continuity of \( t \mapsto e^{-at}P(t)x \), where \( \omega = a \). Concerning the boundedness we have
\[
\|e^{-at}P(t)x\| \leq e^{-at}\|P(t)\|\|x\| \leq e^{-at}Me^{at}\|x\| = M\|x\|,
\]
for all \( t \in [0, \infty) \) and each fixed \( x \in X \). Therefore DEFINITION 3.1 is also a generalisation of the situation of a strongly continuous semigroup.

**REMARK 3.2** We want to point out that for a pre-semigroup which is exponentially tamed, \( M := \sup_{t \geq 0}e^{-\omega t}\|P(t)\| \) exists. This follows directly from the principle of uniform boundedness theorem, since \( \sup_{t \geq 0}\|e^{-\omega t}P(t)x\| \leq M_x \) for all \( x \in X \) by definition.

**DEFINITION 3.3** For an exponentially-tamed pre-semigroup \( \{P(t)\}_{t \geq 0} \), let \( Y \) be the vector space
\[
Y := \{ x \in X : f_x(t) \in P(0)X \quad \forall t \geq 0, \ P(0)^{-1}f_x \in C_b([0, \infty); X) \},
\]
normed by \( \|x\|_Y := \|P(0)^{-1}f_x\|_b = \sup_{t \geq 0}\|e^{-\omega t}P(0)^{-1}P(t)x\| \).
REMARK 3.4 $Y$ is clearly a vector space because of linearity (in $x$) of $P(0)^{-1}f_x$ and since $C_b([0, \infty); X)$ is a vector space.

An element of $Y$ has to fulfill two strong conditions concerning the operator $P(0)^{-1}$. First of all $e^{-\omega t}P(tx)$ has to be in $P(0)X$ for all $t \geq 0$ so that the term is well-defined. Further $P(0)^{-1}$ has to support uniform continuity and boundedness of $f_x : [0, \infty) \to X : t \mapsto e^{-\omega t}P(tx)$. At this point it is not clear how strong these requests are, and how big this restriction for $x$ in $X$ is. We will analyse this later on.

As $\|x\| = \|P(0)^{-1}e^0P(0)x\| \leq \sup_{t \geq 0} \|P(0)^{-1}e^{-\omega t}P(t)x\|$ clearly $\|\cdot\| \leq \|\cdot\|_Y$ on $Y$.

The fact that $C_b([0, \infty); X)$ is a Banach space, gives us even more.

**LEMMA 3.5** Let $\{P(t)\}_{t \geq 0}$ be a exponentially-tamed pre-semigroup. Then the normed space $(Y, \|\cdot\|_Y)$ is a Banach space.

**PROOF:** Let $\{x_n\}$ be a Cauchy sequence in $(Y, \|\cdot\|_Y)$. From $\|\cdot\| \leq \|\cdot\|_Y$ it follows that $\{x_n\}$ is also Cauchy in $X$ and hence has a limit $x \in X$.

From definition of $(Y, \|\cdot\|_Y)$ we know that the sequence of functions $P(0)^{-1}f_{x_n}$ is Cauchy in $C_b([0, \infty); X)$, hence converges to $g \in C_b([0, \infty); X)$, i.e. $P(0)^{-1}f_{x_n} \overset{C_b}{\to} g$. This convergence (in $\|\cdot\|_{C_b}$) especially implies pointwise convergence in $X$, i.e. $P(0)^{-1}f_{x_n}(t) \overset{X}{\to} g(t)$ for all fixed $t \geq 0$. By continuity of $P(0)$ we get

$$P(0)P(0)^{-1}f_{x_n}(t) = f_{x_n}(t) = e^{-\omega t}P(t)x_n \overset{X}{\to} P(0)g(t).$$

Since $P(t)$ (for fixed $t \geq 0$) is continuous, $e^{-\omega t}P(t)$ is continuous and therefore (with $x_n \overset{X}{\to} x$, as mentioned above)

$$f_{x_n}(t) \overset{X}{\to} e^{-\omega t}P(t)x = f_x(t) = P(0)g(t).$$

So $P(0)^{-1}e^{-\omega t}P(t)x = P(0)^{-1}f_x(t) = g(t)$ for all $t \geq 0$, hence $P(0)^{-1}f_x \in C_b([0, \infty); X)$ and $P(0)^{-1}f_{x_n} \overset{C_b}{\to} P(0)^{-1}f_x$. Therefore $x_n$ converges to $x$ in $\|\cdot\|_Y$ and $x \in Y$. [Q.E.D.]

