Refined Convex Sobolev Inequalities

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Abstract

This paper is devoted to refinements of convex Sobolev inequalities in the case of power law relative entropies: a nonlinear entropy–entropy production relation improves the known inequalities of this type. The corresponding generalized Poincaré type inequalities with weights are derived. Optimal constants are compared to the usual Poincaré constant.

Key words and phrases: Sobolev inequality – Poincaré inequality – entropy method – diffusion – logarithmic Sobolev inequality – spectral gap

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1 Introduction and main results

In this paper, we consider convex Sobolev inequalities relating a (non-negative) convex entropy functional

$$e_{\psi}(\rho|\rho_\infty) := \int_{\mathbb{R}^n} \psi\left(\frac{\rho}{\rho_\infty}\right) d\rho_\infty$$

to an entropy production functional

$$I_{\psi}(\rho|\rho_\infty) := -\int_{\mathbb{R}^n} \psi''\left(\frac{\rho}{\rho_\infty}\right) \left|\nabla\left(\frac{\rho}{\rho_\infty}\right)\right|^2 d\rho_\infty, \quad (1.1)$$

where $\rho$ and $\rho_\infty$ belong to $L^1_+(\mathbb{R}^n, dx)$ and satisfy $\|\rho\|_{L^1(\mathbb{R}^n)} = \|\rho_\infty\|_{L^1(\mathbb{R}^n)} = M > 0$. Here we use the notation $d\rho_\infty = \rho_\infty(x) dx$. The generating function $\psi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ of the relative entropy is strictly convex and satisfies $\psi(1) = 0$.

A very efficient method to prove convex Sobolev inequalities has been developed by D. Bakry and M. Emery [3, 4] in probability theory and by A. Arnold, P. Markowich, G. Toscani, A. Unterreiter [2] in the context of partial differential equations. See [1] for a recent review. The main idea goes as follows: We consider $\rho = \rho(x,t)$ depending now on the auxiliary variable $t > 0$ (“time”). For any solution of
\[
\frac{\partial \rho}{\partial t} = \text{div} \left( D \rho_\infty \nabla \left( \frac{\rho}{\rho_\infty} \right) \right), \quad x \in \mathbb{R}^n, \ t > 0, \quad (1.2)
\]

the time evolution of the relative entropy is given by the entropy production:

\[
\frac{d}{dt} e_\psi(\rho(t)|\rho_\infty) = I_\psi(\rho(t)|\rho_\infty) \leq 0.
\]

In (1.1) and (1.2) \( D = D(x) \) denotes a (positive) scalar diffusion coefficient, and we assume \( D \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^n) \). It is also clear that \( \rho_\infty(x) \) is a steady state solution of (1.2).

For \( D \equiv 1 \), the main assumption is that \( A := -\log \rho_\infty \) is a uniformly convex function, i.e.

\[
\lambda_1 := \inf_{x \in \mathbb{R}^n} \left( \xi, \frac{\partial^2 A}{\partial x^2}(x) \xi \right) > 0.
\]

For \( D \not\equiv 1 \) the corresponding assumption reads:

\[
\exists \lambda_1 > 0 \text{ such that for any } x \in \mathbb{R}^n \nabla \right) + \frac{1}{2} \left( \nabla A \otimes \nabla D + \nabla D \otimes \nabla A \right) - \frac{\partial^2 D}{\partial x^2} \geq \lambda_1 \mathbb{I}
\]

(in the sense of positive definite matrices). Here \( \mathbb{I} \) denotes the identity matrix. In these two cases, one can prove the convex Sobolev inequality

\[
e_\psi(\rho|\rho_\infty) \leq \frac{1}{2 \lambda_1} |I_\psi(\rho|\rho_\infty)| \quad \forall \rho \in L^1_+(\mathbb{R}^n) \text{ with } \|\rho\|_{L^1(\mathbb{R}^n)} = M \quad (1.3)
\]

by computing

\[
R_\psi(\rho(t)|\rho_\infty) := \frac{d}{dt} \left[ I_\psi(\rho(t)|\rho_\infty) + 2\lambda_1 e_\psi(\rho(t)|\rho_\infty) \right]
\]

and proving that

\[
R_\psi(\rho(t)|\rho_\infty) \geq 0. \quad (1.4)
\]

Integrating this differential inequality from \( t \) to \( \infty \) then yields (1.3).

