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GLOBAL MARTINGALE SOLUTIONS FOR QUASILINEAR SPDES VIA THE BOUNDEDNESS-BY-ENTROPY METHOD

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Abstract. The existence of global-in-time bounded martingale solutions to a general class of cross-diffusion systems with multiplicative Stratonovich noise is proved. The equations describe multicomponent systems from physics or biology with volume-filling effects and possess a formal gradient-flow or entropy structure. This structure allows for the derivation of almost surely positive lower and upper bounds for the stochastic processes. The existence result holds under some assumptions on the interplay between the entropy density and the multiplicative noise terms. The proof is based on a stochastic Galerkin method, a Wong–Zakai type approximation of the Wiener process, the boundedness-by-entropy method, and the tightness criterion of Brzeźniak and coworkers. Three-species Maxwell–Stefan systems and n-species biofilm models are examples that satisfy the general assumptions.

1. Introduction

Cross-diffusion systems arise in many application areas like fluid dynamics of mixtures, electrochemistry, cell biology, and biofilm modeling. Cross diffusion occurs if the gradient in the concentration of one species induces a flux of another species. In many applications, volume-filling effects need to be taken into account because of the finite size of the species or components, which means that the unknowns are volume fractions which sum up to one. Such cross-diffusion systems with volume filling in deterministic setting were analyzed in, for instance, [7, 11, 19] in the context of gas mixtures or ion transport through membranes. The boundedness-by-entropy method [33] provides a framework for the existence analysis and the proof of positive lower and upper bounds for the concentrations. The aim of this paper is to extend this technique to the stochastic setting. We prove the global-in-time existence of martingale solutions to cross-diffusion systems with volume filling and Stratonovich stochastic forcing.

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Key words and phrases. Cross diffusion, martingale solutions, entropy method, tightness, Skorokhod–Jakubowski theorem, Maxwell–Stefan systems, biofilm model.

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1.1. **Model equations.** The dynamics of the concentration (or volume fraction) vector $u = (u_1, \ldots, u_n)$ is given by

$$\mathrm{d}u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right)\mathrm{d}t = \sum_{j=1}^n \sigma_{ij}(u) \circ \mathrm{d}W_j(t) \quad \text{in } \mathcal{O},\ t > 0, \tag{1}$$

where $i = 1, \ldots, n$ and $\mathcal{O} \subset \mathbb{R}^d$ $(1 \le d \le 3)$ is a bounded domain, supplemented with the no-flux boundary and initial conditions,

$$\sum_{j=1}^n A_{ij}(u)\nabla u_j \cdot \nu = 0 \quad \text{on } \partial\mathcal{O},\ t > 0, \quad u_i(0) = u_i^0 \quad \text{in } \mathcal{O},\ i = 1, \ldots, n. \tag{2}$$

Here, ν is the exterior unit normal vector to $\partial\mathcal{O}$ and u_i^0 is a possibly random initial datum. The concentrations $u_i(\omega, x, t)$ are defined on $\Omega \times \mathcal{O} \times [0, T]$, where $\omega \in \Omega$ represents the stochastic variable, $x \in \mathcal{O}$ the spatial variable, and $t \in [0, T]$ the time. Together with the solvent concentration u_{n+1}, the concentrations fill up the domain, i.e., $\sum_{i=1}^{n+1} u_i = 1$. We call this assumption *volume filling*. The matrix $A(u) = (A_{ij}(u))$ is the diffusion matrix, $\sigma(u) = (\sigma_{ij}(u))$ is the multiplicative noise term, and $W = (W_1, \ldots, W_n)$ is an n-dimensional Wiener process. Details on the stochastic framework will be given in Section 1.3. The stochastic forcing represents external perturbations or a lack of knowledge of certain physical or biological parameters.