Our target is to construct a strongly continuous semigroup on this dedicated space $Y$, where we want the generator of the semigroup to correspond to the generator $A$ of the given pre-semigroup. In this context the phrase part of $A$ in $Y$ will be used (see: Notation, Definitions and Elementary Results). Before, we state a lemma concerning the Laplace transform of a pre-semigroup.

**LEMMA 3.6** For a given exponentially-tamed pre-semigroup $\{P(t)\}_{t \geq 0}$ with generator $A$, the integral

$$L_P(\lambda)x := \int_0^\infty e^{-\lambda t}P(t)x \, dt, \quad (3.1)$$

exists for $\lambda > \omega$ and $x \in X$. The Laplace transform $L_P : (\omega, \infty) \to \mathcal{B}(X) : \lambda \mapsto L(\lambda)$ satisfies

$$L_P(\lambda)[(\lambda I - A)x] = P(0)x, \quad (3.2)$$

for $\lambda > \omega$ and $x \in \text{dom}(A)$. In particular, if $\{P(t)\}_{t \geq 0}$ is semigroup, then $(\omega, \infty) \subset \rho(A)$. 

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PROOF: Because of REMARK 3.2 the following holds true
\[
\|e^{-\lambda t}P(t)x\| \leq e^{-\lambda t}\|P(t)\| \|x\| \leq e^{-\lambda t}Me^{\omega t} \|x\| = M \|x\| e^{(\omega-\lambda)t},
\]
for \(x \in X, t \geq 0\). Therefore the integral \(\int_0^\infty M \|x\| e^{(\omega-\lambda)t}dt = \frac{M}{\lambda-\omega} \|x\|\) clearly exists. Hence and since \((t \mapsto e^{-\lambda t}P(t)x)\) is continuous, (3.1) exists for \(\lambda > \omega\) (see: Notation, Definitions and Elementary Results).

Further we show that \(L_P(\lambda)\) is in \(\mathcal{B}(X)\) for \(\lambda > \omega\). Linearity is trivial from the definition of the integral. As seen above, we have
\[
\|L_P(\lambda)\| = \left\|\int_0^\infty e^{-\lambda t}P(t)x \, dt\right\| \\
\leq \int_0^\infty \|e^{-\lambda t}P(t)x\| \, dt \\
\leq \int_0^\infty M \|x\| e^{(\omega-\lambda)t} \, dt \\
= \frac{M}{\lambda-\omega} \|x\|,
\]
which gives us the boundedness.

Let \(\lambda > \omega\) and \(x \in \text{dom}(A)\). Consider
\[
L_P(\lambda)(\lambda I - A)x = \int_0^\infty e^{-\lambda t}P(t)(\lambda I - A)x \, dt \\
= \int_0^\infty [\lambda e^{-\lambda t}P(t)x - e^{-\lambda t}AP(t)x] \, dt,
\]
where we use that \(A\) and \(P(t)\) commute. With the product rule, LEMMA 1.11, we see that the integrand equals \(-[e^{-\lambda t}P(t)x]'\). This yields
\[
L_P(\lambda)(\lambda I - A)x = -\int_0^\infty [e^{-\lambda t}P(t)x]' \, dt \\
= -\left[\left[e^{-\lambda t}P(t)x\right]^\infty_0\right] \\
= P(0)x.
\]