Actually, these calculations can only be carried out only for admissible relative entropies where \( \psi \in C^4(\mathbb{R}^+) \) has to satisfy

\[
2(\psi''')^2 \leq \psi'' \psi^{IV} \quad \text{on } \mathbb{R}^+.
\]

Typical and the most important – for practical applications – examples are generating functions of the form

\[
\psi_p(\sigma) = \sigma^p - 1 - p(\sigma - 1) \quad \text{for } p \in (1, 2], \quad (1.5)
\]

and

\[
\tilde{\psi}_1(\sigma) = \sigma \log \sigma - \sigma + 1.
\]
which corresponds to the limiting case of $\psi_p/(p-1)$ as $p \to 1$. With $\psi = \psi_1$, Inequality (1.3) is exactly the logarithmic Sobolev inequality found by L. Gross [8, 9], and generalized by many authors later on.

Analyzing the precise form of $R_\psi(\rho|\rho_\infty)$ allows us to identify cases of optimality of (1.3) under the assumption $D \equiv 1$. For $p = 1$ or 2, and for potentials $A$ that are quadratic in at least one coordinate direction (with convexity $\lambda_1$) there exist extremal functions $\rho = \rho_{ex} \neq \rho_\infty$ such that (1.3) becomes an equality, cf. [2]. Some of these optimality results were already noted by E. Carlen [6], M. Ledoux [12], and G. Toscani [14].

The non-optimality of the other cases may have two reasons: either $\lambda_1$ from (A1), (A2) is not the sharp convex Sobolev constant (an example for this is $A(x) = x^4$, $x \in \mathbb{R}$: see §3.3 of [2]), or there exists no extremal function to saturate (1.3), even for the sharp constant $\lambda_1$. This happens for the entropies with $p \in (1, 2)$, and it is due to the fact that the linear relationship of $|I_\psi|$ and $e_\psi$ is then not optimal.

A refinement of (1.3) for $p \in (1, 2)$ is the topic of this paper. In this case, the non-optimality of (1.3) stems from the fact that, for any fixed $D$ and $\rho_\infty$,

$$J(e, e', M) := \inf_{\rho \in L^1_+(\mathbb{R}^n), \|\rho\|_{L^1(\mathbb{R}^n)}=M} \rho, e_\psi(\rho|\rho_\infty) = e, e' \rho|\rho_\infty = e$$

is a positive quantity for $e > 0$ and $e' \leq -2\lambda_1 e$. Here, the $t$-derivatives entering in $R_\psi$ are defined via (1.2). Our main result is based on a lower bound for $J(e, e', M)$:

$$J(e, e', M) \geq \frac{2 - p}{p} \frac{|e'|^2}{M + e},$$

which yields an improvement of (1.3). Finding the minimizers of $J$ (if they exist) is probably difficult.

**Theorem 1** Let $\rho_\infty$ satisfy (A2) for some $\lambda_1 > 0$, and take $\psi = \psi_p$ for some $p \in (1, 2)$. Then

$$k(e) = k \left( \int_{\mathbb{R}^n} \psi \left( \frac{\rho}{\rho_\infty} \right) d\rho_\infty \right) \leq \frac{1}{2\lambda_1} \int_{\mathbb{R}^n} \psi'' \left( \frac{\rho}{\rho_\infty} \right) D \left| \nabla \left( \frac{\rho}{\rho_\infty} \right) \right|^2 d\rho_\infty = \frac{1}{2\lambda_1} |I|$$

(1.6)

holds for any $\rho \in L^1_+(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \rho \, dx = M$, where

$$k(e) := \frac{M}{1 - \kappa} \left( 1 + \frac{e}{M} - \left( 1 + \frac{e}{M} \right)^\kappa \right), \quad \kappa = \frac{2 - p}{p}.$$  

We will show that there are still no extremal functions to saturate the refined convex Sobolev inequality (1.6). Therefore it is not yet known whether the above functional dependence of $|I_\psi|$ and $e_\psi$ is optimal. But it improves upon (1.3) since we have

$$k(e) > e, \quad \forall e > 0,$$

(1.7)

and the best possible constants $\lambda_1$ are shown to be independent of $p$ (see Theorem 4).

Also, the presented method can be extended to the case $\lambda_1 = 0$ (see Proposition 3 below), thus giving a decay rate of $t \mapsto I_\psi(\rho(t)|\rho_\infty)$ for any solution $\rho$ of (1.2), even if $A$ is not uniformly convex. We remark that nonlinear entropy–entropy production
inequalities, or “defective logarithmic Sobolev inequalities,” have been derived for the logarithmic entropy (i.e. $\psi = \tilde{\psi}_1$) and Gaussian measures $\rho_\infty$ (cf. §1.3, §4.3 of [13]).

