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Spin-diffusion model for micromagnetics in the limit of long times

G. Di Fratta, A. Jüngel, D. Praetorius, and V. Slastikov

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
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Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

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SPIN-DIFFUSION MODEL FOR MICROMAGNETICS IN THE LIMIT OF LONG TIMES

GIOVANNI DI FRATTA, ANSGAR JÜNGEL, DIRK PRAETORIUS, AND VALERIY SLASTIKOV

Abstract. In this paper, we consider spin-diffusion Landau–Lifshitz–Gilbert equations (SDLLG), which consist of the time-dependent Landau–Lifshitz–Gilbert (LLG) equation coupled with a time-dependent diffusion equation for the electron spin accumulation. The model takes into account the diffusion process of the spin accumulation in the magnetization dynamics of ferromagnetic multilayers. We prove that in the limit of long times, the system reduces to simpler equations in which the LLG equation is coupled to a nonlinear and nonlocal steady-state equation, referred to as SLLG. As a by-product, the existence of global weak solutions to the SLLG equation is obtained. Moreover, we prove weak-strong uniqueness of solutions of SLLG, i.e., all weak solutions coincide with the (unique) strong solution as long as the latter exists in time. The results provide a solid mathematical ground to the qualitative behavior originally predicted by Zhang, Levy, and Fert in [44] in ferromagnetic multilayers.

Keywords: Micromagnetics, Landau–Lifshitz–Gilbert equation, spin diffusion, asymptotic analysis, existence of solutions, weak-strong uniqueness.

2010 *Mathematics Subject Classification.* 35C20, 35D30, 35G20, 35G25 49S05.

1. Introduction and Physical Motivations

In the last decades, the study of ferromagnetic materials and their magnetization processes has been the focus of considerable research for its application to magnetic recording technology. Below the Curie temperature, ferromagnetic media possess a spontaneous magnetization state (which is the result of a spontaneous alignment of the elementary magnetic moments in the medium) that can be controlled through appropriate external magnetic fields. The magnetic recording technology exploits the magnetization of ferromagnetic media to store information.

The *giant magnetoresistance* effect (GMR), for which Fert and Grünberg have been awarded the Nobel prize in 2007, has introduced new solutions in the design of magnetic random access memories (MRAMs). In a typical MRAM device, the binary information is stored in elementary cells that can be addressed via two perpendicular arrays of parallel conducting lines, called word lines and bit lines. A schematic of the MRAM architecture is depicted in Figure 1.

The GMR allows for a *giant* change in the resistance of a conductor in response to an applied magnetic field, and it is the primary mechanism behind the reading process in MRAMs. Furthermore, the switching (writing) process of an MRAM cell can be achieved by magnetic field pulses produced by the sum of horizontal and vertical currents. The magnetic pulse induces a magnetic torque, whose strength depends on the angle between the field and the magnetization, which permits the switching of the cell.

This behavior is conceptually simple, but it is tough to realize in practice on the nanoscale. One of the fundamental issues connected with the downscaling of magnetic storage devices is the thermal stability of magnetization states. In principle, the problem can be circumvented

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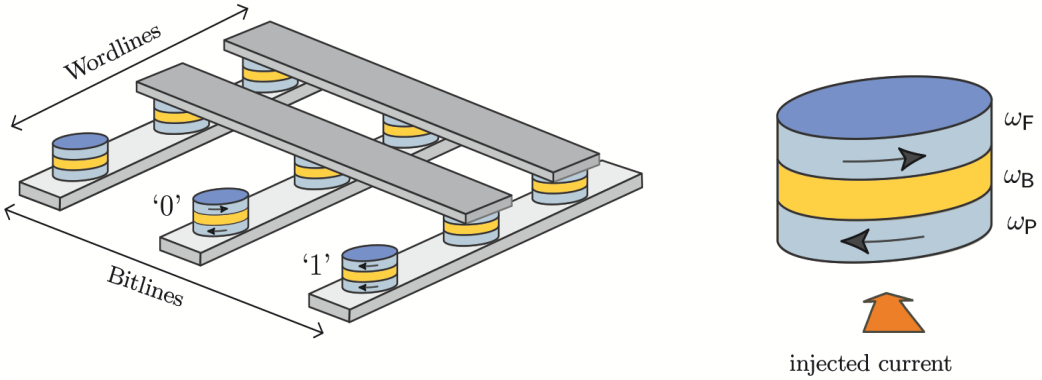

FIGURE 1. (Left) In a typical MRAM device, the binary information is stored in elementary cells that can be addressed via two perpendicular arrays of parallel conducting lines, called word lines and bit lines. (Right) A schematic picture of an MRAM cell. In its simplest form, an MRAM cell consists of two ferromagnetic layers ω_F (the *free layer*) and ω_P (the *pinned layer*), separated by a thin insulator ω_B (the *tunnel barrier*). Such a multilayer structure is compatible with the GMR effect. The logical states '0' and '1' are coded into the direction of the magnetization in the free layer ω_F. Note that MRAM cells usually have an ellipsoidal cross-section. This is quite common in applications because ellipsoidal particles support single-domain magnetization states [4,10,13,17] which allow for a qualitative analysis of the system dynamics in the framework of ODEs.

by increasing the magnetic anisotropy of the material, but as a consequence, higher magnetic fields are required to reverse the magnetization states. However, these magnetic fields typically act on a long-range, whereas it is desired that the field produced by the two currents can switch only the target cell. For this reason, considerable attention has recently been paid to design new strategies of magnetization switching in which the applied field is assisted by additional external actions. Examples of these new approaches are heat and microwave-assisted switchings [12, 28].

Fairly recently, SLONCZEWSKI [40] and BERGER [7] proposed a novel model for magnetization reversal based on the use of spin-polarized currents injected directly in the magnetic free layer. The new mechanism proved extremely valuable to overcome the difficulties imposed by the use of strong magnetic fields. Since its introduction, it has been the object of much research work in the spintronics community as a candidate to assist the switching of the magnetization in MRAMs cells. In this new approach, each MRAM cell hosts a multilayer structure, called *magnetic tunnel junction*, which in its simplest form consists of two ferromagnetic layers ω_F (the so-called *free layer*, where the magnetization can change freely) and ω_P (the *pinned layer*, where the magnetization is pinned by exchange interactions), separated by a thin insulator ω_B (the *tunnel barrier*). A current is injected perpendicular to the multilayer. The electron spin is polarized in the pinned layer ω_P. When the electrons reach the free layer ω_F, the spin exerts an additional torque on the underlying magnetization, which assists the switching process.

The model proposed by SLONCZEWSKI [40] does not take into account the effects of spin diffusion, which have been found to be important in understanding magnetoresistance in magnetic multilayers. This motivated the work of ZHANG, LEVY, and FERT [44] (see also [39]), where a new spin-transfer model for the relaxation of the coupled system spin-magnetization is proposed. Their model includes spatial variations in both spin and magnetization, but is derived under the assumption that the magnetization is uniform in each of the layers and,

therefore, essentially one-dimensional. Later, Garcia-Cervera and Wang (cf. [23], see also [1]) generalized the model to the three-dimensional setting.

1.1. The spin-diffusion Landau–Lifshitz–Gilbert equation. In this section, we introduce the model proposed in [23]. For that, we set $\Omega := \omega_{\mathsf{F}} \cup \omega_{\mathsf{P}} \subset \mathbb{R}^3$, and denote by $\Omega' := \Omega \cup \omega_{\mathsf{B}} \subset \mathbb{R}^3$, the region occupied by the multilayer. The spin accumulation $\mathbf{S}$ is defined on Ω', and the magnetization $\mathbf{M}$ is supported on the region Ω occupied by the two magnetic layers. The magnetization is zero in ω_{B} (i.e., outside the magnetic samples). We assume that the temperature is constant and well below the Curie temperature so that the magnetization $\mathbf{M}$ is of constant modulus in Ω, i.e., $|\mathbf{M}| = M_s$ with $M_s \in \mathbb{R}_+$ being the saturation magnetization in Ω. To shorten notation, we set $\boldsymbol{s} = \mathbf{S}/M_s$ and $\boldsymbol{m} := \mathbf{M}/M_s$.

The spin-diffusion Landau–Lifshitz–Gilbert equation (SDLLG) consists of a quasilinear diffusion equation for the evolution of the spin accumulation coupled to the well-established equation for magnetization dynamics. In strong form, SDLLG reads as (cf. [1, 21, 22])

$$\partial_t \boldsymbol{s} = \mathbf{div}\,[J(\nabla \boldsymbol{s}, \boldsymbol{m}\chi_\Omega)] - \frac{2D_0}{\lambda_{\mathrm{sf}}^2}\boldsymbol{s} - \frac{2D_0}{\lambda_J^2}\boldsymbol{s} \times \boldsymbol{m}\chi_\Omega \quad \text{in } \Omega' \times \mathbb{R}_+, \tag{1}$$

$$\partial_t \boldsymbol{m} = -\gamma_0 \boldsymbol{m} \times (\boldsymbol{h}_{\mathrm{eff}}[\boldsymbol{m}] + j_0 \boldsymbol{s} + \boldsymbol{f}) + \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m} \quad \text{in } \Omega \times \mathbb{R}_+. \tag{2}$$

Here, $\boldsymbol{s} : \Omega' \times \mathbb{R}_+ \to \mathbb{R}^3$ is the spin accumulation, $\boldsymbol{m} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^2$ is the magnetization, and $\boldsymbol{m}\chi_\Omega$ denotes the extension of $\boldsymbol{m}$ by zero to the whole $\mathbb{R}^3$. From now on, to simplify the notation, we will identify $\boldsymbol{m}$ with its extension $\boldsymbol{m}\chi_\Omega$ as long as no ambiguities can arise. The first equation (1) models the spin-polarized transport in the multilayer Ω' as a diffusive process [23, 39, 44]. The second equation (2) is a variant of the Landau–Lifshitz–Gilbert (LLG) equation [24, 29] and describes the relaxation process of the magnetization. Since the modulus of the magnetization is preserved by LLG, we have normalized the magnetization to take values on the 2-sphere $\mathbb{S}^2$. The system (1)–(2) is supplemented with the initial/boundary conditions

$$\begin{cases} \boldsymbol{m}(x,0) = \boldsymbol{m}^*(x) & \text{in} \quad \Omega, \\ \partial_{\boldsymbol{n}} \boldsymbol{m} = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}_+, \end{cases} \qquad \begin{cases} \boldsymbol{s}(x,0) = \boldsymbol{s}^*(x) & \text{in} \quad \Omega', \\ \partial_{\boldsymbol{n}} \boldsymbol{s} = 0 & \text{on} \quad \partial\Omega' \times \mathbb{R}_+, \end{cases}$$

for some given initial states $\boldsymbol{m}^* : \Omega \to \mathbb{S}^2$, $\boldsymbol{s}^* : \Omega' \to \mathbb{R}^3$. A detailed description of the terms involved in SDLLG follows.

The spin-diffusion equation. In the spin-diffusion equation (1), $J(\nabla \boldsymbol{s}, \boldsymbol{m})$ is the spin current, $D_0 \in L^\infty(\Omega)$ is the diffusion coefficient, $\lambda_{\mathrm{sf}} \in \mathbb{R}_+$ is the characteristic length for spin-flip relaxation, and $\lambda_J \in \mathbb{R}_+$ is related to the electron's mean free path. The spin current is given by

$$J(\nabla \boldsymbol{s}, \boldsymbol{m}) := 2D_0\,[\nabla \boldsymbol{s} - \beta\beta'(\nabla \boldsymbol{s} \cdot \boldsymbol{m}) \otimes \boldsymbol{m}] - \frac{\beta \mu_B}{e} \boldsymbol{j}_e \otimes \boldsymbol{m},$$

where $\boldsymbol{j}_e$ is the applied electric current, $0 < \beta, \beta' < 1$ are the dimensionless spin-polarization parameters of the magnetic layers, $\mu_B > 0$ is the Bohr magneton, and $e > 0$ is the charge of the electron. With given $\gamma \in \mathbb{R}_+$, the diffusion coefficient $D_0 \in L^\infty(\Omega)$ satisfies $D_0(x) \geqslant \gamma$ for almost all $x \in \Omega$. The last term in (1) represents the interaction between the spin accumulation and the background magnetization, and it is responsible for the transfer of angular momentum between them.

The Landau–Lifshitz–Gilbert equation. In (2), $\gamma_0 \in \mathbb{R}_+$ is the gyromagnetic ratio, j_0 models the strength of the interaction between the spin and the magnetization, and α is the dimensionless damping parameter. The first term on the right-hand side of (2) describes a precession around the field $\boldsymbol{h}_{\mathrm{eff}}[\boldsymbol{m}] + j_0\boldsymbol{s} + \boldsymbol{f}$, whereas the (phenomenological) second term accounts for

dissipation in the system. The time-dependent vector field $\boldsymbol{f}$ is the so-called applied field, and it is assumed to be unaffected by variations of $\boldsymbol{m}$. The effective field $\boldsymbol{h}_{\text{eff}}$ includes contributions originating from different spatial scales and is defined by the negative first-order variation of the micromagnetic energy functional $\mathcal{G}_\Omega$, i.e.

$$\boldsymbol{h}_{\text{eff}}[\boldsymbol{m}] := -\partial_{\boldsymbol{m}}\mathcal{G}_\Omega. \tag{3}$$

For single-crystal ferromagnets, the micromagnetic energy functional reads as (cf. [9, 27])

$$\mathcal{G}_\Omega(\boldsymbol{m}) := \underbrace{\frac{c_{\text{ex}}}{2}\int_\Omega |\nabla \boldsymbol{m}|^2}_{=:\mathcal{E}_\Omega(\boldsymbol{m})} + \underbrace{\kappa\int_\Omega \phi_{\text{an}}(\boldsymbol{m})}_{=:\mathcal{A}_\Omega(\boldsymbol{m})} - \underbrace{\frac{\mu_0}{2}\int_\Omega \boldsymbol{h}_{\text{d}}\left[\boldsymbol{m}\chi_\Omega\right]\cdot\boldsymbol{m}}_{=:\mathcal{W}_\Omega(\boldsymbol{m})} \tag{4}$$

with $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$, and recall that $\boldsymbol{m}\chi_\Omega$ is the extension by zero of $\boldsymbol{m}$ to $\mathbb{R}^3$. Combining (3) and (4), we obtain the following expression for the effective field:

$$\boldsymbol{h}_{\text{eff}}[\boldsymbol{m}] = c_{\text{ex}}\Delta\boldsymbol{m} - \kappa\nabla\phi_{\text{an}}(\boldsymbol{m}) + \mu_0\boldsymbol{h}_{\text{d}}\left[\boldsymbol{m}\chi_\Omega\right]. \tag{5}$$

In (4), the *exchange energy* $\mathcal{E}_\Omega$ penalizes nonuniformities in the orientation of the magnetization, and $c_{\text{ex}} > 0$ is the exchange stiffness constant. The *magnetocrystalline anisotropy energy* $\mathcal{A}_\Omega$ accounts for the existence of preferred directions of the magnetization: its energy density $\phi_{\text{an}} : \mathbb{S}^2 \to \mathbb{R}^+$ vanishes only on a finite set of directions (the so-called *easy directions*); κ is the anisotropic constant. The third term $\mathcal{W}_\Omega$ is the *magnetostatic self-energy*, i.e., the energy due to the demagnetizing field $\boldsymbol{h}_{\text{d}}$ generated by $\boldsymbol{m}$. The operator $\boldsymbol{h}_{\text{d}} : \boldsymbol{m} \mapsto \boldsymbol{h}_{\text{d}}[\boldsymbol{m}]$ provides, for every $\boldsymbol{m} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, the unique solution in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ of the Maxwell–Ampère equations of magnetostatics [8, 15, 35]:

$$\begin{cases} \operatorname{div}\boldsymbol{b}[\boldsymbol{m}] = 0, \\ \mathbf{curl}\,\boldsymbol{h}_{\text{d}}[\boldsymbol{m}] = 0, \\ \boldsymbol{b}[\boldsymbol{m}] = \mu_0\left(\boldsymbol{h}_{\text{d}}[\boldsymbol{m}] + \boldsymbol{m}\right), \end{cases} \tag{6}$$

where $\boldsymbol{b}[\boldsymbol{m}]$ denotes the magnetic flux density, and μ_0 is the magnetic permeability of vacuum.