Let \(x \in X\). For the term \((\lambda I - A)L_P(\lambda)x\) we consider the strong right derivative of \(P(.)[L_P(\lambda)x]\) at zero. With \(P(s) \in \mathcal{B}(X), s \geq 0\), (COM) and (ADD) it follows,
\[
\frac{1}{h}(P(h) - P(0)) [L_P(\lambda)x] = \frac{1}{h} (P(h) - P(0)) \int_0^\infty e^{-\lambda t}P(t)x \, dt \\
= \frac{1}{h} \left(\int_0^\infty e^{-\lambda t}[P(0)P(h + t) - P(0)P(t)]x \, dt\right) \\
= \frac{1}{h} \int_0^\infty e^{-\lambda(t-h)}P(0)P(h + t)x - \frac{1}{h} \int_0^\infty e^{-\lambda t}P(0)P(t)x \, dt \\
= \frac{e^{h\lambda} - 1}{h} \int_0^\infty e^{-\lambda s}P(0)P(s)x \, ds - \frac{e^{h\lambda}}{h} \int_0^h e^{-\lambda s}P(0)P(s)x \, ds \\
= \frac{e^{h\lambda} - 1}{h} P(0) \int_0^\infty e^{-\lambda s}P(s)x \, ds - \frac{e^{h\lambda}}{h} P(0) \int_0^h e^{-\lambda s}P(s)x \, ds,
\]
where the last equality holds since $P(0)$ is continuous and since
\[
\int_0^\infty e^{-\lambda s} P(s) x \, ds = \lim_{\beta \to \infty} \int_0^\beta e^{-\lambda s} P(s) x \, ds
\]
exists (as showed in beginning of this proof). Consider the limit of the first term in the last equality,
\[
\lim_{h \to 0^+} e^{h\lambda} - \frac{1}{h} P(0) \int_0^\infty e^{-\lambda s} P(s) x \, ds = \lim_{h \to 0^+} e^{h\lambda} - \frac{1}{h} P(0) L_P(\lambda) x
\]
exists
\[
= \lambda P(0) L_P(\lambda) x.
\]
Since $(s \mapsto e^{-\lambda s} P(s) x)$ is continuous, it follows from the fundamental theorem of calculus, that
\[
\lim_{h \to 0^+} \frac{1}{h} \int_0^h e^{-\lambda s} P(s) x \, ds = P(0) x.
\]
Therefore, the limit of the second term is
\[
\lim_{h \to 0^+} e^{h\lambda} h P(0) \int_0^h e^{-\lambda s} P(s) x \, ds = P(0) P(0) x.
\]
Hence,
\[
P^{\prime+}(0)[L_P(\lambda)x] = \lambda P(0) L_P(\lambda) x - P(0) P(0) x.
\]
Therefore the strong right derivative exists and belongs to $P(0)X$. Thus
\[
AL_P(\lambda) x = \lambda L_P(\lambda) x - P(0) x.
\]
This gives us
\[
(\lambda I - A)L_P(\lambda) x = P(0) x,
\]
for $x \in X$. If \{P(t)\}_{t \geq 0} is even a semigroup, hence especially $P(0) = I$. Therefore, we have, with the calculation from above,
\[
(\lambda I - A)L_P(\lambda) x = L_P(\lambda)(\lambda I - A)x = x,
\]
for $x \in \text{dom}(A)$. Hence $(\omega, \infty) \subset \rho(A).$ 

\textbf{THEOREM 3.7} Let $\{P(t)\}_{t \geq 0}$ be an exponentially-tamed pre-semigroup with generator $A$. Then, $\{T(t)\}_{t \geq 0}$ with
\[
T(\cdot) := P(0)^{-1} P(\cdot),
\]
is a strongly continuous semigroup on the Banach space $(Y, \|\cdot\|_Y)$ which satisfies
\[
\|T(t)\|_{B(Y)} \leq e^{\omega t},
\]
for all $t \geq 0$. The generator of $\{T(t)\}_{t \geq 0}$ equals the part of $A$ in $Y$, $A_Y$. 