Equations (1) can be equivalently formulated in the Itô form [20, Section 6.5]:

$$\mathrm{d}u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right)\mathrm{d}t = \sum_{j=1}^n \sigma_{ij}(u)\mathrm{d}W_j(t) + \frac{1}{2}\left(\sum_{k=1}^n\sum_{j=1}^n \sigma_{kj}(u)\frac{\partial\sigma_{ij}}{\partial u_k}(u)\right)\mathrm{d}t,$$

where $i = 1, \ldots, n$, and this formulation will be also used in our analysis. The formulation of (1) in the Stratonovich form comes purely from a modeling viewpoint. In fact, our analysis uses the Wong–Zakai approximation, where we approximate the noise by smooth functions, thus obtaining a system of PDEs, which in turn converge in the limit to stochastic differential equations in the Stratonovich form. Alternatively, we could consider (1) in the Itô form and include the correction term in the formulation. In fact, considering the Itô formulation would enable us to treat more general infinite-dimensional noise but increasing the already involved technicalities. Therefore, this aspect will be discussed in a future work.

Quasilinear stochastic partial differential equations (SPDEs) (e.g. the porous-media equation) have been extensively analyzed using the theory of (locally) monotone operators [3, 25, 40, 44] or approximating the corresponding coefficients by locally monotone ones [28]. Recently, there has been a growing interest in developing various solution concepts for quasilinear SPDEs such as: kinetic [16, 21, 26], strong (in the probabilistic sense) and weak (in the PDE sense) [15, 28], entropy [12], martingale [16, 17] or (pathwise) mild solutions [38]. We mention that solution concepts for certain quasilinear SPDEs have been developed also via rough paths theory [43], paracontrolled calculus [2, 22], or regularity structures [23]. To our best knowledge, the techniques employed in this context heavily rely on the fact that the diffusion matrix is symmetric and/or positive-(semi)definite [15, 16, 28, 30]. However, in many applications, the diffusion matrix does not

satisfy these requirements, i.e., it is neither symmetric nor positive semi-definite. Therefore, most of the techniques available in the literature on quasilinear SPDEs do not apply or allow only local-in-time solutions [30, 38]. The main goal of this work is to prove the *global-in-time existence* of martingale solutions for quasilinear SPDEs whose diffusion matrix is *neither symmetric nor positive semi-definite, but admits a certain structure*, which we precisely describe below.

It turns out that deterministic cross-diffusion systems arising from (thermodynamic) applications often have a special structure, a so-called entropy or formal gradient-flow structure, which can be exploited for the existence analysis. This means that there exists an entropy density $h : [0,\infty)^n \to \mathbb{R}$ such that the deterministic analog of (1) can be formulated in terms of the entropy variables $w_i := \partial h/\partial u_i$ as

$$\partial_t u_i(w) - \operatorname{div}\left(\sum_{j=1}^n B_{ij}(w)\nabla w_j\right) = 0, \quad i = 1,\dots,n, \tag{3}$$

and the so-called Onsager matrix $B(w) = A(u(w))h''(u(w))^{-1}$ is positive semi-definite, where $h''(u)^{-1}$ denotes the inverse of the Hessian of h and $u = u(w) = (h')^{-1}(w)$ is now interpreted as a vector-valued function of w, assuming that the inverse of h' exists. An example is the Boltzmann-type entropy density $h_B(u) = \sum_{i=1}^{n+1}(u_i(\log u_i - 1) + 1)$. Using w_i as a test function in (3), a formal computation leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} h(u)\,\mathrm{d}x + \int_{\mathcal{O}} \nabla u : h''(u)A(u)\nabla u\,\mathrm{d}x = 0, \tag{4}$$

where ":" denotes the Frobenius matrix product. Since $B(w)$ is positive semi-definite, so is $h''(u)A(u)$, and we infer that $t \mapsto \int_{\mathcal{O}} h(u(t))\,\mathrm{d}x$ is a Lyapunov functional along solutions to (3).