Remark 1.1. In what follows we assume that $\Omega = \Omega'$ (see, e.g., [25] and [36]). This permits to simplify the notation and to state the weak-strong uniqueness result in a more elegant form. Everything we are going to prove still holds in the case $\Omega \subset \Omega'$. However, slight modifications may be required depending on the degree of smoothness one agrees to impose to a weak solution in order to elevate it to the rank of a strong solution (cf. Lemma 1 and Remark 3.1).

1.2. Contributions of the present work: SDLLG in the limit of long times. Already Zhang, Levy, and Fert [44] observed that in the non-ballistic regime, the time scales involved in SDLLG (1)–(2) are very different. For the spin accumulation $\boldsymbol{s}$ the characteristic time scales are of the order of $\lambda_{\text{sf}}^2/(2D_0)$ and $\lambda_J^2/(2D_0)$. Given the typical spatial scales involved in spintronic applications, these quantities are of the order of picoseconds. On the other hand, the characteristic time scale for the magnetization dynamics depends on the inverse of the modulus of the precessional term $\gamma_0\boldsymbol{m}_\varepsilon \times (\boldsymbol{h}_{\text{eff}}[\boldsymbol{m}_\varepsilon] + j_0\boldsymbol{s}_\varepsilon + \boldsymbol{f})$. In typical spintronic applications, this time scale is of the order of nanoseconds. Therefore, «*as long as one is interested in the magnetization process of the local moments, one can always treat the spin accumulation in the limit of long times*» [44].

Formally, the observations in [44] can be summarized in the claim that as long as the main interest is in magnetization dynamics, one can forget about the term $\partial_t\boldsymbol{s}$ in (1) and focus on the analysis of the SLLG equation

$$\partial_t\boldsymbol{m} = -\gamma_0\boldsymbol{m}\times(\boldsymbol{h}_{\text{eff}}[\boldsymbol{m}] + j_0\boldsymbol{s} + \boldsymbol{f}) + \alpha\boldsymbol{m}\times\partial_t\boldsymbol{m} \quad \text{in } \Omega\times\mathbb{R}_+,$$

with the spin accumulation $\boldsymbol{s}$ satisfying the *steady-state* equation

$$\mathbf{div}\,[J\,(\nabla \boldsymbol{s}, \boldsymbol{m}\chi_\Omega)] - \frac{2D_0}{\lambda_{\rm sf}^2}\boldsymbol{s} - \frac{2D_0}{\lambda_J^2}\boldsymbol{s}\times\boldsymbol{m}\chi_\Omega = \mathbf{0} \quad \text{in } \Omega\times\mathbb{R}_+,$$

One aim of this paper is to turn this observation into a quantitative statement.

For that, one has to rescale the original domain Ω by a scaling factor ℓ to be of size one (without relabelling) and treat $D_0/\lambda_{\rm sf}^2$, D_0/λ_J^2, $\mu_B\boldsymbol{j}_e/e$, and D_0/ℓ^2 as quantities of the same order. Moreover, we also assume that j_0, $\boldsymbol{f}$, $c_{\rm ex}/\ell^2$ are of the same order and $\gamma_0 c_{\rm ex}/\ell^2 \ll D_0/\ell^2$. We introduce the small parameter $\varepsilon = \gamma_0 c_{\rm ex}/D_0$ and note that in typical spintronic applications $\varepsilon \ll 1$ is of the order $10^{-2}\ldots 10^{-4}$.

After rescaling the time of the system by the order of $\gamma_0 c_{\rm ex}/\ell^2$ (see [1, Section 2.2] for a detailed analysis), one can rewrite the SDLLG equation (1)–(2) as

$$\varepsilon\partial_t\boldsymbol{s}_\varepsilon = \mathbf{div}\,[\mathcal{J}\,(\nabla\boldsymbol{s}_\varepsilon, \boldsymbol{m}_\varepsilon)] - \gamma_1\boldsymbol{s}_\varepsilon - \gamma_2\boldsymbol{s}_\varepsilon\times\boldsymbol{m}_\varepsilon \quad \text{in } \Omega\times\mathbb{R}_+, \tag{7}$$

$$\partial_t\boldsymbol{m}_\varepsilon = -\boldsymbol{m}_\varepsilon\times(\boldsymbol{h}_{\rm eff}[\boldsymbol{m}_\varepsilon] + j_0\boldsymbol{s}_\varepsilon + \boldsymbol{f}_\varepsilon) + \alpha\boldsymbol{m}_\varepsilon\times\partial_t\boldsymbol{m}_\varepsilon \quad \text{in } \Omega\times\mathbb{R}_+, \tag{8}$$

with the spin current given by

$$\mathcal{J}\,(\nabla\boldsymbol{s}, \boldsymbol{m}) := D_0\,[\nabla\boldsymbol{s} - \beta\beta'(\nabla\boldsymbol{s}\cdot\boldsymbol{m})\otimes\boldsymbol{m}] - \frac{\beta}{2}\boldsymbol{j}_e\otimes\boldsymbol{m}. \tag{9}$$

Here, γ_1 and γ_2 are positive quantities of order one, while $\varepsilon \ll 1$ is a small parameter. In (7) and (8), to avoid introducing new notation, we denoted by the very same symbols the rescaled version of the physical quantities appearing in the SLLG equation (1)–(2).

For the diffusion coefficient $D_0 \in L^\infty(\Omega)$ we assume the existence of a positive constant $\gamma \in \mathbb{R}_+$ such that $D_0(x) \geqslant \gamma$ for almost all $x \in \Omega$. However, without loss of generality in our arguments, we assume that γ_1, γ_2 are positive constants.

We note that, formally, if $\boldsymbol{s}_\varepsilon \to \boldsymbol{s}$, $\boldsymbol{m}_\varepsilon \to \boldsymbol{m}$, and $\varepsilon\partial_t\boldsymbol{s}_\varepsilon \to 0$ then we recover the SLLG equation predicted in [44], i.e.,

$$\partial_t\boldsymbol{m} = \boldsymbol{m}\times(\boldsymbol{h}_{\rm eff}[\boldsymbol{m}] + j_0\boldsymbol{s} + \boldsymbol{f}) + \alpha\boldsymbol{m}\times\partial_t\boldsymbol{m} \quad \text{in } \Omega\times\mathbb{R}_+, \tag{10}$$

with the spin accumulation $\boldsymbol{s}$ satisfying the *steady-state* equation

$$\mathbf{div}\,[\mathcal{J}\,(\nabla\boldsymbol{s}, \boldsymbol{m})] - \gamma_1\boldsymbol{s} - \gamma_2\boldsymbol{s}\times\boldsymbol{m} = \mathbf{0} \quad \text{in } \Omega\times\mathbb{R}_+, \tag{11}$$

and the spin current $\mathcal{J}$ given by (9).

The main aim of the paper is to show that this is indeed the case, i.e., that in the limit of long times, the rescaled SDLLG (7)–(8) reduces to the simpler SLLG (10)–(11), in which LLG is coupled to the nonlinear (but still nonlocal) steady-state equation. The result provides a solid mathematical ground to the qualitative analysis of Zhang, Levy, and Fert [44] in the context of magnetic multilayers. Besides, the argument shows the existence (but possibly *nonuniqueness*) of global weak solutions of SLLG.

We complete the paper by proving the weak-strong uniqueness of the solutions of the reduced SDLLG, i.e., weak solutions coincide with the (unique) strong solution as long as the latter exists in time. Weak-strong uniqueness results are of particular relevance in the numerical integration of LLG systems. Indeed, available unconditionally convergent integrators assure that subsequences of the computed discrete solutions converge weakly towards a weak solution of LLG. Weak-strong uniqueness results guarantee that all these numerical schemes will converge towards the same limit (even for the full sequence of computed solutions), at least as long as a strong solution exists. We refer to [3, 6] for some seminal works on the numerical analysis of plain LLG and to [1,16] for the analysis of some coupled LLG systems.

In [2, 38], the observations in [44] are empirically validated through a comparative numerical analysis of SDLLG and SLLG models: It is underlined how SLLG can be more effective in describing magnetization dynamics, since it leads to the same experimental results as for SDLLG, but allows for much larger time steps of the numerical integrator.

1.3. State of the art. Many research works contributed to the study of solutions of LLG. Existence and nonuniqueness of global weak solutions to LLG (2) date back to [5]. In [42] (see also [43]), the Maxwell–Landau–Lifshitz equation coupled with spin accumulation is considered, and a suitable regularization procedure is used to obtain the existence of global weak solutions. For the system we are interested in, the existence (and nonuniqueness) of global weak solutions is proved in [23] (see also [1]). SDLLG (7)–(8) is a rescaled version of the three-dimensional model introduced in [23].

The uniqueness of weak solutions depends on the regularity class they belong to. Indeed, in the class of smooth functions, there exists at most one solution of LLG (see, e.g., [11]). Therefore, a natural question is whether existing smooth and weak solutions coincide, rather than coexist, i.e., whether a weak-strong uniqueness result holds for LLG. Such a question is ubiquitous in the analysis of PDEs since the positive answer given by Leray for the Navier–Stokes equations [34]. For LLG, weak-strong uniqueness has only been investigated recently. In [18], weak-strong uniqueness is proved in the simplified setting where $\Omega = \mathbb{R}^3$ (i.e., possible boundary conditions are neglected) and $\boldsymbol{h}_{\text{eff}}$ consists only of the leading-order exchange contribution. Despite the simplified setting, the proof already involves much tedious algebra. In [14], weak-strong uniqueness for the solutions of LLG is obtained for the full relevant $3d$ setting. In particular, the analysis accounts also for the Dzyaloshinskii–Moriya interaction, which is the primary mechanism behind the emergence of magnetic skyrmions, as well as for the demagnetizing field $\boldsymbol{h}_{\text{d}}$ from (6). However, no coupling of LLG with other nonlinear PDE systems is taken into account. In section 4, we show how the approximation argument in [14] can be used to derive weak-strong uniqueness of solutions of the reduced SLLG.

From the above discussion, there emerges the need for regularity results for LLG. In that regard, we recall that LLG is intimately related to the harmonic map heat flow from Ω into $\mathbb{S}^2$ and stationary harmonic maps, for which one cannot expect to have general regularity results in dimensions higher than two [37]. For sufficiently small initial data, there exists a (unique) strong solution which is global in time [11, 20], whereas for general initial data, even in $2d$, solutions may develop finitely many point singularities in finite time [26]. In [25], the authors prove the existence of global smooth solutions of the spin-polarized transport equation (SDLLG) in $2d$ for small initial data. For general dimensions, partial regularity has been investigated in [31, 33, 41] for LLG, and, more recently, in [36] for SDLLG.

1.4. Outline. The paper is organized as follows. In section 2, we state our main results: Theorem 1 concerns the analysis of SDLLG in the limit of long times, whereas Theorem 2 states the weak-strong uniqueness result for the limiting equation. The proofs of Theorems 1 and 2 are given in sections 3 and 4, respectively.

1.5. Notation. For $\Omega \subset \mathbb{R}^3$ open and bounded, we denote by $H^1(\Omega)^*$ the dual space of $H^1(\Omega)$ and by $\langle \cdot, \cdot \rangle_\Omega$ the corresponding duality pairing, understood in the sense of the Gelfand triple $H^1(\Omega) \subset L^2(\Omega) \subset H^1(\Omega)^*$. In particular, if $u, v \in L^2(\Omega)$, then $\langle u, v \rangle_\Omega$ denotes the usual inner product in $L^2(\Omega)$ and $\| \cdot \|_\Omega$ is the induced $L^2(\Omega)$ norm. Vector-valued functions are denoted in boldface, but we do not embolden the function spaces they belong to; the context will clarify what we mean. Instead, we use the symbol $H^1(\Omega, \mathbb{S}^2)$ to denote the metric subspace of $H^1(\Omega)$ consisting of $\mathbb{S}^2$-valued functions. When dealing with time-dependent

vector fields $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$, we will often use the symbol $\boldsymbol{u}(t)$ to denote the section $\boldsymbol{u}(\cdot, t) : x \in \Omega \mapsto \boldsymbol{u}(x,t) \in \mathbb{R}^3$; again, the context will clarify the meaning. Finally, for every $T \in \mathbb{R}_+$ we set $\Omega_T := \Omega \times (0,T)$.

2. Statement of main results

We recall the definition of a global weak solution of the SDLLG system (7)–(8) as given in [1, 23], where also existence results are shown. The definition naturally extends the notion of global weak solution of LLG introduced in [5]. To simplify the notation, we neglect the contributions from crystalline anisotropy and the external applied field $\boldsymbol{f}$. It is straightforward to include them in the analysis, and details are left to the reader.