Next we consider reformulations of the convex Sobolev inequalities (1.3) and (1.6). We assume $M = 1$ and substitute

$$\frac{\rho}{\rho_\infty} = \frac{|f|^p}{\int_{\mathbb{R}^n} |f|^{\frac{p}{2}} \, d\rho_\infty}$$

in (1.3) to obtain the generalized Poincaré inequalities derived by W. Beckner for Gaussian measures $\rho_\infty$ in [5] and generalized in [2] for log-convex measures:

$$\frac{p}{p-1} \left[ \int_{\mathbb{R}^n} f^2 \, d\rho_\infty - \left( \int_{\mathbb{R}^n} |f|^{2/p} \, d\rho_\infty \right)^p \right] \leq \frac{2}{\lambda_1} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\rho_\infty$$

(1.9)

for all $f \in L^{2/p}(d\rho_\infty)$, $1 < p \leq 2$. In the limit $p \to 1$ this yields the logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^n} f^2 \log \left( \frac{|f|^2}{\|f\|_{L^2(d\rho_\infty)}^2} \right) \, d\rho_\infty \leq \frac{2}{\lambda_1} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\rho_\infty$$

(1.10)

for all $f \in L^2(d\rho_\infty)$. Hence, (1.9) interpolates between the (classical) Poincaré and the logarithmic Sobolev inequalities. A discussion on the interplay between (1.9), (1.10) and additional inequalities “between Poincaré and log. Sobolev” can be found in [11] and in §3 below. In [12] such interpolation inequalities are discussed for the Ornstein–Uhlenbeck process on $\mathbb{R}^n$ and for the heat semigroup on spheres.

Using the transformation (1.8) on the refined Sobolev inequality (1.6) directly yields a refinement of (1.9), which is nothing else than a reformulation of (1.6):

**Theorem 2** Let $\rho_\infty$ satisfy (A2) for some $\lambda_1 > 0$ and assume that $\int_{\mathbb{R}^n} d\rho_\infty = 1$. Then

$$\frac{1}{2} \left( \frac{p}{p-1} \right)^2 \left[ \int_{\mathbb{R}^n} f^2 \, d\rho_\infty - \left( \int_{\mathbb{R}^n} |f|^{2/p} \, d\rho_\infty \right)^{2(p-1)} \cdot \left( \int_{\mathbb{R}^n} f^2 \, d\rho_\infty \right)^{\frac{2}{p}-1} \right]$$

$$\leq \frac{2}{\lambda_1} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\rho_\infty$$

(1.11)

holds for all $f \in L^{2/p}(d\rho_\infty)$, $1 < p \leq 2$ and the limit $p \to 1$ again yields (1.10).

Note that the left hand sides of (1.9) and (1.11) are related by

$$\frac{1}{2} \left( \frac{p}{p-1} \right)^2 \left[ \int_{\mathbb{R}^n} f^2 \, d\rho_\infty - \left( \int_{\mathbb{R}^n} |f|^{2/p} \, d\rho_\infty \right)^{2(p-1)} \cdot \left( \int_{\mathbb{R}^n} f^2 \, d\rho_\infty \right)^{\frac{2}{p}-1} \right]$$

$$\geq \frac{p}{p-1} \left[ \int_{\mathbb{R}^n} f^2 \, d\rho_\infty - \left( \int_{\mathbb{R}^n} |f|^{2/p} \, d\rho_\infty \right)^p \right]$$

(1.12)

as a consequence of (1.7) and (1.8). This can of course be recovered using Hölder’s inequality:

$$\left( \int_{\mathbb{R}^n} |f|^{2/p} \, d\rho_\infty \right)^p \leq \int_{\mathbb{R}^n} |f|^2 \, d\rho_\infty$$

(1.13)
and the inequality: $\frac{1}{2} \frac{p}{p-1} (1 - t^{\frac{2}{p}}(p-1)) \geq 1 - t$ for any $t \in [0,1]$, $p \in (1,2]$. Note that the equality holds in (1.12) if and only if $1 = dI_{\psi}$, shall not go into details here. After a sequence of integrations by parts, since the first part of the proof is identical to the proof of Theorem 1.

The general case $M > 0$ can be estimated below by the case $\lambda = 1$ and the inequality: $1/dI_{\psi}$ is a constant.

Poincaré inequalities are presented in Section 3.

In the next section, we shall prove Theorems 1 and 2 and exploit the method in the case $\lambda_1 = 0$. Further results on best constants, perturbations and connections with Poincaré inequalities are presented in Section 3.

2 Convex Sobolev inequalities for power law entropies

Here and in the sequel we shall assume for simplicity that $\int_{\mathbb{R}^n} \rho_{\infty} \, dx = M = 1$.

The general case $M > 0$ then immediately follows by scaling.  