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PROOF: Let $x$ be in $Y$ and fix $s \geq 0$. Clearly, $P(0)^{-1}e^{-\omega t}P(s)x$ exists by definition of $Y$, hence $T(s)x = P(0)^{-1}P(s)x$ exists by linearity. Further, we have to show that $T(s)x$ is in $Y$, which is $P(0)^{-1}f_{T(s)x} \in C_b([0\infty); X)$. By definition, for $t \geq 0$ we have

$$P(0)^{-1}f_{T(s)x}(t) = P(0)^{-1}e^{-\omega t}P(t)T(s)x = P(0)^{-1}e^{-\omega t}P(t)P(0)^{-1}P(s)x.$$ 

$P(t)P(0)^{-1}P(s)x$ exists by the argument from above. $P(0)^{-1}P(t)P(s)x$ is equal to $P(0)^{-1}P(0)P(t+s)x = P(t+s)x$ by the properties of pre-semigroups (ADD) and therefore exists, too. Hence and because of commutativity (COM) of the operators $P(s), s \in [0, \infty)$, $P(t)$ and $P(0)^{-1}$ commute (see: Notation, Definitions and Elementary Results: LEMMA 0.2) and so

$$P(t)P(0)^{-1}P(s)x = P(t+s)x. \quad (3.3)$$ 

This gives

$$P(0)^{-1}f_{T(s)x}(t) = P(0)^{-1}e^{-\omega t}P(t+s)x = e^{-\omega s}P(0)^{-1}e^{-\omega(t+s)}P(t + s)x.$$ 

Hence $P(0)^{-1}f_{T(s)x}$ belongs to $C_b([0, \infty); X)$ since $t \mapsto P(0)^{-1}e^{-\omega t}P(t)x \in C_b([0\infty); X)$ ($s$ is fixed). So we have shown that $T(s)x$ is in $Y$. As a composition of linear operators, $T(s): Y \to Y$ is also linear. The boundedness of $T(s): Y \to Y$ is proved as follows. By definition, properties of pre-semigroups and (3.3) we get

$$\|T(s)x\|_Y = \sup_{t \geq 0} \left\| P(0)^{-1}e^{-\omega t}P(t)P(0)^{-1}P(s)x \right\|$$

$$\leq \sup_{t \geq 0} \left\| P(0)^{-1}e^{-\omega t}P(t + s)x \right\|$$

$$= e^{\omega s} \sup_{t \geq 0} \left\| P(0)^{-1}e^{-\omega(t+s)}P(t + s)x \right\|$$

$$\leq e^{\omega s} \|x\|_Y.$$ 

This also shows $\|T(t)\|_{B(Y)} \leq e^{\omega t}$ for $t \geq 0$.

The semigroup properties are obviously satisfied by definition: $T(0) = P(0)^{-1}P(0) = I$ and for $x \in Y, s, t \geq 0$

$$T(s + t)x = P(0)^{-1}P(s + t)x$$

$$= P(0)^{-1}P(0)^{-1}P(s)P(t)x$$

$$= P(0)^{-1}P(s)P(0)^{-1}P(t)x$$

$$= T(s)T(t)x,$$ 

where we use that $P(s)$ and $P(0)^{-1}$ commute (as above, see: Notation, Definitions and Elementary Results: LEMMA 0.2).

Next, we verify the $C_0$-property. It suffices to show the strong continuity of $T(.)$ at zero, clearly in the $\|.\|_Y$-norm. For that we consider the uniform continuity of $P(0)^{-1}f_x$ for $x \in Y$. For $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ so that for all $|h| < \delta_\epsilon$

$$\sup_{t \geq 0} \left\| P(0)^{-1}e^{-\omega(t+h)}P(t + h)x - P(0)^{-1}e^{-\omega t}P(t)x \right\| < \epsilon \quad (3.4)$$
With the definition and (3.3), it follows elemtarily
\[
\|T(h)x - x\|_Y = \sup_{t \geq 0} \|P(0)^{-1}e^{-\omega t}P(t)[P(0)^{-1}P(h)x - x]\|
\]
\[
= \sup_{t \geq 0} \|P(0)^{-1}e^{-\omega t}P(t + h)x - P(0)^{-1}e^{-\omega t}P(t)x\|
\]
\[
\leq \sup_{t \geq 0} e^{\omega h} \|P(0)^{-1}e^{-\omega(t + h)}P(t + h)x - P(0)^{-1}e^{-\omega t}P(t)x\| + 
\]
\[
+ \sup_{t \geq 0} \|(e^{\omega h} - 1)P(0)^{-1}e^{-\omega t}P(t)x\|. 
\]