Definition 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $\boldsymbol{m}^* \in H^1(\Omega, \mathbb{S}^2)$ and $\boldsymbol{s}^* \in H^1(\Omega)$. For $\varepsilon > 0$, the pair $(\boldsymbol{m}_\varepsilon, \boldsymbol{s}_\varepsilon) \in L^\infty(\mathbb{R}_+, H^1(\Omega, \mathbb{S}^2)) \times L^\infty(\mathbb{R}_+, L^2(\Omega))$ is called a global weak solution of the* SDLLG *system* (7)–(8) *if for every $T > 0$ the following properties* (i)–(v) *are satisfied:*

(i) $\boldsymbol{m}_\varepsilon \in H^1(\Omega_T, \mathbb{S}^2)$ *and* $\boldsymbol{m}_\varepsilon(0) = \boldsymbol{m}^*$ *in the sense of traces;*

(ii) $\boldsymbol{s}_\varepsilon \in L^2(\mathbb{R}_+, H^1(\Omega)), \partial_t \boldsymbol{s}_\varepsilon \in L^2(\mathbb{R}_+, H^1(\Omega)^*)$, *and* $\boldsymbol{s}_\varepsilon(0) = \boldsymbol{s}^*$ *in the sense of traces;*

(iii) *for every $\boldsymbol{\varphi} \in H^1(\Omega_T)$, there holds*

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{m}_\varepsilon, \boldsymbol{\varphi} \rangle_\Omega &= \int_0^T \alpha \langle \partial_t \boldsymbol{m}_\varepsilon, \boldsymbol{\varphi} \times \boldsymbol{m}_\varepsilon \rangle_\Omega + c_{\mathrm{ex}} \langle \nabla \boldsymbol{m}_\varepsilon, \nabla(\boldsymbol{\varphi} \times \boldsymbol{m}_\varepsilon) \rangle_\Omega \\ &\quad - \mu_0 \int_0^T \langle \boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}_\varepsilon], \boldsymbol{\varphi} \times \boldsymbol{m}_\varepsilon \rangle_\Omega - \int_0^T j_0 \langle \boldsymbol{s}_\varepsilon, \boldsymbol{\varphi} \times \boldsymbol{m}_\varepsilon \rangle_\Omega; \end{aligned} \tag{12}$$

(iv) *for all $\boldsymbol{\varphi} \in L^2(\mathbb{R}_+, H^1(\Omega))$, there holds*

$$\begin{aligned}\varepsilon \int_0^T \langle \partial_t \boldsymbol{s}_\varepsilon, \boldsymbol{\varphi} \rangle_\Omega &= -\int_0^T \langle \mathcal{J}(\nabla \boldsymbol{s}_\varepsilon, \boldsymbol{m}_\varepsilon), \nabla \boldsymbol{\varphi} \rangle_\Omega - \gamma_1 \int_0^T \langle \boldsymbol{s}_\varepsilon, \boldsymbol{\varphi} \rangle_\Omega \\ &\quad - \gamma_2 \int_0^T \langle \boldsymbol{s}_\varepsilon \times \boldsymbol{m}_\varepsilon, \boldsymbol{\varphi} \rangle_\Omega - \frac{\beta}{2} \int_0^T \int_{\partial\Omega} (\boldsymbol{m}_\varepsilon \cdot \boldsymbol{\varphi})(\boldsymbol{j}_e \cdot \boldsymbol{n}), \end{aligned} \tag{13}$$

where $\boldsymbol{j}_e \in L^2(\mathbb{R}_+, H^1(\Omega))$ is a prescribed current and $\mathcal{J}$ is given by (9)*;*

(v) *the following energy inequality holds:*

$$\mathcal{F}_\Omega(\boldsymbol{m}_\varepsilon(T)) + \alpha \int_0^T \|\partial_t \boldsymbol{m}_\varepsilon\|_\Omega^2 \leqslant \mathcal{F}_\Omega(\boldsymbol{m}^*) + \int_0^T \langle \partial_t \boldsymbol{m}_\varepsilon, j_0 \boldsymbol{s}_\varepsilon \rangle_\Omega, \tag{14a}$$

with

$$\mathcal{F}_\Omega(\boldsymbol{m}_\varepsilon(T)) = \frac{c_{\mathrm{ex}}}{2} \int_\Omega |\nabla \boldsymbol{m}_\varepsilon|^2 - \frac{\mu_0}{2} \int_\Omega \boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}_\varepsilon \chi_\Omega] \cdot \boldsymbol{m}_\varepsilon.$$

Remark 2.1. If $(\boldsymbol{m}_\varepsilon, \boldsymbol{s}_\varepsilon)$ is a weak solution of SDLLG, then standard results guarantee that $\boldsymbol{m}_\varepsilon \in C(0,T; L^2(\Omega))$ and $\boldsymbol{s}_\varepsilon \in C(0,T; L^2(\Omega))$, cf., e.g., [19, Section 5.9.2, Theorems 2 and 3].

Remark 2.2. Since the seminal paper [5], the definition of global weak solutions of LLG includes the energy requirement (v), which even holds with equality for strong solutions of LLG. However, in the definition of weak solutions of SDLLG given in [23], the condition (v) has been omitted. It was later shown in [1, Thm. 24] that this requirement can be satisfied.

Remark 2.3. Throughout, $\nabla \boldsymbol{s} = (\partial_i s_j)_{i,j=1}^3$ stands for the matrix whose columns are the gradients of the components of $\boldsymbol{s}$, i.e., the transposed of the Jacobian matrix of $\boldsymbol{s}$. In our notation, the Jacobian matrix is thus denoted by $\nabla^\top \boldsymbol{s}$. The same remark applies to $\nabla \boldsymbol{m}$. The vector **div** operator acts on the columns of $\nabla \boldsymbol{s}$: If $A : \Omega \to \mathbb{R}^{3\times 3}$ is a matrix-valued function, then $\mathbf{div}\, A = \sum_{i=1}^3 \mathrm{div}\,(Ae_i)e_i$, with $(e_i)_{i=1}^3$ the canonical basis of $\mathbb{R}^3$, is the vector having for

components the scalar divergence of the columns of A. In particular, $\mathbf{div}\,\nabla \boldsymbol{s} = \Delta \boldsymbol{s}$. Also, if $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^3$ is a vector field, then

$$\mathbf{div}\, A \cdot \boldsymbol{\varphi} = \sum_{i=1}^{3} \varphi_i \mathrm{div}\,(Ae_i) = \sum_{i=1}^{3} [\mathrm{div}\,(\varphi_i A e_i) - \nabla \varphi_i \cdot Ae_i] = \mathrm{div}\,(A\boldsymbol{\varphi}) - A{:}\nabla \boldsymbol{\varphi}. \tag{15}$$

According to the divergence theorem, this leads to

$$\int_\Omega \mathbf{div}\, A \cdot \boldsymbol{\varphi} = \int_{\partial\Omega} A\boldsymbol{\varphi} \cdot \boldsymbol{n} - \int_\Omega A{:}\nabla \boldsymbol{\varphi}. \tag{16}$$

From the previous considerations, it is clear that (13) is the natural weak formulation of (7). Indeed, applying (16) and (15) to (9), we have

$$\int_\Omega \mathbf{div}\,[(\nabla \boldsymbol{s} \cdot \boldsymbol{m}) \otimes \boldsymbol{m}] \cdot \boldsymbol{\varphi} = -\int_\Omega (\nabla \boldsymbol{s} \cdot \boldsymbol{m}) \otimes \boldsymbol{m}{:}\nabla \varphi + \int_{\partial\Omega} ((\nabla \boldsymbol{s} \cdot \boldsymbol{m}) \otimes \boldsymbol{m})\boldsymbol{\varphi} \cdot \boldsymbol{n}$$

and the last integrand gives

$$((\nabla \boldsymbol{s} \cdot \boldsymbol{m}) \otimes \boldsymbol{m})\boldsymbol{\varphi} \cdot \boldsymbol{n} \;=\; (\boldsymbol{m} \cdot \boldsymbol{\varphi})(\nabla \boldsymbol{s} \cdot \boldsymbol{m}) \cdot \boldsymbol{n} \;=\; (\boldsymbol{m} \cdot \boldsymbol{\varphi})(\partial_{\boldsymbol{n}} \boldsymbol{s} \cdot \boldsymbol{m}) = 0 \quad \text{on } \partial\Omega\,.$$

Similarly, integration by parts of the term $\mathbf{div}\,(\boldsymbol{j}_e \otimes \boldsymbol{m}) \cdot \boldsymbol{\varphi}$ gives the boundary term

$$(\boldsymbol{j}_e \otimes \boldsymbol{m})\boldsymbol{\varphi} \cdot \boldsymbol{n} = (\boldsymbol{m} \cdot \boldsymbol{\varphi})(\boldsymbol{j}_e \cdot \boldsymbol{n}).$$

For future reference, we collect here the mathematical assumptions on the physical parameters of the SDLLG system (12)–(13) that will be assumed throughout the paper:

Assumptions on the physical parameters of the system. In what follows, we assume that $\alpha, c_{\mathrm{ex}}, j_0, \mu_0$ are positive constants (cf. (12)). Also, we assume that $D_0 \in L^\infty(\Omega)$ and that there exists a positive constant $\gamma \in \mathbb{R}_+$ such that $D_0(x) \geqslant \gamma$ for almost all $x \in \Omega$. The coefficients γ_1, γ_2 are assumed to be positive constants (cf. (13)). Finally, we assume that $0 < \beta, \beta' < 1$ (cf. (9)). (H1)

Our first contribution is the following result concerning the behavior of the spin transport equation in the limit $\varepsilon \to 0$. For the sake of clarity, we will often write the equations in strong form, although their weak counterpart is meant.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and assume* (H1)*. For every $\varepsilon > 0$, let $(\boldsymbol{m}_\varepsilon, \boldsymbol{s}_\varepsilon) \in L^\infty(\mathbb{R}_+, H^1(\Omega, \mathbb{S}^2)) \times L^\infty(\mathbb{R}_+, L^2(\Omega))$ be a weak solution of the* SDLLG *system* (12)–(13)*. Then, there exists a vector field $\boldsymbol{m}_0 \in L^2(\mathbb{R}_+, H^1(\Omega, \mathbb{S}^2))$ such that*

$$\boldsymbol{m}_\varepsilon \rightharpoonup \boldsymbol{m}_0 \qquad \text{weakly in } L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1(\Omega, \mathbb{S}^2)),$$

and $\boldsymbol{m}_0 \in H^1(\Omega_T, \mathbb{S}^2)$ for every $T > 0$. Moreover, the vector field $\boldsymbol{m}_0$ satisfies the SLLG *equation*

$$\partial_t \boldsymbol{m}_0 = -\boldsymbol{m}_0 \times (\boldsymbol{h}_{\mathrm{eff}}[\boldsymbol{m}_0] + j_0 \mathcal{H}_s[\boldsymbol{m}_0] + \boldsymbol{f}) + \alpha \boldsymbol{m}_0 \times \partial_t \boldsymbol{m}_0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{17}$$

with

$$\begin{cases} \boldsymbol{m}_0(0) = \boldsymbol{m}^*(x) & \text{in} \quad \Omega, \\ \partial_{\boldsymbol{n}} \boldsymbol{m}_0 = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}_+. \end{cases}$$

Here, $\mathcal{H}_s : \boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2) \mapsto \mathcal{H}_s[\boldsymbol{m}] \in H^1(\Omega, \mathbb{R}^3)$ denotes the nonlinear operator which maps every $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$ to the unique solution $\boldsymbol{s} := \mathcal{H}_s[\boldsymbol{m}] \in H^1(\Omega, \mathbb{R}^3)$ of the stationary spin-diffusion equation

$$-\,\mathbf{div}\,[\mathcal{J}\,(\nabla \boldsymbol{s}, \boldsymbol{m})] + \gamma_1 \boldsymbol{s} + \gamma_2 \boldsymbol{s} \times \boldsymbol{m} = 0 \text{ in } \Omega, \quad \text{subject to } \partial_{\boldsymbol{n}} \boldsymbol{s} = 0 \text{ on } \partial\Omega. \tag{18}$$

We have written the reduced equations (17)–(18) in strong form to improve the readability (their weak formulation is immediate to derive). As for the system (7)–(8), uniqueness of weak solutions is, in general, out of the question. In fact, when $\boldsymbol{j}_e \equiv 0$, the system SLLG reduces to the classical LLG equation for which possible nonuniqueness has been shown in [5].

In the statement of Theorem 1, we assumed that Ω is a Lipschitz domain. Our second contribution is the following weak-strong uniqueness result, for which higher regularity of the domain Ω, as well as regularity assumptions on the diffusion coefficient D_0 become essential.

Theorem 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary, and assume that* (H1) *holds with $D_0 \in C^\infty(\overline{\Omega})$. Let $\boldsymbol{m}^* \in C^\infty(\overline{\Omega}, \mathbb{S}^2)$ and $T > 0$. Suppose that $\boldsymbol{m}_1, \boldsymbol{m}_2 \in L^\infty(\mathbb{R}_+, H^1(\Omega, \mathbb{S}^2))$ are two global weak solutions of* (17)*. If $\boldsymbol{m}_1 \in C^\infty(\overline{\Omega_T}, \mathbb{S}^2)$, then*

$$\boldsymbol{m}_1 \equiv \boldsymbol{m}_2 \quad \textit{a.e. in } \Omega_T.$$

Remark 2.4. A closer look to the proof of Theorem 2 shows that it is sufficient to assume $\boldsymbol{m}^* \in C^3(\overline{\Omega}, \mathbb{S}^2)$, $D_0 \in C^3(\overline{\Omega})$, and $\boldsymbol{m}_1 \in C^3(\overline{\Omega_T}, \mathbb{S}^2)$, but we do not dwell on this. Also, we provide a proof of the weak-strong uniqueness result in a more general form (see section 4.2).

In the proofs of Theorems 1 and 2, we will make use of some properties of the demagnetizing field operator $\boldsymbol{h}_\mathrm{d}$ that we recall here (cf. [15]). If Ω is a bounded domain, $\boldsymbol{m} \in L^2(\Omega, \mathbb{R}^3)$, and $\boldsymbol{h}_\mathrm{d}[\boldsymbol{m}\chi_\Omega] \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ is a solution of the Maxwell–Ampère equations (6) then, by Poincaré's lemma, $\boldsymbol{h}_\mathrm{d}[\boldsymbol{m}\chi_\Omega] = \nabla v_{\boldsymbol{m}}$, where v_m is the unique solution in $H^1(\mathbb{R}^3)$ of the Poisson's equation

$$-\Delta v_{\boldsymbol{m}} = \operatorname{div}(\boldsymbol{m}\chi_\Omega) \quad \text{in } \mathbb{R}^3. \tag{19}$$

Therefore, the demagnetizing field can be described as the map which to every magnetization $\boldsymbol{m} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ associates the distributional gradient of the unique solution of (19) in $H^1(\mathbb{R}^3)$. It is easily seen that the map $-\boldsymbol{h}_\mathrm{d} : \boldsymbol{m} \in L^2(\Omega, \mathbb{R}^3) \mapsto -\nabla v_{\boldsymbol{m}} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ defines a self-adjoint and positive-definite bounded linear operator from $L^2(\mathbb{R}^3, \mathbb{R}^3)$ into itself:

$$-\int_\Omega \boldsymbol{h}_\mathrm{d}[\boldsymbol{m}_1\chi_\Omega] \cdot \boldsymbol{m}_2 = -\int_\Omega \boldsymbol{h}_\mathrm{d}[\boldsymbol{m}_2\chi_\Omega] \cdot \boldsymbol{m}_1 \tag{20}$$

and

$$-\int_\Omega \boldsymbol{h}_\mathrm{d}[\boldsymbol{m}_1\chi_\Omega] \cdot \boldsymbol{m}_1 = \int_{\mathbb{R}^3} |\boldsymbol{h}_\mathrm{d}[\boldsymbol{m}_1\chi_\Omega]|^2 \leqslant \int_\Omega |\boldsymbol{m}_1|^2 \tag{21}$$

for every $\boldsymbol{m}_1, \boldsymbol{m}_2 \in L^2(\Omega, \mathbb{R}^3)$.

3. From SDLLG to SLLG: Proof of Theorem 1

For convenience, we split the proof in three steps.