**Proof of Theorem 1.** Since the first part of the proof is identical to §2.3 of [2], we shall not go into details here. After a sequence of integrations by parts, $dI_{\psi}/dt$ can be written as

$$
\frac{d}{dt} I_{\psi}(\rho(t)|\rho_{\infty}) = 2 \int_{\mathbb{R}^n} \psi''(\mu) D \left[ u^T \nabla \otimes (\nabla AD - \nabla D) u 
+ \frac{1}{2} \Delta D |u|^2 - \frac{1}{2} |u|^2 \nabla D \cdot \nabla A 
+ \frac{1}{4} \frac{\mu}{D} (u \cdot \nabla D)^2 \right] d\rho_{\infty} 
$$

$$
+ \int_{\mathbb{R}^n} \left[ \psi''(\mu) D^2 |u|^4 
+ \psi''(\mu) (4D^2 u \cdot \nabla u + 2D|u|^2 \nabla u \cdot \nabla D) 
+ 2 \psi''(\mu) \sum_{i,j} \left( D \frac{\partial^2 \mu}{\partial x_i \partial x_j} + \frac{1}{2} \frac{\partial D}{\partial x_i} \frac{\partial \mu}{\partial x_j} 
+ \frac{1}{2} \frac{\partial \mu}{\partial x_i} \frac{\partial D}{\partial x_j} - \frac{1}{2} \delta_{ij} \nabla D \cdot \nabla \mu \right)^2 \right] d\rho_{\infty},
$$

where we used the notation $\mu = \frac{\rho}{\rho_{\infty}}$ and $u = \nabla \mu$. Using (A2), the first integral of (2.1) can be estimated below by $-2\lambda_1 I_{\psi}(\rho(t)|\rho_{\infty})$. In the second integral, we now insert $\psi_p$ from (1.5) and write it as a sum of squares. This is the key step in our analysis, where we deviate from the strategy of [2] by using a sharper estimate:

$$
\frac{d}{dt} I_{\psi}(\rho(t)|\rho_{\infty}) \geq -2\lambda_1 I_{\psi}(\rho(t)|\rho_{\infty}) + p(p-1)^2(2-p) \int_{\mathbb{R}^n} \rho_{\infty} \mu^{p-4} D^2 |u|^4 \, d\rho_{\infty} 
$$

$$
+ 2p(p-1) \int_{\mathbb{R}^n} \rho_{\infty} \mu^{p-2} \sum_{i,j} \left( \frac{p-2}{p} \frac{D}{\mu} \frac{\partial \mu}{\partial x_i} \frac{\partial \mu}{\partial x_j} 
+ D \frac{\partial^2 \mu}{\partial x_i \partial x_j} + \frac{1}{2} \frac{\partial D}{\partial x_i} \frac{\partial \mu}{\partial x_j} 
+ \frac{1}{2} \frac{\partial \mu}{\partial x_i} \frac{\partial D}{\partial x_j} - \frac{1}{2} \delta_{ij} \nabla D \cdot \nabla \mu \right)^2 \, d\rho_{\infty} 
$$

$$
\geq -2\lambda_1 I_{\psi}(\rho(t)|\rho_{\infty}) + p(p-1)^2(2-p) \int_{\mathbb{R}^n} \rho_{\infty} \mu^{p-4} D^2 |u|^4 \, d\rho_{\infty}.
$$
In the two limiting cases $p = 1$ (replace $\psi_p$ by $\tilde{\psi}_1$) and $p = 2$, the second term on the r.h.s. of (2.2) disappears. In [2], this term was always disregarded. For $1 < p < 2$, however, it makes it possible to improve (1.3).

Using the Cauchy-Schwarz inequality, we have the estimate
\[
\left( \int_{\mathbb{R}^n} \mu^{p-2} D|u|^2 \, d\rho_\infty \right)^2 \leq \int_{\mathbb{R}^n} \mu^{p-4} D^2|u|^4 \, d\rho_\infty \cdot \int_{\mathbb{R}^n} \mu^p \, d\rho_\infty
\]
and hence
\[
\int_{\mathbb{R}^n} \mu^{p-4} D^2|u|^4 \, d\rho_\infty \geq \left( \frac{I_{\psi}(\rho|\rho_\infty)}{p(p-1)} \right)^2 \cdot \left[ e_{\psi}(\rho|\rho_\infty) + 1 \right]^{-1}.
\]

With the notation $e(t) = e_{\psi}(\rho(t)|\rho_\infty)$, we get from (2.2)
\[
e'' \geq -2\lambda_1 e' + \kappa \frac{|e'|}{1 + e}.
\] (2.3)

From (2.3) we shall now derive
\[
|e'| = -e' \geq 2\lambda_1 k(e),
\] (2.4)
which is the assertion of Theorem 1. We first note that both $I_{\psi}(\rho(t)|\rho_\infty)$ and $e_{\psi}(\rho(t)|\rho_\infty)$ decay exponentially with the rate $-2\lambda_1$. This follows, respectively, from (1.4) and from the usual convex Sobolev inequality (1.3).