The volume-filling condition $\sum_{i=1}^{n+1} u_i = 1$ implies that the solvent concentration can be replaced by the other concentrations $u_i \geq 0$ according to $u_{n+1} = 1 - \sum_{i=1}^n u_i$. This means that the concentration vector $u = (u_1,\dots,u_n)$ is an element of the Gibbs simplex $\mathcal{D} = \{u \in (0,1)^n : \sum_{i=1}^n u_i < 1\}$. If h is invertible on $\mathcal{D}$, we can define $u(w) = (h')^{-1}(w)$, and this function maps $\mathbb{R}^n$ to $\mathcal{D}$. Thus, if $w(x,t)$ is a solution to (3), $u(w(x,t)) \in \mathcal{D}$ is componentwise positive and bounded from above. This provides L^∞ estimates without using a maximum principle which generally cannot be applied to cross-diffusion systems. In this paper, we show that this idea can be extended to the stochastic case, allowing for L^∞ bounds almost surely.

Examples for cross-diffusion systems (1) with volume filling are the Maxwell–Stefan equations and certain biofilm models (see Section 3 for details). For fluid mixtures with three components, the Maxwell–Stefan diffusion matrix equals

$$A(u) = \frac{1}{a(u)}\begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix},$$
$$\text{where } a(u) = d_0 d_1 u_1 + d_0 d_2 u_2 + d_1 d_2 u_3,$$

and $d_i > 0$ for $i = 0, 1, 2$. This matrix is generally non-symmetric and not positive definite, but its eigenvalues are positive (this allows for local smooth deterministic solutions; see [1]). The first global existence result for deterministic Maxwell–Stefan equations was proved in [27] for initial data around the constant equilibrium state. The existence of local classical solutions was shown in [5]. The entropy structure was revealed in [35], and a general global existence result was proved. Other cross-diffusion models with volume filling arise in ion-transport and biofilm modeling [13, 24]. A general class of volume-filling systems was formally derived in [49] from a random walk on a lattice.

In the stochastic setting, we need to overcome some technical obstacles. First, since the diffusion matrix is not symmetric and not positive definite, standard semigroup theory is not applicable. Second, the application of the Itô formula to derive the stochastic analog of the entropy identity (4) requires that the entropy density is an element of $C^2(\overline{\mathcal{D}})$ which is usually not the case. For instance, the Boltzmann-type entropy density satisfies $\partial^2 h_B/\partial u_i^2 = 1/u_i + 1/u_{n+1}$ which is undefined when $u_i = 0$ or $u_{n+1} = 0$. Third, the system (3) is approximated in [33] by the implicit Euler discretization which is not compatible with the stochastic term (neither in Itô nor in Stratonovich form). We point out that the implicit Euler discretization, which is implemented in [33], could be avoided by introducing an additional regularization, hence avoiding the incompatibility issue, but this idea needs to be explored further.

Our key idea is to approximate the noise by a Wong–Zakai type argument and the space by a stochastic Galerkin method. This results in a system of ordinary differential equations which can be treated by the boundedness-by-entropy method [33]. The limit of vanishing Wong–Zakai parameter requires also the existence of solutions to a Galerkin stochastic differential system. This is proved by a fixed-point argument up to a stopping time $\tau_R > 0$, i.e., up to the first time a certain norm of the solution is larger than some $R > 0$. Estimates uniform in the Galerkin dimension $N \in \mathbb{N}$ are derived from an entropy inequality, which needs a regularization h_δ of the entropy density h, such that $h_\delta \in C^2(\overline{\mathcal{D}})$ with $\delta > 0$. The final step are the limits $\delta \to 0$, $R \to \infty$, and $N \to \infty$. Details of this procedure are given in Section 1.4.