Step 1 (Uniform estimates). For $T > 0$, we test (13) against $\boldsymbol{\varphi} = \boldsymbol{s}_\varepsilon$ to obtain

$$\begin{aligned} \varepsilon \int_0^T \langle \partial_t \boldsymbol{s}_\varepsilon, \boldsymbol{s}_\varepsilon \rangle \;=\; & -\int_0^T \int_\Omega \mathcal{J}(\nabla \boldsymbol{s}_\varepsilon, \boldsymbol{m}_\varepsilon) : \nabla \boldsymbol{s}_\varepsilon - \gamma_1 \int_0^T \|\boldsymbol{s}_\varepsilon\|_\Omega^2 \\ & -\frac{\beta}{2} \int_0^T \int_{\partial\Omega} (\boldsymbol{m}_\varepsilon \cdot \boldsymbol{s}_\varepsilon)(\boldsymbol{j}_e \cdot \boldsymbol{n}). \end{aligned} \tag{22}$$

An integration by parts gives

$$\int_0^T \langle \partial_t \boldsymbol{s}_\varepsilon, \boldsymbol{s}_\varepsilon \rangle = \frac{1}{2} \int_0^T \partial_t \|\boldsymbol{s}_\varepsilon\|_\Omega^2 = \frac{1}{2} \|\boldsymbol{s}_\varepsilon(T)\|_\Omega^2 - \frac{1}{2} \|\boldsymbol{s}^*\|_\Omega^2.$$

On the other hand,

$$\begin{aligned}\int_\Omega \mathcal{J}\left(\nabla \boldsymbol{s}_\varepsilon, \boldsymbol{m}_\varepsilon\right):\nabla \boldsymbol{s}_\varepsilon &= \int_\Omega D_0 \left|\nabla \boldsymbol{s}_\varepsilon\right|^2 - D_0\beta\beta' \boldsymbol{m}_\varepsilon \otimes \left(\nabla \boldsymbol{s}_\varepsilon \cdot \boldsymbol{m}_\varepsilon\right):\nabla \boldsymbol{s}_\varepsilon \\ &\qquad -\frac{\beta}{2}(\boldsymbol{j}_e \otimes \boldsymbol{m}_\varepsilon):\nabla \boldsymbol{s}_\varepsilon \\ &= \int_\Omega D_0 \left(\left|\nabla \boldsymbol{s}_\varepsilon\right|^2 - \beta\beta' \left|\nabla \boldsymbol{s}_\varepsilon \cdot \boldsymbol{m}_\varepsilon\right|^2\right) - \frac{\beta}{2}(\boldsymbol{j}_e \otimes \boldsymbol{m}_\varepsilon):\nabla \boldsymbol{s}_\varepsilon. \quad (23)\end{aligned}$$

Recall that $\beta\beta' < 1$ (see [44]). Let $\gamma_\beta := (1-\beta\beta') > 0$ and $\gamma := \operatorname{ess\,inf}_\Omega D_0 > 0$. With

$$\left|\nabla \boldsymbol{s}_\varepsilon\right|^2 - \beta\beta' \left|\nabla \boldsymbol{s}_\varepsilon \cdot \boldsymbol{m}_\varepsilon\right|^2 \geqslant \gamma_\beta \left|\nabla \boldsymbol{s}_\varepsilon\right|^2, \quad (24)$$

the estimates (22) and (23) lead to

$$\begin{aligned}&\frac{\varepsilon}{2}\left\|\boldsymbol{s}_\varepsilon\left(T\right)\right\|_\Omega^2 + \gamma_1 \int_0^T \|\boldsymbol{s}_\varepsilon\|_\Omega^2 + \gamma\gamma_\beta \int_0^T \|\nabla \boldsymbol{s}_\varepsilon\|_\Omega^2 \\ &\qquad \leqslant \frac{\varepsilon}{2}\|\boldsymbol{s}^*\|_\Omega^2 + \frac{\beta}{2}\int_0^T \int_\Omega (\boldsymbol{j}_e \otimes \boldsymbol{m}_\varepsilon):\nabla \boldsymbol{s}_\varepsilon - \frac{\beta}{2}\int_0^T \int_{\partial\Omega} (\boldsymbol{m}_\varepsilon \cdot \boldsymbol{s}_\varepsilon)(\boldsymbol{j}_e \cdot \boldsymbol{n}) \\ &\qquad \leqslant \frac{\varepsilon}{2}\|\boldsymbol{s}^*\|_\Omega^2 + \frac{\beta}{2}\int_0^T \|\boldsymbol{j}_e\|_\Omega \|\nabla \boldsymbol{s}_\varepsilon\|_\Omega - \frac{\beta}{2}\int_0^T \int_{\partial\Omega} (\boldsymbol{m}_\varepsilon \cdot \boldsymbol{s}_\varepsilon)(\boldsymbol{j}_e \cdot \boldsymbol{n}), \quad (25)\end{aligned}$$

The continuous embedding of $H^1(\Omega)$ into $L^2(\partial\Omega)$ implies the existence of $\delta > 0$ such that (we use Young's inequality)

$$\varepsilon \left\|\boldsymbol{s}_\varepsilon\left(T\right)\right\|_\Omega^2 + \gamma_1 \|\boldsymbol{s}_\varepsilon\|_{\Omega_T}^2 + c_\delta \int_0^T \|\nabla \boldsymbol{s}_\varepsilon\|_\Omega^2 \leqslant \varepsilon \|\boldsymbol{s}^*\|_\Omega^2 + \delta\left(\|\boldsymbol{j}_e\|_{\Omega\times\mathbb{R}_+}^2 + \|\boldsymbol{j}_e\|_{\partial\Omega\times\mathbb{R}_+}^2\right) \quad (26)$$

for some constant $c_\delta > 0$ which depends only on $\delta, \beta, \gamma, \gamma_\beta$, and Ω. Taking the supremum over $T > 0$, we infer that

$$\gamma_1 \|\boldsymbol{s}_\varepsilon\|_{\Omega\times\mathbb{R}_+}^2 + c_\delta \int_{\mathbb{R}_+} \|\nabla \boldsymbol{s}_\varepsilon\|_\Omega^2 \leqslant \varepsilon \|\boldsymbol{s}^*\|_\Omega^2 + \delta\left(\|\boldsymbol{j}_e\|_{\Omega\times\mathbb{R}_+}^2 + \|\boldsymbol{j}_e\|_{\partial\Omega\times\mathbb{R}_+}^2\right). \quad (27)$$

Therefore, $(\boldsymbol{s}_\varepsilon)$ is *uniformly bounded* in $L^2(\mathbb{R}_+, H^1(\Omega))$.

Step 2 (The steady-state limit). The uniform bound on $(\boldsymbol{s}_\varepsilon)$ implies the existence of a (not relabeled) subsequence $(\boldsymbol{s}_\varepsilon)$ in $L^2(\mathbb{R}_+; H^1(\Omega))$ such that $\boldsymbol{s}_\varepsilon \rightharpoonup \boldsymbol{s}_0$ weakly in $L^2(\mathbb{R}_+, H^1(\Omega))$ as $\varepsilon \to 0$. In particular, for every $T > 0$, we have $\boldsymbol{s}_\varepsilon \rightharpoonup \boldsymbol{s}_0$ weakly in $L^2(\Omega_T, \mathbb{R}^3)$ as $\varepsilon \to 0$. Next, for every $\boldsymbol{\varphi} \in C^\infty(\overline{\Omega_T}, \mathbb{R}^3)$, equation (13) reads, in expanded form, as

$$\begin{aligned}&\int_{\Omega_T} \mathcal{J}\left(\nabla \boldsymbol{s}_\varepsilon, \boldsymbol{m}_\varepsilon\right):\nabla \boldsymbol{\varphi} + \gamma_1 \int_{\Omega_T} \boldsymbol{s}_\varepsilon \cdot \boldsymbol{\varphi} + \gamma_2 \int_{\Omega_T} (\boldsymbol{s}_\varepsilon \times \boldsymbol{m}_\varepsilon) \cdot \boldsymbol{\varphi} \\ &\quad + \frac{\beta}{2}\int_0^T \int_{\partial\Omega} (\boldsymbol{m}_\varepsilon \cdot \boldsymbol{\varphi})(\boldsymbol{j}_e \cdot \boldsymbol{n}) \overset{(13)}{=} \varepsilon \int_0^T \langle \boldsymbol{s}_\varepsilon, \partial_t \boldsymbol{\varphi}\rangle - \varepsilon\Big[\langle \boldsymbol{s}_\varepsilon(T), \boldsymbol{\varphi}(T)\rangle - \langle \boldsymbol{s}^*, \boldsymbol{\varphi}(0)\rangle\Big] \quad (28)\end{aligned}$$

with

$$\begin{aligned}\int_\Omega \mathcal{J}\left(\nabla \boldsymbol{s}_\varepsilon, \boldsymbol{m}_\varepsilon\right):\nabla \boldsymbol{\varphi} &\overset{(9)}{=} \int_\Omega D_0 \left(\nabla \boldsymbol{s}_\varepsilon:\nabla \boldsymbol{\varphi} - \beta\beta' \left(\nabla \boldsymbol{s}_\varepsilon \cdot \boldsymbol{m}_\varepsilon\right) \otimes \boldsymbol{m}_\varepsilon : \nabla \boldsymbol{\varphi}\right) \\ &\qquad - \frac{\beta}{2}\int_\Omega (\boldsymbol{j}_e \otimes \boldsymbol{m}_\varepsilon):\nabla \boldsymbol{\varphi}. \quad (29)\end{aligned}$$

Now, we use the energy inequality (14a). By Young's inequality, we can absorb a part of $|\langle \partial_t \boldsymbol{m}_\varepsilon, \boldsymbol{s}_\varepsilon\rangle|$ into the left-hand side of (14a). With positive constants $\alpha_0, \alpha_1 > 0$, we find that

$$\mathcal{F}_\Omega\left(\boldsymbol{m}_\varepsilon\left(T\right)\right) + \alpha_0 \int_0^T \|\partial_t \boldsymbol{m}_\varepsilon\|_\Omega^2 \overset{(14a)}{\leqslant} \mathcal{F}_\Omega(\boldsymbol{m}^*) + \alpha_1 \|\boldsymbol{s}_\varepsilon\|_{\Omega\times\mathbb{R}_+}^2. \quad (30)$$

Thus, from the uniform bound (27) on $\|\boldsymbol{s}_\varepsilon\|^2_{\Omega\times\mathbb{R}_+}$ and (21), we infer a uniform bound on the family $(\boldsymbol{m}_\varepsilon)$ in $L^\infty(\mathbb{R}_+, H^1(\Omega,\mathbb{S}^2))$, as well as a uniform bound on $(\partial_t \boldsymbol{m}_\varepsilon)$ in $L^2(\mathbb{R}_+, L^2(\Omega))$. By the Aubin–Lions–Simon lemma, there hence exists $\boldsymbol{m}_0 \in L^\infty(\mathbb{R}_+, H^1(\Omega,\mathbb{S}^2))$ such that, up to a subsequence,

$$\begin{aligned} \boldsymbol{m}_\varepsilon &\to \boldsymbol{m}_0 && \text{strongly in } C^0(0,T;L^2(\Omega,\mathbb{S}^2)), && (31)\\ \partial_t \boldsymbol{m}_\varepsilon &\rightharpoonup \partial_t \boldsymbol{m}_0 && \text{weakly in } L^2(\mathbb{R}_+, L^2(\Omega)). && (32)\end{aligned}$$

In particular, $\boldsymbol{m}_\varepsilon \to \boldsymbol{m}_0$ strongly in $L^2(\Omega_T)$, from which it follows that $\boldsymbol{m}_\varepsilon \otimes \boldsymbol{m}_\varepsilon \to \boldsymbol{m}_0 \otimes \boldsymbol{m}_0$ strongly in $L^2(\Omega_T)$. Indeed, $|\boldsymbol{m}_\varepsilon| = 1 = |\boldsymbol{m}_0|$ a.e. in Ω_T guarantees that

$$\begin{aligned} |\boldsymbol{m}_\varepsilon \otimes \boldsymbol{m}_\varepsilon - \boldsymbol{m}_0 \otimes \boldsymbol{m}_0| &\leqslant |\boldsymbol{m}_\varepsilon \otimes (\boldsymbol{m}_\varepsilon - \boldsymbol{m}_0)| + |(\boldsymbol{m}_\varepsilon - \boldsymbol{m}_0) \otimes \boldsymbol{m}_0| \\ &\leqslant 2|\boldsymbol{m}_\varepsilon - \boldsymbol{m}_0|. \qquad (33)\end{aligned}$$

Hence, for every $\boldsymbol{\varphi} \in C^\infty(\overline{\Omega_T}, \mathbb{R}^3)$, we have

$$\int_0^T \left(\mathcal{J}\left(\nabla \boldsymbol{s}_\varepsilon, \boldsymbol{m}_\varepsilon\right), \nabla \boldsymbol{\varphi}\right)_\Omega \to \int_0^T \left(\mathcal{J}\left(\nabla \boldsymbol{s}_0, \boldsymbol{m}_0\right), \nabla \boldsymbol{\varphi}\right)_\Omega . \qquad (34)$$

Overall, using that $\boldsymbol{m}_\varepsilon \rightharpoonup \boldsymbol{m}_0$ weakly in $L^2(0,T;L^2(\partial\Omega))$, we obtain

$$\begin{aligned} -\lim_{\varepsilon\to 0} \varepsilon \left(\boldsymbol{s}_\varepsilon(T), \boldsymbol{\varphi}(T)\right)_\Omega \overset{(28)}{=} & \int_0^T \left(\mathcal{J}\left(\nabla \boldsymbol{s}_0, \boldsymbol{m}_0\right), \nabla \boldsymbol{\varphi}\right)_\Omega + \int_0^T (\gamma_1 \boldsymbol{s}_0 + \gamma_2 \boldsymbol{s}_0 \times \boldsymbol{m}_0, \boldsymbol{\varphi})_\Omega \\ & + \frac{\beta}{2}\int_0^T \int_{\partial\Omega} (\boldsymbol{m}_0 \cdot \boldsymbol{\varphi})(\boldsymbol{j}_e \cdot \boldsymbol{n}), \qquad (35)\end{aligned}$$

since $\lim_{\varepsilon\to 0} \int_0^T \langle \boldsymbol{s}_\varepsilon, \partial_t \boldsymbol{\varphi}\rangle = \int_0^T \langle \boldsymbol{s}_0, \partial_t \boldsymbol{\varphi}\rangle$.

It remains to compute the limit on the left-hand side of (35). To this end, we observe that (26) leads to $\|\boldsymbol{s}_\varepsilon(T)\|^2_\Omega \leqslant \|\boldsymbol{s}^*\|^2_\Omega + \varepsilon^{-1}\delta(\|\boldsymbol{j}_e\|^2_{\Omega\times\mathbb{R}_+} + \|\boldsymbol{j}_e\|^2_{\partial\Omega\times\mathbb{R}_+})$. This gives the estimate

$$\varepsilon \|\boldsymbol{s}_\varepsilon(T)\|_\Omega \leqslant \left[\varepsilon^2 \|\boldsymbol{s}^*\|^2_\Omega + \varepsilon\delta\left(\|\boldsymbol{j}_e\|^2_{\Omega\times\mathbb{R}_+} + \|\boldsymbol{j}_e\|^2_{\partial\Omega\times\mathbb{R}_+}\right)\right]^{\frac{1}{2}}, \qquad (36)$$

from which it follows that

$$\varepsilon \left|(\boldsymbol{s}_\varepsilon(T), \boldsymbol{\varphi}(T))_\Omega\right| \leqslant \varepsilon \|\boldsymbol{s}_\varepsilon(T)\|_\Omega \|\boldsymbol{\varphi}(T)\|_\Omega \to 0.$$

Summarizing, for every $\boldsymbol{\varphi} \in C^\infty(\overline{\Omega_T}, \mathbb{R}^3)$, the family $(\boldsymbol{s}_\varepsilon, \boldsymbol{m}_\varepsilon)_{\varepsilon\in\mathbb{R}_+}$ converges, for $\varepsilon \to 0$, to a solution of the equation

$$\begin{aligned} \int_0^T \left(\mathcal{J}\left(\nabla \boldsymbol{s}_0, \boldsymbol{m}_0\right), \nabla \boldsymbol{\varphi}\right)_\Omega + \int_0^T (\gamma_1 \boldsymbol{s}_0 + \gamma_2 \boldsymbol{s}_0 \times \boldsymbol{m}_0, \boldsymbol{\varphi})_\Omega \\ + \frac{\beta}{2}\int_0^T \int_{\partial\Omega} (\boldsymbol{m}_0 \cdot \boldsymbol{\varphi})(\boldsymbol{j}_e \cdot \boldsymbol{n}) = 0. \qquad (37)\end{aligned}$$

By density, the previous relation holds for every $\boldsymbol{\varphi} \in L^2(\mathbb{R}_+, H^1(\Omega))$. This gives the limit equation (18).

