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Some polymeric fluid flows models: steady states & large-time-convergence

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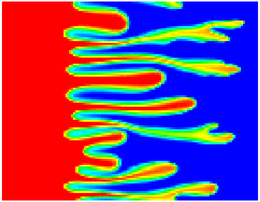
Dilute polymer suspension – applications

- multi-grade motor oil:
  polymer additives to improve/tune viscosity $\nu(p, T)$
- enhanced (tertiary) oil recovery:
  strong fingering at oil/water interface
  $\rightarrow$ front stabilization with polymer additives (increase viscosity of water)



- food industry:
  polymer additives to thicken sauces, ...

Macro Model: fluid flow

- dilute solution of polymers in homogeneous fluid
- coupled micro-macro model
- incompressible Navier-Stokes for macro flow $u(t,x)$:

$$\begin{aligned} u_t + (u \cdot \nabla_x) u &= \Delta_x u - \nabla_x p + \mathrm{div}_x \tau, \ \Omega \subset \mathbb{R}^d \\ \mathrm{div}_x u &= 0 \end{aligned}$$

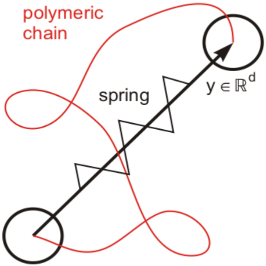
- coupling to polymer-model via extra stress tensor:

$$\tau(t,x) = \int_{\mathbb{R}^d} (y \otimes \nabla_y \Pi(y)) \psi(t,x,y) \, dy$$

(all parameters :=1)

Micro Model: polymer distribution

- dumbbell model for polymeric chains: $y \in \mathbb{R}^d$ … extension, orientation



- micro-distribution (probility densitity in y) at each $x \in \Omega : \psi(t, x, y)$ in Fokker-Planck equ.:

$$\psi_t + \underbrace{u \cdot \nabla_x \psi}_{\substack{\text{transport} \\ \text{in flow}}} = \frac{1}{2}\mathrm{div}_y\Big(\big[\underbrace{\nabla_y \Pi(y)}_{\substack{\text{spring force} \\ \text{in dumbbells}}} \;-2\; \underbrace{(\nabla_x \otimes u)^T \cdot y}_{\substack{\text{drag force of} \\ \text{inhom. flow field } u}}\big]\psi\Big) + \frac{1}{2}\Delta_y \psi$$

Results from [Jourdain-LeBris-Lelièvre-Otto, ARMA 2006]

linear FP: $\psi_t = \frac{1}{2}\mathrm{div}_y([\nabla_y \Pi(y) - 2\kappa y]\psi + \nabla\psi)$, $y \in \mathbb{R}^d$; const: $\kappa \in \mathbb{R}^{d\times d}$

1. Hookean, $\Pi = \frac{1}{2}|y|^2$:

Theorem

κ symmetric with $\lambda_j(\kappa) < \frac{1}{2}$, or κ anti-symmetric:
$\Rightarrow \exists!$ steady state ψ_∞; exp. convergence of $\psi(t)$, $t \to \infty$; general κ open !

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3. coupled nonlin. model ($\to$ log. relative entropy of $\psi + \|u - u_\infty\|^2_{L^2}$)

Theorem

FENE, if $|\kappa^s| < \frac{1}{2}$, $Tr\,\kappa = 0$, $[\kappa, \kappa^T]$ small (κ from BC/u_∞-steady state)
$\Rightarrow$ *exp. convergence of $(u, \psi) \stackrel{t\to\infty}{\longrightarrow} (u_\infty, \psi_\infty)$; Hookean open !*

Outline:

1. linear Fokker-Planck: steady state & large time convergence (entropy method)
   a) non-symmetric Fokker-Planck: steady state, entropy decay
   b) Hookean dumbbells
   c) FENE dumbbells [=finite extensibility nonlinear elasticity]
2. coupled micro-macro model (Hookean)

1. Linear Fokker-Planck equ.

assume: given homogeneous flow with $u(x) = \kappa x, \kappa \in \mathbb{R}^{d\times d}$