1.2. **Notation and stochastic framework.** Let $\mathcal{O} \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain. The usual Lebesgue and Sobolev spaces are denoted by $L^p(\mathcal{O})$ and $W^{k,p}(\mathcal{O})$, respectively, where $p \in [1, \infty]$, $k \in \mathbb{N}$, and we set $H^k(\mathcal{O}) = W^{k,2}(\mathcal{O})$. The norm of a function $u = (u_1, \ldots, u_n) \in L^2(\mathcal{O}; \mathbb{R}^n)$ is understood as $\|u\|_{L^2(\mathcal{O})}^2 = \sum_{i=1}^n \|u_i\|_{L^2(\mathcal{O})}^2$, and we use this notation also for other vector- or matrix-valued functions. We write $\langle u, v\rangle$ for the dual product between $H^3(\mathcal{O})'$ and $H^3(\mathcal{O})$. We use the same notation if u, $v \in L^2(\mathcal{O})$, and in this case, $\langle u, v\rangle = \int_\mathcal{O} uv \,\mathrm{d}x$. In the vector-valued case, we have $\langle u, v\rangle = \sum_{i=1}^n \int_\mathcal{O} u_i v_i \,\mathrm{d}x$ for u, $v \in L^2(\mathcal{O}; \mathbb{R}^n)$. The set $\mathcal{D} = \{u \in (0, 1)^n : \sum_{i=1}^n u_i < 1\}$ is the Gibbs simplex in $\mathbb{R}^n$, and we set $u_{n+1} := 1 - \sum_{i=1}^n u_i > 0$ if $u \in \mathcal{D}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a complete right-continuous filtration $\mathbb{F} = (\mathcal{F})_{t\geq 0}$ and let H be a Hilbert space. The space $L^2(\Omega; H)$ consists of all H-valued

random variables u such that

$$\mathbb{E}\|u\|_H^2 := \int_\Omega \|u(\omega)\|_H^2 \mathbb{P}(\mathrm{d}\omega) < \infty.$$

Let $(\tilde{\eta}_k)_{k=1}^n$ be the canonical basis of $\mathbb{R}^n$. We denote by

$$\mathcal{L}_2(\mathbb{R}^n; L^2(\mathcal{O})) := \left\{ L : \mathbb{R}^n \to L^2(\mathcal{O}) \text{ linear continuous: } \sum_{k=1}^n \|L\tilde{\eta}_k\|_{L^2(\mathcal{O})}^2 < \infty \right\}$$

the space of Hilbert–Schmidt operators from $\mathbb{R}^n$ to $L^2(\mathcal{O})$ endowed with the norm

$$\|L\|_{\mathcal{L}_2(\mathbb{R}^n;L^2(\mathcal{O}))}^2 := \sum_{k=1}^n \|L\tilde{\eta}_k\|_{L^2(\mathcal{O})}^2.$$

The multiplicative noise term $\sigma : \Omega \times [0,T] \times \mathbb{R}^n \ni (\omega, t, u) \to \sigma(\omega, t, u) \in \mathbb{R}^{n\times n}$ with $\sigma = (\sigma_{ij})_{i,j=1,\dots,n}$ is assumed to be $\mathcal{B}(L^2(\mathcal{O}) \times [0,T]) \otimes \mathcal{F}; \mathcal{B}(\mathcal{L}_2(\mathbb{R}^n; L^2(\mathcal{O})))$-measurable and $\mathbb{F}$-adapted.

1.3. **Assumptions and main result.** We impose the following assumptions.

(A1) Domain: $\mathcal{O} \subset \mathbb{R}^d$ $(d \le 3)$ is a bounded domain with Lipschitz boundary.

(A2) Initial datum: $u^0 \in L^2(\Omega; L^\infty(\mathcal{O}))$ is $\mathcal{F}_0$-measurable and $u_i^0(x) \in \mathcal{D}$ for a.e. $x \in \mathcal{O}$ $\mathbb{P}$-a.s., $i = 1, \dots, n$.

(A3) Diffusion matrix: $A = (A_{ij}) \in C^0(\overline{\mathcal{D}}; \mathbb{R}^{n\times n})$ is Lipschitz continuous.

(A4) Multiplicative noise $\sigma : L^2(\mathcal{O}) \to \mathcal{L}_2(\mathbb{R}^n; L^2(\mathcal{O}))$ satisfies for some constant $C_\sigma > 0$ and any $u \in L^2(\mathcal{O})$, $i, j, k = 1, \dots, n$,

$$\left\| \frac{\partial \sigma_{ij}}{\partial u^k}(u) \right\|_{\mathcal{L}(L^2(\mathcal{O});L^2(\mathcal{O}))} \le C_\sigma.$$