Finally, (17) follows by a standard application of the convergence relations (31) and (32) to the weak formulation of LLG given in (12).

Step 3 (Unique solvability of the limit spin diffusion equation (18)**.** The proof is completed as soon as we show that for every $\boldsymbol{m} \in H^1(\Omega,\mathbb{S}^2)$ there exists a *unique* solution $\boldsymbol{s} := \mathcal{H}_s[\boldsymbol{m}]$ of the stationary spin-diffusion equation (18). This is the content of the next lemma, which also provides some details on the regularity of the operator $\mathcal{H}_s$ that is exploited in the proof of the weak-strong uniqueness theorem.

Lemma 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. For any $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$ and $\mathcal{J}$ given by (9), there exists a unique solution $\boldsymbol{s} := \mathcal{H}_s[\boldsymbol{m}] \in H^1(\Omega, \mathbb{R}^3)$ of the stationary spin-diffusion equation*

$$-\,\mathbf{div}\,[\mathcal{J}\,(\nabla \boldsymbol{s}, \boldsymbol{m})] + \gamma_1 \boldsymbol{s} + \gamma_2 \boldsymbol{s} \times \boldsymbol{m} = 0 \text{ in } \Omega, \quad \text{subject to } \partial_{\boldsymbol{n}} \boldsymbol{s} = 0 \text{ on } \partial\Omega. \tag{38}$$

Moreover, the operator $\mathcal{H}_s : H^1(\Omega, \mathbb{S}^2) \to H^1(\Omega, \mathbb{R}^3)$ maps the space $C^{k+1}(\overline{\Omega}, \mathbb{S}^2)$ into the space $C^k(\bar{\Omega}, \mathbb{R}^3)$ provided that $D_0, \boldsymbol{j}_e \in C^{k+1}(\overline{\Omega})$ and Ω is of class $C^{k+1,1}$.

Remark 3.1. Lemma 1 is the only point where a difference arise if one assumes that Ω is strictly included in Ω'. In this case, since $\boldsymbol{m} \equiv 0$ in $\Omega' \setminus \Omega$, a similar result on the regularity in Ω (but up to the boundary $\partial\Omega$) of the solutions of (38) cannot be inferred due to the jump discontinuity of the $\boldsymbol{m}$-dependent coefficients. This is the reason why, when dealing with partial regularity results for weak solutions of the SDLLG equation, one assumes that $\Omega = \Omega'$ (cf. [25, 36]). That said, everything we state still works in the case $\Omega \subset \Omega'$ as soon as one agrees that strong solutions have a smooth induced spin accumulation on the interface $\partial\Omega \cap \Omega'$.

Proof. First, we note that the spin-diffusion equation can be rearranged in a more convenient form. The weak formulation of (18) gives the relation

$$\int_\Omega D_0\,[\nabla \boldsymbol{s} - \beta\beta'(\nabla \boldsymbol{s} \cdot \boldsymbol{m}) \otimes \boldsymbol{m}] : \nabla \boldsymbol{\varphi} + \int_\Omega (\gamma_1 \boldsymbol{s} + \gamma_2 \boldsymbol{s} \times \boldsymbol{m}) \cdot \boldsymbol{\varphi} = -\frac{\beta}{2}\,\langle \mathbf{div}\,(\boldsymbol{j}_e \otimes \boldsymbol{m}), \boldsymbol{\varphi}\rangle_\Omega \tag{39}$$

for every $\boldsymbol{\varphi} \in H^1(\Omega)$. We observe that

$$\begin{aligned} D_0\,[\nabla \boldsymbol{s} - \beta\beta'(\nabla \boldsymbol{s} \cdot \boldsymbol{m}) \otimes \boldsymbol{m}] : \nabla \boldsymbol{\varphi} &= D_0 \left[\nabla^\top \boldsymbol{s} - \beta\beta'(\boldsymbol{m} \otimes \boldsymbol{m}) \nabla^\top \boldsymbol{s}\right] : \nabla^\top \boldsymbol{\varphi} \\ &= D_0 \left[(I - \beta\beta'(\boldsymbol{m} \otimes \boldsymbol{m}))\, \nabla^\top \boldsymbol{s}\right] : \nabla^\top \boldsymbol{\varphi}. \end{aligned} \tag{40}$$

Therefore, (18) can be written as

$$-\sum_{i=1}^{3} \partial_i \left(D_0\,(I - \beta\beta'(\boldsymbol{m} \otimes \boldsymbol{m}))\, \partial_i \boldsymbol{s}\right) + \gamma_1 \boldsymbol{s} + \gamma_2 \boldsymbol{s} \times \boldsymbol{m} = -\frac{\beta}{2} \mathbf{div}\,(\boldsymbol{j}_e \otimes \boldsymbol{m}). \tag{41}$$

For every $\boldsymbol{\xi} \in \mathbb{R}^3$, it holds that

$$[I - \beta\beta'(\boldsymbol{m} \otimes \boldsymbol{m})]\boldsymbol{\xi} \cdot \boldsymbol{\xi} = |\boldsymbol{\xi}|^2 - \beta\beta'(\boldsymbol{m} \cdot \boldsymbol{\xi})^2 = (1 - \beta\beta')|\boldsymbol{\xi}|^2 + \beta\beta'|\boldsymbol{\xi} \times \boldsymbol{m}|^2\,. \tag{42}$$

Therefore, the matrix $A_{\boldsymbol{m}} := D_0[I - \beta\beta'(\boldsymbol{m} \otimes \boldsymbol{m})]$ is uniformly positive definite, i.e.,

$$A_{\boldsymbol{m}}(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant \gamma(1 - \beta\beta')|\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^3 \tag{43}$$

with $\gamma := \operatorname{ess\,inf}_\Omega D_0 > 0$. Hence, if we set $\boldsymbol{f}_{\boldsymbol{m}} := -(\beta/2)\mathbf{div}\,(\boldsymbol{j}_e \otimes \boldsymbol{m})$ and denote by $K_{\boldsymbol{m}}$ the matrix in $\mathbb{R}^{3\times 3}$ such that $K_{\boldsymbol{m}}\boldsymbol{\eta} = \gamma_1 \boldsymbol{\eta} + \gamma_2 \boldsymbol{\eta} \times \boldsymbol{m}$ for every $\boldsymbol{\eta} \in \mathbb{R}^3$, we can rearrange (41) into the form of a strongly elliptic system:

$$-\sum_{i=1}^{3} \partial_i (A_{\boldsymbol{m}} \partial_i \boldsymbol{s}) + K_{\boldsymbol{m}} \boldsymbol{s} = \boldsymbol{f}_{\boldsymbol{m}}. \tag{44}$$

Note that $K_{\boldsymbol{m}}\boldsymbol{\eta} \cdot \boldsymbol{\eta} \geqslant 0$ for every $\boldsymbol{\eta} \in \mathbb{R}^3$. Therefore, the bilinear form on the left-hand side of (44) is uniformly elliptic in the sense of the Lax–Milgram lemma. It follows that, for every $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$, there exists a unique $\boldsymbol{s}_{\boldsymbol{m}} \in H^1(\Omega)$ that satisfies the elliptic system (44) and, therefore, (39). Moreover, by elliptic regularity [32, Thm. 4.18, p. 137], we obtain $\boldsymbol{s}_{\boldsymbol{m}} \in H^{k+2}(\Omega)$ provided that $A_{\boldsymbol{m}}, K_{\boldsymbol{m}} \in C^{k,1}(\overline{\Omega})$ and $\boldsymbol{f}_{\boldsymbol{m}} \in H^k(\Omega)$. Thus, if $\boldsymbol{m}, \boldsymbol{j}_e \in C^{k+1}(\overline{\Omega})$ then $\boldsymbol{f}_{\boldsymbol{m}} \in C^k(\overline{\Omega})$ and $\boldsymbol{s}_{\boldsymbol{m}} \in H^{k+2}(\Omega)$. Eventually, by Morrey's inequality in dimension three

[30, Thm. 12.55 p. 384], we conclude that $\boldsymbol{s_m} \in C^k(\overline{\Omega})$ provided that $\boldsymbol{m}, \boldsymbol{j}_e, D_0 \in C^{k+1}(\overline{\Omega})$. This completes the proof. □

4. Weak-Strong uniqueness of solutions (proof of Theorem 2)

4.1. A regularity result. The proof of the weak-strong uniqueness of the solutions of LLG is given in section 4.2. Our argument allows us to obtain weak-strong uniqueness for a class of nonlinearities more general than the one introduced by $\mathcal{H}_s$. A precise definition of the type of nonlinearities covered by our result is given in the next section and is motived by the following result.

Lemma 2. *For $\boldsymbol{m}_1 \in C^2(\overline{\Omega}, \mathbb{S}^2)$ and $\boldsymbol{m}_2 \in H^1(\Omega, \mathbb{S}^2)$, the following estimate holds:*

$$\|\mathcal{H}_s[\boldsymbol{m}_1] - \mathcal{H}_s[\boldsymbol{m}_2]\|^2_{H^1(\Omega)} \leqslant c_L^2 \left(\|\mathcal{H}_s[\boldsymbol{m}_1]\|^2_{C^1(\overline{\Omega})} + \|\boldsymbol{j}_e\|_{C^0(\overline{\Omega})}\right) \|\boldsymbol{m}_1 - \boldsymbol{m}_2\|^2_{H^1(\Omega)}, \tag{45}$$

with c_L^2 depending only on $D_0, \beta, \beta', \gamma_1, \gamma_2$, and Ω.

Proof. With $\boldsymbol{s}_1 := \mathcal{H}_s[\boldsymbol{m}_1]$ and $\boldsymbol{s}_2 := \mathcal{H}_s[\boldsymbol{m}_2]$, the following relations hold in a weak sense:

$$\begin{aligned} -\mathbf{div}\ [D_0 \nabla \boldsymbol{s}_1] + \beta\beta' \mathbf{div}\, [D_0(\nabla \boldsymbol{s}_1 \cdot \boldsymbol{m}_1) \otimes \boldsymbol{m}_1] \\ + \gamma_1 \boldsymbol{s}_1 + \gamma_2 \boldsymbol{s}_1 \times \boldsymbol{m}_1 = -\frac{\beta}{2} \mathbf{div}\, (\boldsymbol{j}_e \otimes \boldsymbol{m}_1)\,, \end{aligned} \tag{46}$$

$$\begin{aligned} -\mathbf{div}\ [D_0 \nabla \boldsymbol{s}_2] + \beta\beta' \mathbf{div}\, [D_0(\nabla \boldsymbol{s}_2 \cdot \boldsymbol{m}_2) \otimes \boldsymbol{m}_2] \\ + \gamma_1 \boldsymbol{s}_2 + \gamma_2 \boldsymbol{s}_2 \times \boldsymbol{m}_2 = -\frac{\beta}{2} \mathbf{div}\, (\boldsymbol{j}_e \otimes \boldsymbol{m}_2)\,. \end{aligned} \tag{47}$$

We recall that the **div** operator acts on columns. In a weak sense, it follows that

$$\begin{aligned} -\mathbf{div}\ (D_0 \nabla(\boldsymbol{s}_1 - \boldsymbol{s}_2)) + \beta\beta' \mathbf{div}\, [D_0((\nabla \boldsymbol{s}_1 \cdot \boldsymbol{m}_1) \otimes \boldsymbol{m}_1 - (\nabla \boldsymbol{s}_2 \cdot \boldsymbol{m}_2) \otimes \boldsymbol{m}_2)] \\ + \gamma_1(\boldsymbol{s}_1 - \boldsymbol{s}_2) + \gamma_2(\boldsymbol{s}_1 \times \boldsymbol{m}_1 - \boldsymbol{s}_2 \times \boldsymbol{m}_2) = -\frac{\beta}{2} \mathbf{div}\, [\boldsymbol{j}_e \otimes (\boldsymbol{m}_1 - \boldsymbol{m}_2)]. \end{aligned} \tag{48}$$

We note that $\boldsymbol{s}_1 \times \boldsymbol{m}_1 - \boldsymbol{s}_2 \times \boldsymbol{m}_2 = \boldsymbol{s}_1 \times (\boldsymbol{m}_1 - \boldsymbol{m}_2) + (\boldsymbol{s}_1 - \boldsymbol{s}_2) \times \boldsymbol{m}_2$, where the last term disappears when dot multiplied by $(\boldsymbol{s}_1 - \boldsymbol{s}_2)$. Hence, we have

$$\begin{aligned} \int_\Omega |[\boldsymbol{s}_1 \times \boldsymbol{m}_1 - \boldsymbol{s}_2 \times \boldsymbol{m}_2] \cdot (\boldsymbol{s}_1 - \boldsymbol{s}_2)| &= \int_\Omega |\boldsymbol{s}_1 \cdot [(\boldsymbol{m}_1 - \boldsymbol{m}_2) \times (\boldsymbol{s}_1 - \boldsymbol{s}_2)]| \\ &\leqslant \|\boldsymbol{s}_1\|_{C(\bar{\Omega})} \|\boldsymbol{m}_1 - \boldsymbol{m}_2\|_\Omega \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|_\Omega \\ &\leqslant \frac{\|\boldsymbol{s}_1\|_{C(\bar{\Omega})}}{2\delta^2} \|\boldsymbol{m}_1 - \boldsymbol{m}_2\|^2_\Omega + \frac{\delta^2}{2} \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^2_\Omega. \end{aligned} \tag{49}$$

Also, we have

$$\begin{aligned} &[(\nabla \boldsymbol{s}_1 \cdot \boldsymbol{m}_1) \otimes \boldsymbol{m}_1 - (\nabla \boldsymbol{s}_2 \cdot \boldsymbol{m}_2) \otimes \boldsymbol{m}_2]^\top = (\boldsymbol{m}_1 \otimes \boldsymbol{m}_1)\nabla^\top \boldsymbol{s}_1 - (\boldsymbol{m}_2 \otimes \boldsymbol{m}_2)\nabla^\top \boldsymbol{s}_2 \\ &= ((\boldsymbol{m}_1 - \boldsymbol{m}_2) \otimes \boldsymbol{m}_1)\nabla^\top \boldsymbol{s}_1 + (\boldsymbol{m}_2 \otimes \boldsymbol{m}_1)\nabla^\top \boldsymbol{s}_1 - (\boldsymbol{m}_2 \otimes \boldsymbol{m}_2)\nabla^\top \boldsymbol{s}_2 \\ &= ((\boldsymbol{m}_1 - \boldsymbol{m}_2) \otimes \boldsymbol{m}_1)\nabla^\top \boldsymbol{s}_1 + (\boldsymbol{m}_2 \otimes (\boldsymbol{m}_1 - \boldsymbol{m}_2))\nabla^\top \boldsymbol{s}_1 + (\boldsymbol{m}_2 \otimes \boldsymbol{m}_2)\nabla^\top (\boldsymbol{s}_1 - \boldsymbol{s}_2). \end{aligned} \tag{50}$$