The function
\[
k(e) = \frac{1}{1 - \kappa} \left( 1 + e - (1 + e)^\kappa \right)
\]
is the solution of
\[
k' = 1 + \kappa \frac{k(e)}{1 + e}, \quad k(0) = 0.
\]

Let
\[
y(t) = \left[ e'(t) + 2\lambda_1 k(e(t)) \right] \cdot e^{-\kappa \int_0^t \frac{e'(s)}{1 + e(s)} \, ds}.
\]
For any $t \geq 0$, we calculate
\[
y'(t) = \left( e''(t) + 2\lambda_1 e'(t) - \kappa \frac{|e'(t)|^2}{1 + e(t)} \right) \cdot e^{-\kappa \int_0^t \frac{e'(s)}{1 + e(s)} \, ds}.
\]
Since
\[
|y(t)| \leq |e'(t) + 2\lambda_1 k(e(t))| \cdot e^{-\kappa \int_0^t \frac{e'(s)}{1 + e(s)} \, ds} = |e'(t) + 2\lambda_1 k(e(t))| e^{-\kappa |e(t)|} \to 0
\]
as $t \to +\infty$, we conclude that $y(t) \leq 0$, which proves (2.4).

As we had to expect, one recovers the usual convex Sobolev inequality (1.3) in the limiting cases $p = 1$ (take the limit $p \to 1$ after dividing (1.6) by $p - 1$) and $p = 2$ (this gives $\kappa = 0$ and $k(e) = e$).

For $1 < p < 2$, we notice that
\[
\lim_{e \to 0^+} \frac{k(e)}{e} = 1.
\]
Hence, the estimate of Theorem 1 does not improve the asymptotic convergence rate of the solution of Equation (1.2) except for $\lambda_1 = 0$:
Proposition 3 With the above notations, let $\lambda_1 = 0$ and $1 < p < 2$. Any solution of Equation (1.2) satisfies

$$|I_\psi(\rho(t)|\rho_\infty)| \leq \frac{I_0}{1 + \alpha t} \quad \forall t > 0$$

with $I_0 = |I_\psi(\rho(0)|\rho_\infty)|$ and $\alpha = \kappa \frac{I_0}{1 + e(\rho(0)|\rho_\infty)}$.

Proof. Inequality (2.3) can be rewritten in the form

$$- \frac{|e'|'}{|e'|^2} \geq \frac{\kappa}{1 + e} \geq \frac{\kappa}{1 + e(0)} ,$$

thus proving the result. $\square$

Next we address the question of saturation of the refined convex Sobolev inequality (1.6), for simplicity only for the case $D \equiv 1$. Using the strategy from [2] we rewrite (2.1) as

$$e'' = -2\lambda_1 e' + \kappa \frac{|e'|^2}{1 + e} + r_\psi(\rho(t)) ,$$

where the remainder term is

$$r_\psi(\rho(t))$$

$$= 2 \int_{\mathbb{R}^n} \psi''(\mu) u^+ \left( \frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbb{I} \right) u \, d\rho_\infty$$

$$+ 2 p(p - 1) \int_{\mathbb{R}^n} \mu^{2-p} \sum_{i,j} \left( \frac{\partial z_i}{\partial x_j} \right)^2 d\rho_\infty$$

$$+ \frac{p(p - 1)^2(2 - p)}{e + 1} \left[ \int_{\mathbb{R}^n} \mu^p d\rho_\infty \cdot \int_{\mathbb{R}^n} \mu^{p-4} |u|^4 d\rho_\infty - \left( \int_{\mathbb{R}^n} \mu^{p-2} |u|^2 d\rho_\infty \right)^2 \right] \geq 0 ,$$

with the notation $z = \mu^{p-2} \nabla \mu$. Using the notation from the proof of Theorem 1, we have

$$y'(t) = r_\psi(\rho(t)) e^{-\kappa \int_0^t \frac{\psi'(\mu)}{1 + e(\mu)|\rho_\infty|} \, ds} ,$$

and an integration with respect to $t$ gives

$$-y(0) = |e'(0)| - \lambda_1 k(e(0)) = \int_0^\infty r_\psi(\rho(t)) e^{-\kappa \int_0^t \frac{\psi'(\mu)}{1 + e(\mu)|\rho_\infty|} \, ds} \, dt \geq 0 .$$

Hence we conclude that (2.4) becomes an equality, for $\rho = \rho(0)$, if and only if the remainder vanishes along the whole trajectory of $\rho(t)$, i.e.