$\rightarrow$ FP equ. for dumbbell distribution $\psi(t,y)$ in micro-variable $y \in \mathbb{R}^d$

$$\begin{cases} \psi_t = L\psi := \frac{1}{2}\mathrm{div}_y\big(\underbrace{[\nabla_y \Pi(y) - 2\kappa y]}_{\text{not gradient!}}\psi + \nabla_y \psi\big) \\ \psi(0,y) = \psi_0(y) \geq 0 \end{cases}$$

$$\psi(t,y) \geq 0, \quad \int_{\mathbb{R}^d} \psi(t,y)\, dy = \int_{\mathbb{R}^d} \psi_0\, dy = 1$$

a) non-symmetric Fokker-Planck equ.

symmetric FP:

$$\begin{aligned} \psi_t &= L\psi = \text{div}(\nabla A(y)\psi + \nabla\psi) = \text{div}(\psi_\infty \nabla \frac{\psi}{\psi_\infty}); \\ \psi_\infty &= e^{-A} \ \ldots \text{ steady state, normalized} \\ L & \ \ldots \ \text{symmetric in } L^2(\psi_\infty^{-1}\, dy) \end{aligned}$$

relative entropy: $e(\psi|\psi_\infty) := \int_{\mathbb{R}^d} \psi(y) \ln \frac{\psi}{\psi_\infty}\, dy \geq 0$

entropy dissipation: $\frac{d}{dt} e(\psi(t)|\psi_\infty) = -\frac{1}{2}\int \left|\nabla \frac{\psi(t)}{\psi_\infty}\right|^2 \frac{\psi_\infty^2}{\psi(t)}\, dy \leq 0$

If A uniformly convex, $\frac{\partial^2 A}{\partial y^2} \geq \lambda I$ (Bakry-Emery condition):
$\Rightarrow$ exp. decay of relative entropy $e(\psi(t)|\psi_\infty)$ with $e^{-\lambda t}$ (at least)

a) non-symmetric Fokker-Planck equ.

non-symmetric FP

$$\psi_t = L\psi = \mathrm{div}([\nabla A + \vec{F}]\psi + \nabla\psi)$$

$\vec{F}$ given with "orthogonality condition" $\mathrm{div}_y(\vec{F}\psi_\infty) = 0 \ \forall y$

$\Rightarrow \quad \psi_\infty = \mathrm{e}^{-A}$ still steady state

$\Rightarrow \quad$ same entropy dissipation, same decay estimate as for symm. FP

$L^{as}\psi := \mathrm{div}(\vec{F}\psi) \cdots$ skew-symmetric in $L^2(\psi_\infty^{-1}\, dy)$

$\rightarrow$ decomposition of $L = L^s + L^{as}$

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general situation:

$$\psi_t = \mathrm{div}(\vec{G}\psi + \nabla\psi)$$

$\longrightarrow$ find decomposition $\vec{G} = \nabla A + \vec{F},\ \mathrm{div}(\psi_\infty \vec{F}) = 0,\ \psi_\infty = \mathrm{e}^{-A}$;

is equivalent to find ψ_∞ (similar to Helmholtz-Hodge decomposition).

Remark for Hookean $A = |y|^2/2$: $\mathrm{div}(\vec{F}) = 0$ (a-posteriori)

b) polymer model with Hookean dumbbells: $\Pi(y)=\frac{|y|^2}{2}$

$$\psi_t = L\psi = \frac{1}{2}\mathrm{div}([y - 2\kappa y]\psi + \nabla\psi), \quad y \in \mathbb{R}^d, \ t \geq 0 \tag{1}$$

Theorem (steady state [AA-Carrillo-Manzini, 2010])

Let $-(I - 2\kappa)$ be stable (otherwise no confinement pot.), i.e. $Re\lambda_j(\kappa) < \frac{1}{2}$:

$\Rightarrow \exists!$ *normalized steady state of* (1)*:*

$$\psi_\infty(y) = c\ exp(-\frac{1}{2}y^T\Sigma^{-1}y),$$

$$0 < \Sigma = \Sigma^T = 2\int_0^\infty e^{-(I-2\kappa)s}e^{-(I-2\kappa^T)s}\,ds$$

if κ normal: $\Sigma = (I - 2\kappa^s)^{-1}$. $\exists$ standard algorithms to compute Σ from κ

Proof.