From (3.4) we get
\[
\|T(h)x - x\|_Y \leq e^{\omega h} \epsilon + (e^{\omega h} - 1)\|x\|_Y. 
\]

This gives us strong continuity: For fixed \(x \in Y\) and a given \(\epsilon'\) choose \(\epsilon < \epsilon'\) in (3.4). Clearly, \(e^{\omega h}\) converges to 1 for \(h \to 0\), hence \(e^{\omega h} \epsilon \to \epsilon\) and \((e^{\omega h} - 1)\|x\|_Y \to 0\) and therefore \(\|T(h)x - x\|_Y < \epsilon'\) holds true for sufficiently small \(h\).

It remains to show that the generator \(A_T\) of the semigroup \(T(.)\) equals, the part \(A_Y\) of \(A\) in \(Y\). Let \(x \in \text{dom}(A_T) \subseteq Y\), then \(\lim_{t \to 0} \frac{1}{t}[T(h)x - x]\) exists in \(\|\|_Y\), hence in \(\|\|\) due to the fact that \(\|\|_Y \geq \|\|\). By definition of \(T\) we have \(P(.) = P(0)T(.)\). \(P(0)\) is in \(\mathcal{B}(X)\) and, therefore, (following limits are in \(\|\|)\)
\[
\lim_{h \to 0} \frac{1}{h}[P(h)x - P(0)x] = \lim_{h \to 0} P(0)^{-1} \frac{1}{h}[P(h)x - x] 
\]
\[
= P(0) \lim_{t \to 0} \frac{1}{t}[T(h)x - x] 
\]
\[
= P(0)A_Tx. 
\]

Hence \(P^t(0)\) exists and lies in \(P(0)X\). So \(x \in \text{dom}(A)\) and \(Ax = P(0)^{-1}P(0)A_Tx = A_Tx\).

Thus \(A_T \subseteq A\), and even \(A_T \subseteq A_Y\) since \(A_T\) is defined on a subset of \(Y\).

We regard the operator \((\lambda I - A)\) for an aproiate \(\lambda \in \mathbb{R}\): From LEMMA 3.6 we know that for \(\lambda > \omega\) the Laplace transform \(L_P(\lambda) \in \mathcal{B}(X)\) exists. The relation (3.2), \(L_P(\lambda)(\lambda I - A)x = P(0)x\), implies that if \((\lambda I - A)x = 0\), then \(P(0)x = 0\) and therefore \(x = 0\) since \(P(0)\) is injective. Consequently \((\lambda I - A)\) is injective, hence \((\lambda I - A_Y)\) is also injective.

LEMMMA 3.6 can also be applied with the semigroup \(T(.)\) (on \(Y\)). Thus \(\lambda > \omega\) is in \(\rho(A_T)\) which means that \((\lambda I - A_T)^{-1} : X \to \text{dom}(A_T)\) is bijective. In particular \((\lambda I - A_T)\) is surjective. We know already \(A_T \subseteq A_Y\), hence
\[
\lambda I - A_T \subseteq \lambda I - A_Y, 
\]

where the map on the left hand side is surjective and injective on the right hand side. Such a relation implies already that
\[
\lambda I - A_T = \lambda I - A_Y, 
\]

(see: Notation, Definitions and Elementary Results LEMMA 0.3). Obviously this is equivalent to
\[
A_T = A_Y, 
\]
which completes the proof.

**REMARK 3.8** This result is surprising in a way. But how big is the space $Y$?! We have already mentioned in REMARK 3.4 that the elements of $Y$ have to fulfill some strong requirements and therefore depending especially on the operator $P(0)$. Let us consider the special case where $P(0)$ is bijective. By the Closed Graph theorem, $P(0)^{-1} \in \mathcal{B}(X)$ and therefore $P(0)^{-1}f_x$ is clearly bounded and uniformly continuous for all $x \in X$ since $f_x \in C_b([0, \infty); X)$ by definition. That is, $Y = X$. 

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Bibliography