$$r_\psi(\rho(t)) = 0 , \quad t \in \mathbb{R}^+ \text{ a.e.}$$

However, no extremal function can simultaneously annihilate the second integral and the square bracket of (2.5): to make the second integral vanish, the function $\mu$ has to be of the form $\mu(x) = (C_1 + C_2 \cdot x)^{1/(1-p)}$ (whenever $\mu(x) \neq 0$), and for the last term it would have to be $\mu(x) = e^{C_1 + C_2 \cdot x}$. Hence, (2.4) does not admit extremal functions.
3 Further results and comments

In the previous sections we derived convex Sobolev inequalities (corresponding to power law entropies) for steady state measures $\rho_\infty = e^{-A(x)}$, whose potential $A(x)$ satisfies the Bakry–Emery condition (A2). However, such inequalities hold also in much more general situations: As soon as $\rho_\infty$ gives rise to a (classical) Poincaré inequality (cf. (3.1) below), convex Sobolev inequalities of type (1.3), (1.6), (1.9), and (1.11) hold for $p \in (1, 2]$. Note that this condition is much weaker than the assumption (A2).

3.1 Spectral gap, Poincaré and convex Sobolev inequalities

Using the Poincaré constant

$$\Lambda_2 := \inf_{w \in D(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} D[|\nabla w|^2 \, d\rho_\infty]}{\int_{\mathbb{R}^n} |w|^2 \, d\rho_\infty}$$

we shall now give an estimate on the sharp constant in the refined Sobolev inequality (1.6) and its reformulation (1.11):

**Theorem 4** Let $D = D(x) > 0$ and assume that $\rho_\infty \in L^1_+ (\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \rho_\infty \, dx =: M = 1$ is such that $\Lambda_2 > 0$. Then, refined convex Sobolev inequalities of type (1.6) hold for any $p \in (1, 2]$. And the optimal constant

$$\Lambda_p := \frac{1}{2} \inf_{\rho_\infty \in L^1_+ (\mathbb{R}^n), \int_{\mathbb{R}^n} \rho_\infty \, dx = M} \frac{|I_{\psi_p}(\rho_\infty)|}{k (e^{\psi_p}(\rho_\infty))}$$

satisfies the estimate

$$4 \left( \frac{p-1}{p} \right)^2 \Lambda_2 \leq \Lambda_p \leq \Lambda_2. \tag{3.2}$$

**Proof.** The r.h.s. of this inequality is proved by contradiction: Assume that $\Lambda_p > \Lambda_2$ and substitute $\frac{\rho_\infty}{\rho_\infty^2} = |f|^2/p \left( \int_{\mathbb{R}^n} |f|^{2/p} \, d\rho_\infty \right)^{-1}$ (cf. (1.8)). A standard linearization argument (put $f^2 = 1 + \varepsilon w$ and take the limit $\varepsilon \to 0$) then implies a Poincaré inequality with the constant $\Lambda_p$ which would contradict the sharpness of $\Lambda_2$ in (3.1).

For the l.h.s. of inequality (3.2) we estimate (using twice Jensen’s inequality and then the Poincaré inequality):

$$\int_{\mathbb{R}^n} f^2 \, d\rho_\infty - \left( \int_{\mathbb{R}^n} f \, d\rho_\infty \right)^2 \left( \int_{\mathbb{R}^n} f^2 \, d\rho_\infty \right)^{-\frac{2}{p}} \leq \int_{\mathbb{R}^n} f^2 \, d\rho_\infty - \left( \int_{\mathbb{R}^n} f \, d\rho_\infty \right)^2 \left( \int_{\mathbb{R}^n} f^2 \, d\rho_\infty \right)^{2(2(p-1))} \left( \int_{\mathbb{R}^n} f \, d\rho_\infty \right)^{2(2(p-1))} \leq \frac{1}{\Lambda_2} \int_{\mathbb{R}^n} D[|\nabla f|^2 \, d\rho_\infty].$$

This reformulation of (1.6) (just like in (1.11)) shows that $4 \left( \frac{p-1}{p} \right)^2 \Lambda_2 \leq \Lambda_p. \tag{3.2}$

Next we shall show that the validity of a logarithmic Sobolev inequality implies the convex Sobolev inequalities (1.9) and (1.11). Part (i) of the following corollary is mainly due to Latała and Oleszkiewicz (Corollary 1 of [11]), with an improved constant for $\frac{3}{2} < p < 2$. 