(A5) Entropy density: (i) There exists a convex function $h \in C^2(\mathcal{D}; [0,\infty)) \cap C^0(\overline{\mathcal{D}}; [0,\infty))$ such that its derivative $h' : \mathcal{D} \to \mathbb{R}^n$ is invertible; (ii) there exist $c_h > 0$, $0 \le m < 1$ such that for all $u \in \mathcal{D}$, $z \in \mathbb{R}^n$,

$$z^\top h''(u) A(u) z \ge c_h \sum_{i=1}^n \frac{z_i^2}{u_i^{2m}}.$$

(A6) Interaction of entropy density and noise: There exists $C_h > 0$ such that for all $u \in \mathcal{D}$,

$$\max_{j=1,\dots,n} \left| \sum_{i=1}^n \frac{\partial h}{\partial u_i}(u) \sigma_{ij}(u) \right| + \left| \sum_{i,j,k=1}^n \sigma_{kj}(u) \frac{\partial \sigma_{ij}}{\partial u_k}(u) \frac{\partial h}{\partial u_i}(u) \right| + \left| \sum_{i,j,k=1}^n \sigma_{ik}(u) \frac{\partial^2 h(u)}{\partial u_i \partial u_j} \sigma_{jk}(u) \right| \le C_h.$$

(A7) Approximation of the entropy density: Let

$$[u_i]_\delta = \frac{u_i + \delta/n}{1+\delta} \quad \text{for } i = 1, \ldots, n$$

and set $[u]_\delta = ([u_1]_\delta, \ldots, [u_n]_\delta)$ for $u \in \overline{\mathcal{D}}$. It holds that for all $u \in \mathcal{D}$ and $z \in \mathbb{R}^n$,

$$z^\top h''([u]_\delta) A(u) z - c_h \sum_{i=1}^n \frac{z_i^2}{[u_i]_\delta^{2m}} \geq z^\top R_\delta(u) z,$$

where $R_\delta(u) \in \mathbb{R}^{n\times n}$ is a correction matrix that appears as a result of the compatibility of the regularized entropy $h([u]_\delta)$ with Assumption (A5ii), and it holds that $R_\delta(u) \to 0$ as $\delta \to 0$ uniformly in $\overline{\mathcal{D}}$.

Remark 1 (Discussion of the assumptions)**.** Assumptions (A1)–(A3), (A5) are essentially the same conditions imposed in the (deterministic) boundedness-by-entropy method [33]. We assume additionally that the diffusion matrix is Lipschitz continuous, which is needed to apply classical existence results for stochastic differential equations (see, e.g., [44]). Assumption (A5ii) means that the Onsager matrix is positive definite but not necessarily uniformly in u. It provides gradient estimates for u_i^{1-m}, i.e., the diffusion matrix has a fast-diffusion-type degeneracy. Assumption (A4) implies global Lipschitz continuity for the multiplicative noise term, which is a standard condition for SPDEs; see ,e.g.,[44]. Assumption (A6) allows us to deal with the stochastic part when we derive the entropy estimate. The idea is that the multiplicative noise is chosen in order to compensate possible singularities of $h'(u)$ and $h''(u)$. Finally, Assumption (A7) is needed since generally h is not a $C^2(\overline{\mathcal{D}})$ function and cannot be used in the Itô lemma, whereas its regularization $h_\delta(u) = h([u]_\delta)$ is a $C^2(\overline{\mathcal{D}})$ function and therefore admissible in the Itô lemma. We suppose that h_δ is compatible with Assumption (A5ii). We present two examples from applications fulfilling Assumptions (A3)–(A7) in Section 3. □

Remark 2 (Extensions)**.** Our setting can be slightly generalized in different directions. The space dimension d can be arbitrarily large. The condition $d \leq 3$ is needed to conclude the continuous embedding $H^3(\mathcal{O}) \hookrightarrow W^{1,\infty}(\mathcal{O})$. For general $d \geq 1$, we need to work with $H^s(\mathcal{O})$ with $s > 1 + d/2$ instead of $H^3(\mathcal{O})$. We may include a nonlinear source term $F(u)$ (satisfying standard local Lipschitz continuity assumptions) which additionally interacts with the corresponding entropy density [34, Assumption H3, p. 86], namely

$$\int_\mathcal{O} F(u) \cdot h'(u)\,\mathrm{d}x \leq C_F \left(1 + \int_\mathcal{O} h(u)\,\mathrm{d}x\right).$$