Fourier transform of steady state equ: $\operatorname{div}([y - 2\kappa y]\psi + \nabla\psi) = 0$

$$\xi^T (I - 2\kappa)\nabla_\xi \hat{\psi}(\xi) = -|\xi|^2 \hat{\psi}(\xi)$$

Use ansatz $\hat{\psi}(\xi) = c \exp(-\frac{1}{2}\xi^T \Sigma \xi)$:

$$\Rightarrow \; 0 = -(I - 2\kappa)\Sigma - \Sigma(I - 2\kappa)^T + 2I \; \ldots \text{ "continuous Lyapunov equ"}$$

Since $-(I - 2\kappa)$ stable; $2I$ pos.def, symm $\Rightarrow \exists! \; \Sigma$ □

$A(y) = -\ln \psi_\infty = \frac{1}{2} y^T \Sigma^{-1} y + c \; \ldots$ uniformly convex potential of symmetric part of L in $L^2(\psi_\infty^{-1})$

Theorem (convergence in rel. entropy [ACM]:)

Let $-(I-2\kappa)$ be stable

$$\Rightarrow \; e(\psi(t)|\psi_\infty) \le e^{-\lambda_{min}(\Sigma^{-1})t}\, e(\psi_0|\psi_\infty),\; t \ge 0,$$

with λ_{min} computable

Proof.

entropy method [AMTU,2001], Bakry-Emery cond. for symm. part of L:

$$\frac{\partial^2 A}{\partial y^2} \ge \lambda_{\min}(\Sigma^{-1})$$

□

Rem:

decay is sharp for quadratic potentials, also for non-symmetric Fokker-Planck equ. $\forall \vec{F}$ (same entropy dissipation for "optimal functions")

c) FENE - dumbbells [finite extensibility nonlinear elasticity]

$$\begin{cases} \psi_t = \frac{1}{2}\,\mathrm{div}([\nabla\Pi - 2\kappa y]\psi + \nabla\psi),\ \underbrace{|y| < \sqrt{b}}_{\mathcal{B}},\ b \geq 2,\ t > 0 \\ \Pi(y) = -\frac{b}{2}\ln(1 - \frac{|y|^2}{b}) \quad \to \text{r.h.s. degenerate elliptic} \\ \psi\big|_{\partial\mathcal{B}} = 0 \end{cases}$$

steady state for κ normal: $\psi_\infty(y) = ce^{-\Pi(y)+y^T\kappa^s y}$;
in general not explicit

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perturbation result:

$$L\psi = L_1\psi + L_2\psi := \frac{1}{2}\operatorname{div}\big([\nabla\Pi - 2\underbrace{\kappa_1}_{\text{normal !}}y]\psi + \nabla\psi\big) - \operatorname{div}(\kappa_2 y\psi)$$

ex.: $\kappa_1 = \kappa^s; \kappa_2 = \kappa^{as}$

$$\mu(y) := ce^{-\Pi(y)+y^T\kappa_1^s y}, \qquad \int_{\mathcal{B}}\mu\,dy = 1$$

$L_1\mu = 0$; pos. spectral gap $\lambda_1 > 0$ ($\exists$ estimates); L_1 symm. on $L^2(\mu^{-1}\,dy)$

solve steady state equ ($\to$ eigenvalue problem for ψ):

$$L\psi = 0 \text{ with } \int_{\mathcal{B}} \psi \, dy = 1 \text{ in } \mathcal{V} = \left\{ \frac{\psi}{\mu} \in H^1(\mu \, dy) \right\} \Rightarrow \psi|_{\partial \mathcal{B}} = 0 \qquad (2)$$

alternative formulation: $\Phi := \psi - \mu \in \mathcal{V}^{\perp} = \{\Phi \in \mathcal{V} \mid \int \Phi \, dy = 0\} \perp \mu$

$\to$ solve for $\Phi \in \mathcal{V}^{\perp}$:

$$L\Phi = -L\mu = -L_2\mu$$

Theorem (FENE - small κ_2 [ACM, 2009])

if $\sqrt{b}\|\kappa_2\|_2 < \frac{\sqrt{\lambda_1}}{2} \Rightarrow \exists!$ *normalized steady state of* (2)

Proof.

quadratic form of L_1 is coercive on $\mathcal{V}^{\perp}$,
quadratic form of L_2 is (small) bounded perturbation □

- extension $\forall \, \kappa$ [Chupin, 2009]

Theorem (FENE - arbitrary κ [AA-Bardos-Catto, 2009])

$$0 = L\psi = L_1\psi + L_2\psi := \frac{1}{2}\, div\big([\nabla\Pi - 2\kappa^s y]\psi + \nabla\psi\big) - div(\kappa^{as} y\psi)$$

has a normalized sol. $0 < \psi_\infty \in \mathcal{V} = \left\{ \frac{\psi}{\mu} \in H^1(\mu\, dy) \right\};\ \mu = ce^{-\Pi + y^T \kappa^s y}$.