Multiplying (50) by $\boldsymbol{\varphi} := \boldsymbol{s}_1 - \boldsymbol{s}_2$ and applying the Young inequality shows for any $\delta > 0$ that

$$\begin{aligned} &|(\nabla \boldsymbol{s}_1 \cdot \boldsymbol{m}_1) \otimes \boldsymbol{m}_1 - (\nabla \boldsymbol{s}_2 \cdot \boldsymbol{m}_2) \otimes \boldsymbol{m}_2] : \nabla(\boldsymbol{s}_1 - \boldsymbol{s}_2)| \\ &\qquad \leqslant 2\, \|\nabla \boldsymbol{s}_1\|_{C(\bar{\Omega})} |\boldsymbol{m}_1 - \boldsymbol{m}_2|\, |\nabla(\boldsymbol{s}_1 - \boldsymbol{s}_2)| + |\nabla(\boldsymbol{s}_1 - \boldsymbol{s}_2)|^2 \\ &\qquad \leqslant \frac{\|\nabla \boldsymbol{s}_1\|^2_{C(\bar{\Omega})}}{\delta^2} |\boldsymbol{m}_1 - \boldsymbol{m}_2|^2 + (1 + \delta^2)\, |\nabla(\boldsymbol{s}_1 - \boldsymbol{s}_2)|^2\,. \end{aligned} \tag{51}$$

It follows that

$$
\begin{aligned}
&|\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)|^2-\beta\beta'[(\nabla\boldsymbol{s}_1\cdot\boldsymbol{m}_1)\otimes\boldsymbol{m}_1-(\nabla\boldsymbol{s}_2\cdot\boldsymbol{m}_2)\otimes\boldsymbol{m}_2]{:}\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)\\
&\qquad\geqslant|\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)|^2-\frac{\beta\beta'}{\delta^2}\left\|\nabla\boldsymbol{s}_1\right\|^2_{C(\bar\Omega)}|\boldsymbol{m}_1-\boldsymbol{m}_2|^2-\beta\beta'(1+\delta^2)\,|\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)|^2\\
&\qquad=(1-\beta\beta'(1+\delta^2))\,|\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)|^2-\frac{\beta\beta'}{\delta^2}\left\|\nabla\boldsymbol{s}_1\right\|^2_{C(\bar\Omega)}|\boldsymbol{m}_1-\boldsymbol{m}_2|^2\,.
\end{aligned}
$$

The first term on the right-hand side is positive for δ^2 sufficiently small. Also, the following estimate holds:

$$
\begin{aligned}
&|\langle\mathbf{div}\,[\boldsymbol{j}_e\otimes(\boldsymbol{m}_1-\boldsymbol{m}_2)],\boldsymbol{s}_1-\boldsymbol{s}_2\rangle_\Omega|\\
&\quad\leqslant\int_\Omega|\boldsymbol{j}_e\otimes(\boldsymbol{m}_1-\boldsymbol{m}_2){:}\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)|+\int_{\partial\Omega}|(\boldsymbol{j}_e\cdot\boldsymbol{n})(\boldsymbol{m}_1-\boldsymbol{m}_2)\cdot(\boldsymbol{s}_1-\boldsymbol{s}_2)|\\
&\quad\leqslant\|\boldsymbol{j}_e\|_{C(\bar\Omega)}\|\boldsymbol{m}_1-\boldsymbol{m}_2\|_\Omega\,\|\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)\|_\Omega+\|\boldsymbol{j}_e\|_{C(\bar\Omega)}\|\boldsymbol{m}_1-\boldsymbol{m}_2\|_{\partial\Omega}\|\boldsymbol{s}_1-\boldsymbol{s}_2\|_{\partial\Omega}\\
&\quad\leqslant\frac{\|\boldsymbol{j}_e\|^2_{C(\bar\Omega)}}{\delta^2}\|\boldsymbol{m}_1-\boldsymbol{m}_2\|^2_{H^1(\Omega)}+c^2_{\partial\Omega}\delta^2\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2_{H^1(\Omega)}
\end{aligned}
$$

for some positive constant $c^2_{\partial\Omega}$ arising from the continuity of the trace operator.

Next, we observe that equation (48) gives

$$
\begin{aligned}
\int_\Omega D_0\Big(|\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)|^2-\beta\beta'[(\nabla\boldsymbol{s}_1\cdot\boldsymbol{m}_1)\otimes\boldsymbol{m}_1-(\nabla\boldsymbol{s}_2\cdot\boldsymbol{m}_2)\otimes\boldsymbol{m}_2]{:}\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)\Big)\\
+\gamma_1\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2_\Omega+\gamma_2\int_\Omega\boldsymbol{s}_1\cdot[(\boldsymbol{m}_1-\boldsymbol{m}_2)\times(\boldsymbol{s}_1-\boldsymbol{s}_2)]\\
=-\frac{\beta}{2}\langle\mathbf{div}\,[\boldsymbol{j}_e\otimes(\boldsymbol{m}_1-\boldsymbol{m}_2)],\boldsymbol{s}_1-\boldsymbol{s}_2\rangle\,.
\end{aligned}
$$

Taking into account estimates (49) and (51) and recalling that $\inf_\Omega D_0\geqslant\gamma$,

$$
\begin{aligned}
&\gamma\,(1-\beta\beta'(1+\delta^2))\,\|\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)\|^2_\Omega-\|D_0\|_{C(\bar\Omega)}\frac{\beta\beta'}{\delta^2}\left\|\nabla\boldsymbol{s}_1\right\|^2_{C(\bar\Omega)}\|\boldsymbol{m}_1-\boldsymbol{m}_2\|^2_\Omega+\gamma_1\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2_\Omega\\
&\qquad\leqslant\frac{\beta}{2}\left[\frac{\|\boldsymbol{j}_e\|^2_{C(\bar\Omega)}}{\delta^2}\|\boldsymbol{m}_1-\boldsymbol{m}_2\|^2_{H^1(\Omega)}+c^2_{\partial\Omega}\delta^2\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2_{H^1(\Omega)}\right]\\
&\qquad\qquad+|\gamma_2|\left[\frac{\|\boldsymbol{s}_1\|^2_{C(\bar\Omega)}}{2\delta^2}\|\boldsymbol{m}_1-\boldsymbol{m}_2\|^2_\Omega+\frac{\delta^2}{2}\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2_\Omega\right].
\end{aligned}\tag{52}
$$

Collecting the terms in the previous expression, we have

$$
\begin{aligned}
&\left[\gamma\,(1-\beta\beta'(1+\delta^2))-\delta^2\frac{\beta}{2}c^2_{\partial\Omega}\right]\|\nabla(\boldsymbol{s}_1-\boldsymbol{s}_2)\|^2_\Omega+\left(\gamma_1-\delta^2\left(\frac{\beta}{2}c^2_{\partial\Omega}+|\gamma_2|\right)\right)\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2_\Omega\\
&\qquad\leqslant\frac{1}{\delta^2}\left(\frac{|\gamma_2|\|\boldsymbol{s}_1\|^2_{C(\bar\Omega)}}{2}+\|D_0\|_{C(\bar\Omega)}\beta\beta'\left\|\nabla\boldsymbol{s}_1\right\|^2_{C(\bar\Omega)}+\frac{\beta}{2}|\boldsymbol{j}_e|^2_\infty\right)\|\boldsymbol{m}_1-\boldsymbol{m}_2\|^2_{H^1(\Omega)}.
\end{aligned}\tag{53}
$$

Since $0<\beta\beta'<1$, there exists $\delta>0$ such that

$$
\|\boldsymbol{s}_1-\boldsymbol{s}_2\|^2_{H^1(\Omega)}\leqslant c^2_L\left(\|\boldsymbol{s}_1\|^2_{C(\bar\Omega)}+\|\nabla\boldsymbol{s}_1\|^2_{C(\bar\Omega)}+\|\boldsymbol{j}_e\|^2_{C(\bar\Omega)}\right)\|\boldsymbol{m}_1-\boldsymbol{m}_2\|^2_{H^1(\Omega)}\tag{54}
$$

with c^2_L depending only on $D_0,\beta,\beta',\gamma_1,\gamma_2$, and $c_{\partial\Omega}$. This concludes the proof. □

4.2. Weak-strong uniqueness (energy estimate). Our proof of the weak-strong uniqueness of solutions of LLG relies on Lemma 3 stated below. Our argument permits us to prove a more general form of the weak-strong uniqueness result that we explain now. First, we deduce from the self-adjointness of $\boldsymbol{h}_{\mathrm{d}}$ (cf. (20)) that

$$\frac{1}{2}\int_0^T \partial_t \langle \boldsymbol{m}, \boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}]\rangle_\Omega = \int_0^T \langle \partial_t \boldsymbol{m}, \boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}]\rangle_\Omega \,. \tag{55}$$

Hence, integrating by parts (in time) the energy inequality (14a), we infer that weak solutions of SLLG (17) satisfy, for every $T > 0$, the following form of the energy inequality:

$$\begin{aligned}\mathcal{E}[\boldsymbol{m}](T) := \frac{c_{\mathrm{ex}}}{2}\,\|\nabla \boldsymbol{m}\,(T)\|_\Omega^2 + \int_0^T \alpha\|\partial_t \boldsymbol{m}\|_\Omega^2 \\ \leqslant \frac{c_{\mathrm{ex}}}{2}\,\|\nabla \boldsymbol{m}^*\|_\Omega^2 + \int_0^T \langle \partial_t \boldsymbol{m}, \mu_0 \boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}] + j_0 \mathcal{H}_s[\boldsymbol{m}]\rangle_\Omega \,.\end{aligned} \tag{56}$$

Second, due to Lemma 2 and (21), we know that if $\boldsymbol{m}_1 \in C^2(\overline{\Omega}, \mathbb{S}^2)$ and $\boldsymbol{m}_2 \in H^1(\Omega, \mathbb{S}^2)$, then the nonlinear operator $\boldsymbol{\pi}[\boldsymbol{m}] := \mu_0 \boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}] + j_0 \mathcal{H}_s[\boldsymbol{m}]$ satisfies the Lipschitz-type condition

$$\|\boldsymbol{\pi}[\boldsymbol{m}_1] - \boldsymbol{\pi}[\boldsymbol{m}_2]\|_{L^2(\Omega)}^2 \;\leqslant\; c_{\boldsymbol{\pi}}^2 \|\boldsymbol{m}_1 - \boldsymbol{m}_2\|_{H^1(\Omega)}^2. \tag{57}$$

We stress the fact that here $c_{\boldsymbol{\pi}}$ may depend on the smooth vector field $\boldsymbol{m}_1$ (other than the physical parameters of the system) and, therefore, condition (57) is weaker than the classical Lipschitz condition.

Our proof of weak-strong uniqueness works in this more general setting. Therefore, for the rest of the paper, we will assume that our LLG equation has the more general form

$$\partial_t \boldsymbol{m} = -\boldsymbol{m} \times \mathcal{K}[\boldsymbol{m}] \tag{58}$$

with $\mathcal{K}[\boldsymbol{m}] := c_{\mathrm{ex}} \Delta \boldsymbol{m} + \boldsymbol{\pi}[\boldsymbol{m}] - \alpha \partial_t \boldsymbol{m}$, and $\boldsymbol{\pi} : H^1(\Omega, \mathbb{S}^2) \to L^2(\Omega, \mathbb{R}^3)$ a nonlinear operator satisfying the following two properties:

(a) The operator $\boldsymbol{\pi}$ maps $C^\infty(\overline{\Omega}, \mathbb{S}^2)$ into $C^\infty(\overline{\Omega}, \mathbb{R}^3)$.
(b) For every $\boldsymbol{m}_1 \in C^\infty(\overline{\Omega}, \mathbb{S}^2)$ and $\boldsymbol{m}_2 \in H^1(\Omega, \mathbb{S}^2)$ the Lipschitz-type condition (57) holds for some $c_{\boldsymbol{\pi}}$ which may depend on $\boldsymbol{m}_1$ but not on $\boldsymbol{m}_2$.

It is important to stress that condition (57) is not symmetric in $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ because of the special role played by the *smooth* vector field $\boldsymbol{m}_1$.

It has been already pointed out that the operator $\mu_0 \boldsymbol{h}_{\mathrm{d}}[\boldsymbol{m}] + j_0 \mathcal{H}_s$ satisfies (b). But it also satisfies (a): this is a consequence of Lemma 2 for what concerns $\mathcal{H}_s$, and of [14, Lemma 8] for what concerns $\boldsymbol{h}_{\mathrm{d}}$. Other than for SLLG, the present framework covers in particular the case in which, as in (5), the effective field comprises an external applied field $\boldsymbol{f}$ and the crystal anisotropy contribution $\kappa \nabla \phi_{\mathrm{an}}(\boldsymbol{m})$.

Generally speaking, the precise form of the LLG equation depends on the type of interactions involved in the magnetic system. Such interactions are accounted for through the effective field $\boldsymbol{h}_{\mathrm{eff}}$. In its basic form, the effective field takes into account exchange interactions only, and further interactions can be described by suitable linear/nonlinear/local/nonlocal terms into the effective field. In this respect, the current setting promotes a unified treatment of weak-strong uniqueness results for LLG.

We give the following definition.

Definition 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, and let $\boldsymbol{\pi} : H^1(\Omega, \mathbb{S}^2) \to L^2(\Omega, \mathbb{R}^3)$ be a nonlinear operator satisfying the properties* (a) *and* (b) *above. Let $\boldsymbol{m}^* \in H^1(\Omega, \mathbb{S}^2)$ be the initial value of the magnetization at time $t = 0$. We say that $\boldsymbol{m} \in L^\infty(\mathbb{R}_+, H^1(\Omega, \mathbb{S}^2))$ is*

a weak solution of the LLG *equation* (58)*, if the following properties* (i)–(iii) *hold for every* $T > 0$*:*

(i) $\boldsymbol{m} \in H^1(\Omega_T, \mathbb{S}^2)$ *and* $\boldsymbol{m}(0) = \boldsymbol{m}^*$ *in the sense of traces,*
(ii) *for every* $\boldsymbol{\varphi} \in H^1(\Omega_T)$*,*

$$\int_0^T \langle \partial_t \boldsymbol{m}, \boldsymbol{\varphi} \rangle_\Omega = \int_0^T \alpha \langle \partial_t \boldsymbol{m}, \boldsymbol{\varphi} \times \boldsymbol{m} \rangle_\Omega + c_{\mathrm{ex}} \langle \nabla \boldsymbol{m}, \nabla(\boldsymbol{\varphi} \times \boldsymbol{m}) \rangle_\Omega - \int_0^T \langle \boldsymbol{\pi}[\boldsymbol{m}], \boldsymbol{\varphi} \times \boldsymbol{m} \rangle_\Omega \,, \tag{59}$$

(iii) *the following energy inequality holds:*

$$\mathcal{E}[\boldsymbol{m}](T) := \frac{c_{\mathrm{ex}}}{2} \|\nabla \boldsymbol{m}\,(T)\|_\Omega^2 + \int_0^T \alpha \|\partial_t \boldsymbol{m}\|_\Omega^2 \leqslant \frac{c_{\mathrm{ex}}}{2} \|\nabla \boldsymbol{m}^*\|_\Omega^2 + \int_0^T \langle \partial_t \boldsymbol{m}, \boldsymbol{\pi}[\boldsymbol{m}] \rangle_\Omega \,. \tag{60}$$

We prove Theorem 2 in this more general setting. The main ingredient is contained in the next result.