Moreover, we may allow for random initial data, i.e., we may prescribe an initial probability measure instead of a given initial data. We refer to [17, Remark 18] for details. We consider only finite-dimensional Wiener processes instead of infinite-dimensional ones because we need to quantify the interaction of the entropy density and noise terms in Assumption (A6). Our technique also works with (trace-class) Q-Wiener processes but the proof becomes very technical without introducing new ideas, which is the reason why we restrict ourselves to the finite-dimensional case. □

Our main result is the global-in-time existence of martingale solutions to (1)–(2). First, we make precise the definition of martingale solutions.

Definition 1 (Global martingale solution)**.** *For any fixed $T > 0$, the triple $(\widetilde{U}, \widetilde{W}, \widetilde{u})$ is a* global martingale solution *to* (1)–(2) *if $\widetilde{U} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{F}})$ is a stochastic basis with filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$, $\widetilde{W}$ is an $\mathbb{R}^n$-valued Wiener process on this filtered probability space, and $\widetilde{u}(t) = (\widetilde{u}_1(t), \dots, \widetilde{u}_n(t))$ is a progressively measurable stochastic process for all $t \in [0,T]$ such that for all $i = 1, \dots, n$,*

$$\widetilde{u}_i \in L^2(\widetilde{\Omega}; C^0([0,T]; L^2_w(\mathcal{O}))) \cap L^2(\widetilde{\Omega}; L^2(0,T; H^1(\mathcal{O}))),$$

the law of $\widetilde{u}_i(0)$ is the same as for u_i^0, and $\widetilde{u}$ satisfies for all $\phi \in H^1(\mathcal{O})$ and $i = 1, \dots, n$,

$$\int_{\mathcal{O}} \widetilde{u}_i(t)\phi \,\mathrm{d}x = \int_{\mathcal{O}} \widetilde{u}_i(0)\phi \,\mathrm{d}x + \sum_{j=1}^n \int_0^t \int_{\mathcal{O}} A_{ij}(\widetilde{u}(s))\nabla \widetilde{u}_j(s) \cdot \nabla\phi \,\mathrm{d}x \,\mathrm{d}s$$
$$+ \sum_{j=1}^n \int_{\mathcal{O}} \bigg(\int_0^t \sigma_{ij}(\widetilde{u}(s)) \circ \mathrm{d}\widetilde{W}_j(s) \bigg) \phi \,\mathrm{d}x.$$

Here, $C^0([0,T]; L^2_w(\mathcal{O}))$ is the space of weakly continuous functions $u : [0,T] \to L^2(\mathcal{O})$ such that $\sup_{0<t<T} \|u(t)\|_{L^2(\mathcal{O})} < \infty$.

Theorem 3 (Existence of a global martingale solution)**.** *Let Assumptions (A1)–(A7) hold and let $T > 0$. Then there exists a global martingale solution to* (1)–(2) *satisfying $\widetilde{u}(x,t) \in \overline{\mathcal{D}}$ for a.e. $(x,t) \in \mathcal{O} \times (0,T)$ $\widetilde{\mathbb{P}}$-a.s. and $\widetilde{u}_i \in L^p(\widetilde{\Omega}; L^\infty(0,T; L^\infty(\mathcal{O})))$ for any $p < \infty$.*

1.4. **Key ideas.** We explain the strategy of the proof of Theorem 3. The approximation procedure combines the techniques of [17] and [33] and is illustrated in Figure 1.

Step 1: Stochastic Galerkin approximation. Equations (1) are projected on a Galerkin space with finite dimension $N \in \mathbb{N}$. The existence of a unique solution $u^{(N)}$ to the stochastic differential system up to a stopping time τ_R is shown by Banach's fixed-point theorem, exploiting the Lipschitz continuity of the nonlinearities. We recall that $R > 0$ is a previously chosen parameter in the definition of *the* stopping time τ_R, describing the upper bound of a certain norm. Since the contraction constant depends on R, we cannot pass to the limit $R \to \infty$. For global solutions, we need a priori estimates which can be derived in principle from the entropy inequality, similar to (4). However, this requires that the solution is positive and bounded, which cannot be deduced from this technique. We need the boundedness-by-entropy method.