Proof.

- for large $\lambda > 0$: $K_\lambda := (\lambda - L)^{-1}$ compact on $\mathcal{H} := L^2(\mathcal{B}; \mu^{-1} dy)$.
- for large $\lambda > 0$: $K_\lambda : \mathcal{H}^+ := \{\psi \in \mathcal{H} \big| \psi \geq 0\} \to \mathcal{H}^+$, using:

$$K_\lambda \psi = \int_0^\infty e^{-\lambda t} \left[\lim_{n\to\infty} \left(e^{\frac{t}{n} L_1} e^{\frac{t}{n} L_2} \right)^n \psi \right] dt$$

- Krein-Rutman: spectral radius $r(K_\lambda) > 0$ is eigenvalue of K_λ; $\psi \in \mathcal{H}^+$
- $(\lambda - L)\psi = \frac{1}{r}\psi \ \Rightarrow\ (\lambda - \frac{1}{r}) \underbrace{\int_\mathcal{B} \psi\, dy}_{>0} = \int_\mathcal{B} L\psi\, dy \overset{\text{div form}}{=} 0 \ \Rightarrow\ \lambda = \frac{1}{r}$
  $\Rightarrow\ L\psi = 0$
- $\psi_\infty(y) > 0$ on $\mathcal{B}$ by min-principle on $B_\rho(0)$, $\rho < \sqrt{b}$.

2. Coupled micro-macro model for Hookean dumbbells: $t \longrightarrow \infty$ convergence

Navier-Stokes for $u(t,x)$ on Ω :
Choose BC $u\big|_{\partial\Omega} = \kappa x$ for some $\kappa \in \mathbb{R}^{d\times d} \Rightarrow$

$$u_\infty = \kappa x,\ \psi_\infty = c\,\mathrm{e}^{-\frac{1}{2}y^T\Sigma^{-1}y} \quad \text{is steady state.}$$

$\longrightarrow$ decay of "relative entropy" (formal; if solution $\exists$):

$$E(t) := \frac{1}{2}\int_\Omega |u(t) - u_\infty|^2\,dx + \int_\Omega \int_{\mathbb{R}^d} \psi(t) \ln \frac{\psi(t)}{\psi_\infty}\,dy\,dx$$

Theorem ([ACM])

Let $\|\kappa^s\|_2, \; \sup_t \|\underbrace{\nabla_x \otimes u^s(t,\cdot)}_{deformation\ matrix}\|_{L^\infty(\Omega)}, \; \| \int_{\mathbb{R}^d} |y|^4 \psi_0(x,y)\, dy \|_{L^\infty(\Omega)}$ *be small;*

let $Re\, \lambda_j(\kappa) < \frac{1}{2}$.

$\Rightarrow \; E(t) \searrow 0$ *exponentially*

Proof.

- differential inequality between $\frac{dE}{dt}$, $E(t)$
- logarithmic Sobolev inequality for $\psi_\infty(y)$ ... Gaussian
- new weighted Csiszár-Kullback inequ. (was "missing link" in [JLLO])

□

Lemma ([ACM])

$\psi, \phi \in L^1_+(\mathbb{R}^d)$, $\int \psi = \int \phi = 1$, $|y|^4(\psi + \phi) \in L^1(\mathbb{R}^d)$

$$\Rightarrow \ \| |y|^2(\psi - \phi)\|^2_{L^1} \leq 2\, e\,(\psi|\phi) \cdot \max\Big(\int |y|^4 \psi\, dy\,,\ \int |y|^4 \phi\, dy\Big)$$

Proof.

$$e\,(\psi|\phi) = \int_{\mathbb{R}^d} \frac{\psi}{\phi} \ln \frac{\psi}{\phi}\, \phi\, dy \ \overset{2^{nd}\ \text{Taylor}}{=} \ \frac{1}{2} \int_{\mathcal{A}} \frac{1}{\zeta(y)} (\psi - \phi)^2\, dy$$

with $\mathcal{A} := \{\psi(y) \neq \phi(y)\}$; ζ = some intermediate value in (ψ, ϕ)

$$\int_{\mathcal{A}} |y|^2 |\psi - \phi|\, dy \overset{\text{Hölder}}{\leq} \Big(\int_{\mathcal{A}} \frac{1}{\zeta} (\psi - \phi)^2\, dy\Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathcal{A}} |y|^4 \zeta\, dy\Big)^{\frac{1}{2}}$$

□