Lemma 3. *Let* $\boldsymbol{m}_2$ *be a global weak solution of the* LLG *equation* (*in the sense of* Definition 2)*. Let* $\boldsymbol{m}_1 \in C^\infty(\overline{\Omega}_{T^*}, \mathbb{S}^2)$ *be a strong solution of the* LLG *on* $\overline{\Omega_{T^*}}$ *for some* $T^* > 0$*. Set* $\boldsymbol{v} := \boldsymbol{m}_2 - \boldsymbol{m}_1$*. For a.e.* $T \in \mathbb{R}$ *such that* $0 \leqslant T < T^*$*, it holds that*

$$\mathcal{E}[\boldsymbol{v}](T) \leqslant -c_{\mathrm{ex}} \int_0^T \langle \boldsymbol{v} \times \nabla \mathcal{K}[\boldsymbol{m}_1], \nabla \boldsymbol{v} \rangle_\Omega + \int_0^T \langle \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}] - \alpha \partial_t \boldsymbol{v}, \boldsymbol{v} \times \mathcal{K}[\boldsymbol{m}_1] \rangle_\Omega + \int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}] \rangle_\Omega \,. \tag{61}$$

Here, $\boldsymbol{v} \times \nabla \mathcal{K}[\boldsymbol{m}_1] := (\boldsymbol{v} \times \partial_1 \mathcal{K}[\boldsymbol{m}_1], \boldsymbol{v} \times \partial_2 \mathcal{K}[\boldsymbol{m}_1], \boldsymbol{v} \times \partial_3 \mathcal{K}[\boldsymbol{m}_1])$*,* $\mathcal{K}[\boldsymbol{m}] := c_{\mathrm{ex}} \Delta \boldsymbol{m} + \boldsymbol{\pi}[\boldsymbol{m}] - \alpha \partial_t \boldsymbol{m}$*, and* $\boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}] := \boldsymbol{\pi}[\boldsymbol{m}_1 + \boldsymbol{v}] - \boldsymbol{\pi}[\boldsymbol{m}_1]$*.*

Proof. Note that $\boldsymbol{m}_2 = \boldsymbol{m}_1 + \boldsymbol{v}$. Since $\boldsymbol{\pi}$ is nonlinear, it is convenient to write

$$\boldsymbol{\pi}[\boldsymbol{m}_1 + \boldsymbol{v}] = \boldsymbol{\pi}[\boldsymbol{m}_1] + \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}] \,. \tag{62}$$

By the energy inequality (60) as well as the boundary condition $\partial_{\boldsymbol{n}} \boldsymbol{m}_1 = 0$ on $\partial\Omega$, it holds for every $T > 0$ that

$$\begin{aligned} \mathcal{E}[\boldsymbol{v}](T) &= \mathcal{E}[\boldsymbol{m}_1](T) + \mathcal{E}[\boldsymbol{m}_2](T) - c_{\mathrm{ex}} \langle \nabla \boldsymbol{m}_1\,(T)\,, \nabla \boldsymbol{m}_2\,(T) \rangle_\Omega - 2 \int_0^T \alpha \langle \partial_t \boldsymbol{m}_1, \partial_t \boldsymbol{m}_2 \rangle_\Omega \\ &\overset{(60)}{\leqslant} -c_{\mathrm{ex}} \langle \nabla \boldsymbol{m}_1\,(T)\,, \nabla \boldsymbol{m}_2\,(T) \rangle_\Omega + c_{\mathrm{ex}} \|\nabla \boldsymbol{m}_*\|_\Omega^2 + \int_0^T \langle \partial_t \boldsymbol{m}_1, \boldsymbol{\pi}[\boldsymbol{m}_1] \rangle_\Omega \\ &\quad + \langle \partial_t \boldsymbol{m}_2, \boldsymbol{\pi}[\boldsymbol{m}_2] \rangle_\Omega - 2 \int_0^T \alpha \langle \partial_t \boldsymbol{m}_1, \partial_t \boldsymbol{m}_2 \rangle_\Omega. \end{aligned} \tag{63}$$

Let $0 < T < T^*$. Since $\boldsymbol{m}_2 \in H^1(\Omega_T, \mathbb{S}^2)$, there exists a family of vector-valued functions $\boldsymbol{m}_\varepsilon \in C^\infty(\overline{\Omega}_T, \mathbb{R}^3)$ such that $\boldsymbol{m}_\varepsilon \to \boldsymbol{m}_2$ strongly in $H^1(\Omega_T, \mathbb{R}^3)$ as $\varepsilon \to 0$. Since both $\boldsymbol{m}_1$ and $\boldsymbol{m}_\varepsilon$ are smooth, integration by parts yields that

$$\begin{aligned} c_{\mathrm{ex}} \langle \Delta \partial_t \boldsymbol{m}_1, \boldsymbol{m}_\varepsilon \rangle_\Omega &= -c_{\mathrm{ex}} \langle \nabla \partial_t \boldsymbol{m}_1, \nabla \boldsymbol{m}_\varepsilon \rangle_\Omega \\ &= b_{1,\varepsilon} + c_{\mathrm{ex}} \langle \partial_t \boldsymbol{m}_1, \Delta \boldsymbol{m}_\varepsilon \rangle_\Omega \,. \end{aligned} \tag{64}$$

where $b_{1,\varepsilon} := -c_{\mathrm{ex}}\langle \partial_t \boldsymbol{m}_1, \partial_{\boldsymbol{n}} \boldsymbol{m}_\varepsilon\rangle_{\partial\Omega}$. Hence, the first two terms on the right-hand side of (63) read as

$$\begin{aligned} c_{\mathrm{ex}} \int_0^T \partial_t \langle \Delta \boldsymbol{m}_1, \boldsymbol{m}_2\rangle_\Omega &= -c_{\mathrm{ex}} \langle \nabla \boldsymbol{m}_1(T), \nabla \boldsymbol{m}_2(T)\rangle_\Omega + c_{\mathrm{ex}} \|\nabla \boldsymbol{m}_*\|_\Omega^2 \\ &= \int_0^T \langle \partial_t \boldsymbol{m}_2, c_{\mathrm{ex}} \Delta \boldsymbol{m}_1\rangle_\Omega + c_{\mathrm{ex}} \lim_{\varepsilon\to 0} \int_0^T \langle \Delta \partial_t \boldsymbol{m}_1, \boldsymbol{m}_\varepsilon\rangle_\Omega \\ &\overset{(64)}{=} \int_0^T \langle \partial_t \boldsymbol{m}_2, c_{\mathrm{ex}} \Delta \boldsymbol{m}_1\rangle_\Omega + \lim_{\varepsilon\to 0} \int_0^T \left(\langle \partial_t \boldsymbol{m}_1, c_{\mathrm{ex}} \Delta \boldsymbol{m}_\varepsilon\rangle_\Omega + b_{1,\varepsilon}\right). \end{aligned}$$

Thus, we can rearrange (63) to obtain

$$\begin{aligned} \mathcal{E}[\boldsymbol{v}](T) \overset{(63)}{\leqslant} &\int_0^T \langle \partial_t \boldsymbol{m}_2, c_{\mathrm{ex}} \Delta \boldsymbol{m}_1\rangle_\Omega + \lim_{\varepsilon\to 0} \int_0^T \left(\langle \partial_t \boldsymbol{m}_1, c_{\mathrm{ex}} \Delta \boldsymbol{m}_\varepsilon\rangle_\Omega + b_{1,\varepsilon}\right) \\ &+ \int_0^T \langle \partial_t \boldsymbol{m}_1, \boldsymbol{\pi}[\boldsymbol{m}_1]\rangle_\Omega + \langle \partial_t \boldsymbol{m}_2, \boldsymbol{\pi}[\boldsymbol{m}_2]\rangle_\Omega - 2\int_0^T \alpha \langle \partial_t \boldsymbol{m}_1, \alpha \partial_t \boldsymbol{m}_2\rangle_\Omega. \end{aligned} \tag{65}$$

Collecting the terms in the previous expression, we arrive at

$$\begin{aligned} \mathcal{E}[\boldsymbol{v}](T) \overset{(65)}{\leqslant} &\lim_{\varepsilon\to 0} \int_0^T \left(\langle \partial_t \boldsymbol{m}_1, c_{\mathrm{ex}} \Delta \boldsymbol{m}_\varepsilon + \boldsymbol{\pi}[\boldsymbol{m}_1] - \alpha \partial_t \boldsymbol{m}_2\rangle_\Omega + b_{1,\varepsilon}\right) \\ &+ \int_0^T \langle \partial_t \boldsymbol{m}_2, c_{\mathrm{ex}} \Delta \boldsymbol{m}_1 + \boldsymbol{\pi}[\boldsymbol{m}_2] - \alpha \partial_t \boldsymbol{m}_1\rangle_\Omega. \end{aligned} \tag{66}$$

Define $\tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon] := c_{\mathrm{ex}} \Delta \boldsymbol{m}_\varepsilon + \boldsymbol{\pi}[\boldsymbol{m}_2] - \alpha \partial_t \boldsymbol{m}_2$, where the tilde script is used to emphasize that the Laplacian acts on $\boldsymbol{m}_\varepsilon$ and not on $\boldsymbol{m}_2$. Since $\partial_t \boldsymbol{m}_1$ satisfies the strong form of LLG (58) with $\mathcal{K}[\boldsymbol{m}_1] = c_{\mathrm{ex}} \Delta \boldsymbol{m}_1 + \boldsymbol{\pi}[\boldsymbol{m}_1] - \alpha \partial_t \boldsymbol{m}_1$ and $\boldsymbol{\pi}[\boldsymbol{m}_1] = \boldsymbol{\pi}[\boldsymbol{m}_2] - \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]$, the previous estimate leads to

$$\begin{aligned} \mathcal{E}[\boldsymbol{v}](T) \overset{(66)}{\leqslant} &\lim_{\varepsilon\to 0} \int_0^T \left(\langle \partial_t \boldsymbol{m}_1, c_{\mathrm{ex}} \Delta \boldsymbol{m}_\varepsilon + \boldsymbol{\pi}[\boldsymbol{m}_2] - \alpha \partial_t \boldsymbol{m}_2 - \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega + b_{1,\varepsilon}\right) \\ &\qquad + \int_0^T \langle \partial_t \boldsymbol{m}_2, \mathcal{K}[\boldsymbol{m}_1] + \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega \\ = &\lim_{\varepsilon\to 0} \int_0^T \left(\langle \partial_t \boldsymbol{m}_1, \tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon] - \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega + b_{1,\varepsilon}\right) \\ &\qquad + \int_0^T \langle \partial_t \boldsymbol{m}_2, \mathcal{K}[\boldsymbol{m}_1] + \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega, \\ = &\lim_{\varepsilon\to 0} \int_0^T \left(\langle -\boldsymbol{m}_1 \times \mathcal{K}[\boldsymbol{m}_1], \tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon]\rangle_\Omega + \langle \partial_t \boldsymbol{m}_2, \mathcal{K}[\boldsymbol{m}_1]\rangle_\Omega + b_{1,\varepsilon}\right) \\ &\qquad + \int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega, \end{aligned} \tag{67}$$

On the other hand, $\boldsymbol{m}_2$ is a weak solution of LLG (59). From the relation

$$c_{\mathrm{ex}} \langle \nabla \boldsymbol{m}_\varepsilon, \nabla(\boldsymbol{\varphi} \times \boldsymbol{m}_\varepsilon)\rangle_\Omega = -c_{\mathrm{ex}} \langle \boldsymbol{m}_\varepsilon \times \Delta \boldsymbol{m}_\varepsilon, \boldsymbol{\varphi}\rangle_\Omega + b_{2,\varepsilon}[\boldsymbol{\varphi}], \tag{68}$$

where $b_{2,\varepsilon}[\boldsymbol{\varphi}] := c_{\mathrm{ex}}\langle \boldsymbol{m}_\varepsilon \times \partial_{\boldsymbol{n}} \boldsymbol{m}_\varepsilon, \boldsymbol{\varphi}\rangle_{\partial\Omega}$, we infer that

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{m}_2, \boldsymbol{\varphi}\rangle_\Omega &\overset{(59)}{=} \lim_{\varepsilon\to 0}\int_0^T \Big(\langle \alpha\partial_t \boldsymbol{m}_2, \boldsymbol{\varphi}\times \boldsymbol{m}_\varepsilon\rangle_\Omega - \langle \boldsymbol{\pi}[\boldsymbol{m}_2], \boldsymbol{\varphi}\times\boldsymbol{m}_\varepsilon\rangle_\Omega \\ &\qquad\qquad + c_{\mathrm{ex}}\left\langle \nabla \boldsymbol{m}_\varepsilon, \nabla(\boldsymbol{\varphi}\times\boldsymbol{m}_\varepsilon)\right\rangle_\Omega\Big) \\ &= \lim_{\varepsilon\to 0}\int_0^T \Big(\langle \boldsymbol{m}_\varepsilon \times (\alpha\partial_t\boldsymbol{m}_2 - c_{\mathrm{ex}}\Delta\boldsymbol{m}_\varepsilon - \boldsymbol{\pi}[\boldsymbol{m}_2]), \boldsymbol{\varphi}\rangle_\Omega + b_{2,\varepsilon}[\boldsymbol{\varphi}]\Big) \\ &= \lim_{\varepsilon\to 0}\int_0^T \Big(\langle -\boldsymbol{m}_\varepsilon \times \tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon], \boldsymbol{\varphi}\rangle_\Omega + b_{2,\varepsilon}[\boldsymbol{\varphi}]\Big). \end{aligned} \tag{69}$$

To further estimate the energy $\mathcal{E}[\boldsymbol{v}](T)$, we use $\boldsymbol{\varphi} = \mathcal{K}[\boldsymbol{m}_1]$ in (69) and plug the result in (67). To shorten the notation, we collected the two boundary terms into the term

$$\begin{aligned} b_\varepsilon &:= b_{1,\varepsilon} + b_{2,\varepsilon}[\mathcal{K}[\boldsymbol{m}_1]] \\ &= -c_{\mathrm{ex}}\langle \partial_t \boldsymbol{m}_1, \partial_{\boldsymbol{n}}\boldsymbol{m}_\varepsilon\rangle_{\partial\Omega} + c_{\mathrm{ex}}\langle \boldsymbol{m}_\varepsilon\times\partial_{\boldsymbol{n}}\boldsymbol{m}_\varepsilon, \mathcal{K}[\boldsymbol{m}_1]\rangle_{\partial\Omega}. \end{aligned} \tag{70}$$

With $\boldsymbol{v}_\varepsilon := \boldsymbol{m}_\varepsilon - \boldsymbol{m}_1$ (and observing that $\boldsymbol{v}_\varepsilon \to \boldsymbol{v}$ strongly in $H^1(\Omega_T, \mathbb{R}^3)$), we find that

$$\begin{aligned}\mathcal{E}[\boldsymbol{v}](T) &\overset{(67)}{\leqslant} \lim_{\varepsilon\to 0}\int_0^T \Big(\langle -\boldsymbol{m}_1\times\mathcal{K}[\boldsymbol{m}_1], \tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon]\rangle_\Omega + \langle -\boldsymbol{m}_\varepsilon\times\tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon], \mathcal{K}[\boldsymbol{m}_1]\rangle_\Omega \\ &\qquad\qquad + \langle \partial_t\boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega + b_\varepsilon\Big) \\ &= \lim_{\varepsilon\to 0}\int_0^T \Big(\langle -\boldsymbol{m}_1\times\mathcal{K}[\boldsymbol{m}_1], \tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon]\rangle_\Omega + \langle \boldsymbol{m}_\varepsilon\times\mathcal{K}[\boldsymbol{m}_1], \tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon]\rangle_\Omega \\ &\qquad\qquad + \langle \partial_t\boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega + b_\varepsilon\Big) \\ &= \lim_{\varepsilon\to 0}\int_0^T \left(\langle \boldsymbol{v}_\varepsilon\times\mathcal{K}[\boldsymbol{m}_1], \tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon]\rangle_\Omega + \langle \partial_t\boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega + b_\varepsilon\right) \\ &= \lim_{\varepsilon\to 0}\int_0^T \left(\langle \boldsymbol{v}_\varepsilon\times\mathcal{K}[\boldsymbol{m}_1], \tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon] - \mathcal{K}[\boldsymbol{m}_1]\rangle_\Omega + \langle \partial_t\boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega + b_\varepsilon\right). \end{aligned} \tag{71}$$