□
Corollary 5 Let $D(x) > 0$ and let $\mu$ be a probability measure on $\mathbb{R}^d$ that gives rise to a logarithmic Sobolev inequality:

$$
\int f^2 \log \left( \frac{f^2}{\|f\|_{L^2(d\mu)}^2} \right) d\mu \leq \frac{2}{\Lambda_1} \int D|\nabla f|^2 d\mu \quad \forall f \in L^2(d\mu).
$$

Then:

(i) a convex Sobolev inequality holds for any $p \in (1,2]$:

$$
\int f^2 d\mu - \left( \int |f|^{2/p} d\mu \right)^p \leq \frac{\min\{2(p-1), 1\}}{\Lambda_1} \int D|\nabla f|^2 d\mu \quad \forall f \in L^2(d\mu).
$$

(ii) a refined convex Sobolev inequality holds for any $p \in (1,2]$:

$$
\int f^2 d\mu - \left( \int |f|^{2/p} d\mu \right)^{2(p-1)} \left( \int f^2 d\mu \right)^{\frac{2-p}{p}} \leq \frac{1}{\Lambda_1} \int D|\nabla f|^2 d\mu.
$$

Proof. The function $p \mapsto \alpha(p) := p \log \left( \int |f|^{2/p} d\mu \right)$ is convex:

$$
\alpha''(p) = \frac{4}{p^3} \left( \int |f|^{2/p} (\log |f|)^2 d\mu \right) \left( \int |f|^{2/p} d\mu \right) \left( \int |f|^{2/p} d\mu \right)^2 \geq 0.
$$

Thus $p \mapsto e^{\alpha(p)}$ is also convex and

$$
p \mapsto \varphi(p) := \frac{e^{\alpha(1)} - e^{\alpha(p)}}{p-1}
$$

is nonincreasing:

$$
\varphi(p) \leq \lim_{q \to 1} \varphi(q) = \int f^2 \log \left( \frac{f^2}{\|f\|_{L^2(d\mu)}^2} \right) d\mu.
$$

This proves that

$$
\int f^2 d\mu - \left( \int |f|^{2/p} d\mu \right)^p \leq \frac{2(p-1)}{\Lambda_1} \int D|\nabla f|^2 d\mu.
$$

On the other hand, using the linearization from the proof of Theorem 4 for (3.3) and using Hölder’s inequality, $(\int f d\mu)^2 \leq (\int |f|^{2/p} d\mu)^p$, we also get

$$
\int f^2 d\mu - \left( \int |f|^{2/p} d\mu \right)^p \leq \int f^2 d\mu - \left( \int f d\mu \right)^2 \leq \frac{1}{\Lambda_1} \int D|\nabla f|^2 d\mu.
$$

Similarly, since the logarithmic Sobolev inequality (3.3) implies a classical Poincaré inequality, (ii) follows directly from Theorem 4. \qed
3.2 Holley-Stroock type perturbations

In Section 1 we presented the refined convex Sobolev inequality (1.6) for steady state measures $\rho_\infty = e^{-A(x)}$, whose potential $A(x)$ satisfies the Bakry–Émery condition (A2). We shall now extend that inequality for potentials $\tilde{A}(x)$ that are bounded perturbations of such a potential $A(x)$. Our result generalizes the perturbation lemma of Holley and Stroock (cf. [10] for the logarithmic entropy $\psi_1$ and §3.3 of [2] for general admissible entropies).

For our subsequent calculations it is convenient to rewrite (1.6) as

$$k \left( \int_{\mathbb{R}^n} \psi \left( \frac{f^2}{\|f\|_{L^2(\rho_\infty)}} \right) \, d\rho_\infty \right) \leq \frac{2}{\lambda_1} \int_{\mathbb{R}^n} \frac{f^2}{\|f\|_{L^2(\rho_\infty)}} \psi'' \left( \frac{f^2}{\|f\|_{L^2(\rho_\infty)}} \right) \, D[\nabla f]^2 \, d\rho_\infty,$$

(3.5)

where we substituted

$$\frac{\rho}{\rho_\infty} = \frac{f^2}{\int_{\mathbb{R}^n} f^2 \, d\rho_\infty}.$$

**Theorem 6** Assume that $\psi = \psi_p$ with some $1 < p < 2$ is a fixed entropy generator. Let $\rho_\infty(x) = e^{-A(x)}$, $\tilde{\rho}_\infty(x) = e^{-\tilde{A}(x)} \in L^1_+(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \rho_\infty \, dx = \int_{\mathbb{R}^n} \tilde{\rho}_\infty \, dx = M$ and

$$\tilde{A}(x) = A(x) + v(x), \quad 0 < a \leq e^{-v(x)} \leq b < \infty, \quad x \in \mathbb{R}^n. \quad (3.6)$$