We now expand the quantity $\tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon] - \mathcal{K}[\boldsymbol{m}_1]$. Since $\boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}] = \boldsymbol{\pi}[\boldsymbol{m}_2] - \boldsymbol{\pi}[\boldsymbol{m}_1]$, we have

$$\begin{aligned}\tilde{\mathcal{K}}[\boldsymbol{m}_\varepsilon] - \mathcal{K}[\boldsymbol{m}_1] &= c_{\mathrm{ex}}\Delta\boldsymbol{m}_\varepsilon + \boldsymbol{\pi}[\boldsymbol{m}_2] - \alpha\partial_t\boldsymbol{m}_2 - c_{\mathrm{ex}}\Delta\boldsymbol{m}_1 - \boldsymbol{\pi}[\boldsymbol{m}_1] + \alpha\partial_t\boldsymbol{m}_1 \\ &= c_{\mathrm{ex}}\Delta\boldsymbol{v}_\varepsilon - \alpha\partial_t\boldsymbol{v} + \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]. \end{aligned}$$

Summarizing, we have reached the following estimate:

$$\begin{aligned}\mathcal{E}[\boldsymbol{v}](T) &\overset{(71)}{\leqslant} \lim_{\varepsilon\to 0}\int_0^T \left(\langle \boldsymbol{v}_\varepsilon\times\mathcal{K}[\boldsymbol{m}_1], c_{\mathrm{ex}}\Delta\boldsymbol{v}_\varepsilon - \alpha\partial_t\boldsymbol{v} + \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega + b_\varepsilon\right) \\ &\qquad + \int_0^T \langle \partial_t\boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega \\ &= \lim_{\varepsilon\to 0}\int_0^T \left(\langle \boldsymbol{v}_\varepsilon\times\mathcal{K}[\boldsymbol{m}_1], c_{\mathrm{ex}}\Delta\boldsymbol{v}_\varepsilon\rangle_\Omega + b_\varepsilon\right) \\ &\qquad + \int_0^T \langle \boldsymbol{v}\times\mathcal{K}[\boldsymbol{m}_1], -\alpha\partial_t\boldsymbol{v} + \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega + \int_0^T \langle \partial_t\boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega. \end{aligned} \tag{72}$$

Now we have to take care of the boundary terms in b_ε defined in (70). An integration by parts gives

$$\langle \boldsymbol{v}_\varepsilon\times\mathcal{K}[\boldsymbol{m}_1], c_{\mathrm{ex}}\Delta\boldsymbol{v}_\varepsilon\rangle_\Omega = \quad -c_{\mathrm{ex}}\left\langle \nabla(\boldsymbol{v}_\varepsilon\times\mathcal{K}[\boldsymbol{m}_1], \nabla\boldsymbol{v}_\varepsilon\right\rangle_\Omega + c_{\mathrm{ex}}\int_{\partial\Omega}\partial_{\boldsymbol{n}}\boldsymbol{v}_\varepsilon\cdot(\boldsymbol{v}_\varepsilon\times\mathcal{K}[\boldsymbol{m}_1]). \tag{73}$$

Next, we expand the boundary term in the previous expression. Recalling that $\partial_{\boldsymbol{n}}\boldsymbol{m}_1 = 0$, we find that

$$\begin{aligned}\int_{\partial\Omega} \partial_{\boldsymbol{n}}\boldsymbol{v}_\varepsilon \cdot (\boldsymbol{v}_\varepsilon \times \mathcal{K}[\boldsymbol{m}_1]) &= \int_{\partial\Omega} \partial_{\boldsymbol{n}}\boldsymbol{m}_\varepsilon \cdot ((\boldsymbol{m}_\varepsilon - \boldsymbol{m}_1) \times \mathcal{K}[\boldsymbol{m}_1]) \\ &= \int_{\partial\Omega} \mathcal{K}[\boldsymbol{m}_1] \cdot (\partial_{\boldsymbol{n}}\boldsymbol{m}_\varepsilon \times \boldsymbol{m}_\varepsilon) - \int_{\partial\Omega} \mathcal{K}[\boldsymbol{m}_1] \cdot (\partial_{\boldsymbol{n}}\boldsymbol{m}_\varepsilon \times \boldsymbol{m}_1) \\ &= \int_{\partial\Omega} \mathcal{K}[\boldsymbol{m}_1] \cdot (\partial_{\boldsymbol{n}}\boldsymbol{m}_\varepsilon \times \boldsymbol{m}_\varepsilon) - \int_{\partial\Omega} \partial_{\boldsymbol{n}}\boldsymbol{m}_\varepsilon \cdot (\boldsymbol{m}_1 \times \mathcal{K}[\boldsymbol{m}_1]) \\ &\overset{(58)}{=} -\int_{\partial\Omega} (\boldsymbol{m}_\varepsilon \times \partial_{\boldsymbol{n}}\boldsymbol{m}_\varepsilon) \cdot \mathcal{K}[\boldsymbol{m}_1] + \int_{\partial\Omega} \partial_{\boldsymbol{n}}\boldsymbol{m}_\varepsilon \cdot \partial_t \boldsymbol{m}_1 \\ &\overset{(70)}{=} -b_\varepsilon / c_{\mathrm{ex}}. \end{aligned} \tag{74}$$

By (73) and (74), we can rewrite (72) into the form

$$\begin{aligned}\mathcal{E}[\boldsymbol{v}](T) \overset{(72)}{\leqslant} \lim_{\varepsilon\to 0} \int_0^T & -c_{\mathrm{ex}} \langle \nabla(\boldsymbol{v}_\varepsilon \times \mathcal{K}[\boldsymbol{m}_1], \nabla \boldsymbol{v}_\varepsilon\rangle_\Omega \\ &+ \int_0^T \langle \boldsymbol{v} \times \mathcal{K}[\boldsymbol{m}_1], -\alpha\partial_t \boldsymbol{v} + \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega + \int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega . \end{aligned} \tag{75}$$

Finally, we observe that, as $\varepsilon \to 0$,

$$\langle \nabla(\boldsymbol{v}_\varepsilon \times \mathcal{K}[\boldsymbol{m}_1], \nabla \boldsymbol{v}_\varepsilon\rangle_\Omega = \langle \boldsymbol{v}_\varepsilon \times \nabla\mathcal{K}[\boldsymbol{m}_1], \nabla \boldsymbol{v}_\varepsilon\rangle_\Omega \to \langle \boldsymbol{v} \times \nabla\mathcal{K}[\boldsymbol{m}_1], \nabla \boldsymbol{v}\rangle_\Omega . \tag{76}$$

This concludes the proof. □

4.3. Proof of Theorem 2. Thanks to the energy estimate stated in Lemma 3, the weak-strong uniqueness result follows from a classical argument based on the Gronwall lemma and the Poincaré inequality. We start by the energy inequality (61). In expanded form, it reads as follows:

$$\begin{aligned}\frac{c_{\mathrm{ex}}}{2} \|\nabla \boldsymbol{v}\,(T)\|_\Omega^2 + \int_0^T \alpha \|\partial_t \boldsymbol{v}\|_\Omega^2 \leqslant \int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega - c_{\mathrm{ex}} \int_0^T \langle \boldsymbol{v} \times \nabla\mathcal{K}[\boldsymbol{m}_1], \nabla \boldsymbol{v}\rangle_\Omega \\ + \int_0^T \langle \boldsymbol{v} \times \mathcal{K}[\boldsymbol{m}_1], \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega - \int_0^T \langle \boldsymbol{v} \times \mathcal{K}[\boldsymbol{m}_1], \alpha\partial_t \boldsymbol{v}\rangle_\Omega. \end{aligned} \tag{77}$$

We recall that by hypothesis

$$\|\boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\|_\Omega = \|\boldsymbol{\pi}[\boldsymbol{m}_1 + \boldsymbol{v}] - \boldsymbol{\pi}[\boldsymbol{m}_1]\|_\Omega \overset{(57)}{\leqslant} c_{\boldsymbol{\pi}} \|\boldsymbol{v}\|_{H^1(\Omega)} . \tag{78}$$

Setting $c_1 := \|\mathcal{K}[\boldsymbol{m}_1]\|_{C^1(\overline{\Omega_T})}$, we obtain, for every $0 < \delta < 1$, the following estimates:

$$|\langle \partial_t \boldsymbol{v}, \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega| \leqslant \frac{\delta^2}{2} \|\partial_t \boldsymbol{v}\|_\Omega^2 + \frac{c_{\boldsymbol{\pi}}^2}{2\delta^2} \|\boldsymbol{v}\|_\Omega^2 + \frac{c_{\boldsymbol{\pi}}^2}{2\delta^2} \|\nabla \boldsymbol{v}\|_\Omega^2 , \tag{79}$$

$$|\langle \boldsymbol{v} \times \mathcal{K}[\boldsymbol{m}_1], \alpha\partial_t \boldsymbol{v}\rangle_\Omega| \leqslant \frac{\alpha\delta^2}{2} \|\partial_t \boldsymbol{v}\|_\Omega^2 + \frac{c_1^2}{2\alpha\delta^2} \|\boldsymbol{v}\|_\Omega^2,$$

$$|\langle \boldsymbol{v} \times \mathcal{K}[\boldsymbol{m}_1], \boldsymbol{\pi}_{\boldsymbol{m}_1}[\boldsymbol{v}]\rangle_\Omega| \leqslant c_1 c_{\boldsymbol{\pi}} \|\boldsymbol{v}\|_\Omega \|\boldsymbol{v}\|_{H^1(\Omega)} \leqslant c_1 c_{\boldsymbol{\pi}} \|\boldsymbol{v}\|_\Omega^2 + \frac{1}{2} c_1 c_{\boldsymbol{\pi}} \|\nabla \boldsymbol{v}\|_\Omega^2, \tag{80}$$

$$|\langle \boldsymbol{v} \times \nabla\mathcal{K}[\boldsymbol{m}_1], \nabla \boldsymbol{v}\rangle_\Omega| \leqslant \frac{c_1}{2} \|\boldsymbol{v}\|_\Omega^2 + \frac{c_1}{2} \|\nabla \boldsymbol{v}\|_\Omega^2 . \tag{81}$$

We conclude from (77) and the previous bounds that

$$\begin{aligned} 0 &\leqslant \frac{c_{\mathrm{ex}}}{2} \left\|\nabla \boldsymbol{v}\left(T\right)\right\|_{\Omega}^{2} + \alpha_{\delta} \int_{0}^{T} \left\|\partial_t \boldsymbol{v}\right\|_{\Omega}^{2} \\ &\leqslant c_{(1,\boldsymbol{\pi},\alpha,\delta)}^{2} \int_{0}^{T} \left\|\boldsymbol{v}\right\|_{\Omega}^{2} + \left(\frac{c_1}{2} + \frac{c_{\boldsymbol{\pi}}^{2}}{2\delta^2} + \frac{1}{2} c_1 c_{\boldsymbol{\pi}}\right) \int_{0}^{T} \left\|\nabla \boldsymbol{v}\right\|_{\Omega}^{2}, \end{aligned} \tag{82}$$

with $c_{(1,\boldsymbol{\pi},\alpha,\delta)}^{2} := \frac{c_{\boldsymbol{\pi}}^{2}}{2\delta^2} + \frac{c_1^2}{2\alpha\delta^2} + \frac{c_1}{2} + c_1 c_{\boldsymbol{\pi}}$ and $\alpha_{\delta}^{2} := \alpha\left(1 - \frac{\delta^2}{2}\right) - \frac{\delta^2}{2}$. We assume δ small enough so that $\alpha_{\delta}^{2} > 0$. We infer from the Poincaré inequality that

$$\frac{c_{\mathrm{ex}}}{2} \left\|\nabla \boldsymbol{v}\left(T\right)\right\|_{\Omega}^{2} + \alpha_{\delta}^{2} \int_{0}^{T} \left\|\partial_t \boldsymbol{v}\right\|_{\Omega}^{2} \leqslant c_{(1,\boldsymbol{\pi},\alpha,\delta)}^{2} T^2 \int_{0}^{T} \left\|\partial_t \boldsymbol{v}\right\|_{\Omega}^{2} + \left(\frac{c_1}{2} + \frac{c_{\boldsymbol{\pi}}^{2}}{2\delta^2} + \frac{1}{2} c_1 c_{\boldsymbol{\pi}}\right) \int_{0}^{T} \left\|\nabla \boldsymbol{v}\right\|_{\Omega}^{2}. \tag{83}$$

Thus, for any T sufficiently small, say for any $T < T_* < T^*$ with T_* depending only on $c_{(1,\boldsymbol{\pi},\alpha,\delta)}^{2}$, we have $\alpha_{\delta}^{2} > c_{(1,\boldsymbol{\pi},\alpha,\delta)}^{2} T^2$. Hence,

$$\frac{c_{\mathrm{ex}}}{2} \left\|\nabla \boldsymbol{v}\left(T\right)\right\|_{\Omega}^{2} \leqslant \left(\frac{c_1}{2} + \frac{c_{\boldsymbol{\pi}}^{2}}{2\delta^2} + \frac{1}{2} c_1 c_{\boldsymbol{\pi}}\right) \int_{0}^{T} \left\|\nabla \boldsymbol{v}\right\|_{\Omega}^{2}. \tag{84}$$

According to the Gronwall lemma, we infer that $\left\|\nabla \boldsymbol{v}\left(T\right)\right\|_{\Omega}^{2} = 0$ for any $T < T_*$. But then, for every $T < T_*$ relation (83) reduces to

$$\left(\alpha_{\delta}^{2} - c_{(1,\boldsymbol{\pi},\alpha,\delta)}^{2} T^2\right) \int_{0}^{T} \left\|\partial_t \boldsymbol{v}\right\|_{\Omega}^{2} \leqslant 0. \tag{85}$$

This implies that $\left\|\partial_t \boldsymbol{v}\right\|_{\Omega}^{2} = 0$ in $(0, T_*)$ and hence $\boldsymbol{m}_1 \equiv \boldsymbol{m}_2$ in $(0, T_*)$. Since T_* depends essentially only on T^* (more precisely, on the C^3 norm of the smooth function $\boldsymbol{m}_1$, and the physical parameters of the problem), we can repeat the argument *finitely many times* to conclude that $\boldsymbol{m}_1 \equiv \boldsymbol{m}_2$ in $(0, T^*)$.

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Institute for Analysis and Scientific Computing, TU Wien, Wiedner Hauptstrae 8-10, 1040 Wien, Austria

Email address: giovanni.difratta@asc.tuwien.ac.at

Institute for Analysis and Scientific Computing, TU Wien, Wiedner Hauptstrae 8-10, 1040 Wien, Austria

Email address: ansgar.juengel@tuwien.ac.at

Institute for Analysis and Scientific Computing, TU Wien, Wiedner Hauptstrae 8-10, 1040 Wien, Austria

Email address: dirk.praetorius@asc.tuwien.ac.at

School of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom

Email address: valeriy.slastikov@bristol.ac.uk