Let the given diffusion $D(x)$ be such that the convex Sobolev inequality (3.5) holds for all $f \in L^2(\rho_\infty)$. Then a convex Sobolev inequality also holds for the perturbed measure $\tilde{\rho}_\infty$:

$$\frac{1}{a^{p-1}} k \left( \int_{\mathbb{R}^n} \psi \left( \frac{\tilde{f}^2}{\|\tilde{f}\|_{L^2}} \right) \, d\tilde{\rho}_\infty \right) \leq \frac{2}{\lambda_1} \int_{\mathbb{R}^n} \frac{\tilde{f}^2}{\|\tilde{f}\|_{L^2}} \psi'' \left( \frac{\tilde{f}^2}{\|\tilde{f}\|_{L^2}} \right) \, D[\nabla \tilde{f}]^2 \, d\tilde{\rho}_\infty,$$

(3.7)

for all nontrivial $\tilde{f} \in L^2(\tilde{\rho}_\infty) = L^2(\rho_\infty)$. Here $\|\tilde{f}\|_{L^2}^2$ stands for $\|f\|_{L^2(\rho_\infty)}^2$.

Note that the normalization of $\rho_\infty$ and $\tilde{\rho}_\infty$ implies $a \leq 1$ and $b \geq 1$.

**Proof.** First we introduce the notations

$$\chi(x) := \frac{f^2(x)}{\|f\|_{L^2(\rho_\infty)}^2}, \quad \tilde{\chi}(x) := \frac{\tilde{f}^2(x)}{\|\tilde{f}\|_{L^2}^2}, \quad \gamma := \frac{\chi}{\tilde{\chi}} = \frac{\|f\|_{L^2(\rho_\infty)}^2}{\|\tilde{f}\|_{L^2}^2},$$

and because of (3.6) we have $a \leq \gamma \leq b$.

We adapt the idea of [10, 2] and define for a fixed $f \in L^2(\rho_\infty)$ the function

$$g(s) := s^p \int_{\mathbb{R}^n} \psi \left( \frac{\tilde{f}^2}{s} \right) \, d\tilde{\rho}_\infty.$$

Since $g$ attains its minimum at $s = \|f\|_{L^2(\rho_\infty)}^2$, by differentiating w.r.t. $s$, we have

$$\|f\|_{L^2(\rho_\infty)}^{2p} \int_{\mathbb{R}^n} \psi(\tilde{\chi}) \, d\tilde{\rho}_\infty = g(\|f\|_{L^2(\rho_\infty)}^2) \leq \|f\|_{L^2(\rho_\infty)}^2 \leq b \|f\|_{L^2(\rho_\infty)}^{2p} \int_{\mathbb{R}^n} \psi(\chi) \, d\rho_\infty,$$

where we used the estimate (3.6).
Using the monotonicity of $k$ and Assumption (3.5), this yields:

\[
k \left( \frac{\gamma^p}{b} \int_{\mathbb{R}^n} \psi(\tilde{\chi}) \, d\tilde{\rho}_\infty \right) \leq k \left( \int_{\mathbb{R}^n} \psi(\chi) \, d\rho_\infty \right)
\leq \frac{2}{\lambda_1} \int_{\mathbb{R}^n} \frac{f^2}{\|f\|_{L^2(\rho_\infty)}} \psi''(\chi) D|\nabla f|^2 \, d\rho_\infty
\leq \frac{2}{\lambda_1} \frac{\gamma^p}{a} \int_{\mathbb{R}^n} \frac{f^2}{\|f\|_{L^2(\rho_\infty)}} \psi''(\tilde{\chi}) D|\nabla f|^2 \, d\tilde{\rho}_\infty ,
\]

where we again used (3.6) in the last estimate.

Since $\gamma/a \geq 1$, the convexity of $k$ and $k(0) = 0$ imply:

\[
\frac{\gamma^p}{a^p} k \left( \frac{a^p b}{\int_{\mathbb{R}^n} \psi(\tilde{\chi}) \, d\tilde{\rho}_\infty} \right) \leq k \left( \frac{\gamma^p}{b} \int_{\mathbb{R}^n} \psi(\tilde{\chi}) \, d\tilde{\rho}_\infty \right).
\]

Together with (3.8), this finishes the proof. \(\square\)

Note that a Holley–Stroock perturbation of the usual convex Sobolev inequality (1.3) would lead – under the assumptions of Theorem 6 – to the inequality

\[
\frac{a}{b} 2\lambda_1 e \leq |e'|
\]

(cf. [2]). Since

\[
\frac{a}{b} e < \frac{1}{a^{p-1}} k \left( \frac{a^p b}{e} \right) \quad \forall \frac{a^p b}{e} > 0 ,
\]

Inequality (3.7) certainly improves (3.9).

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References