Step 2: Wong–Zakai approximation. In order to obtain the uniform boundedness for the solutions, we regularize the noise in the sense of the Wong–Zakai approximation with parameter $\eta > 0$, giving a system of ordinary differential equations, which is parametrized by the stochastic variable. The existence of a solution $u^{(N,\eta)}$ follows from the boundedness-by-entropy method [33]. A consequence of this technique is the nonnegativity and boundedness of $u^{(N,\eta)}(x,t)$ $\mathbb{P}$-a.s. We also obtain estimates from an entropy inequality, but they depend on η and are therefore cannot be further applied. The Wong–Zakai theory allows us to pass to the limit $\eta \to 0$, showing that $u^{(N,\eta)}$ converges to the solution $u^{(N)}$ obtained in Step

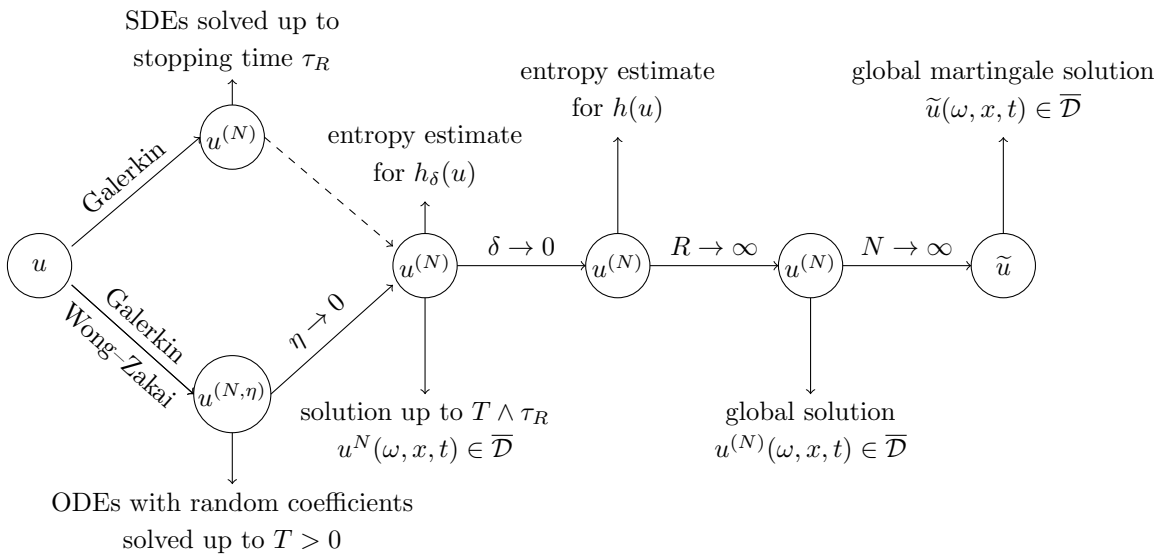

FIGURE 1. Steps of the proof of the existence theorem.

1. Since this solution is unique, we deduce that $u^{(N)}$ is nonnegative and bounded, more precisely $u^{(N)}(x,t)\in\overline{\mathcal{D}}$ for a.e. $(x,t)\in\mathcal{O}\times(0,T)$ $\mathbb{P}$-a.s.

Step 3: Entropy estimates. Gradient estimates uniform in N are obtained from the entropy inequality, which is derived in the deterministic setting by using the test function $h'(u^{(N)})$. Since the entropy density h generally does not belong to $C^2(\overline{\mathcal{D}})$, we cannot use the Itô lemma. We need to regularize the entropy density by a function h_δ (with parameter $\delta>0$) which belongs to $C^2(\overline{\mathcal{D}})$. Itô's lemma then allows us to derive entropy estimates which are uniform in δ, R, and N. After passing to the limit $\delta\to 0$, we infer the following entropy estimates uniform in the Galerkin dimension N:

$$\mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(t))\,\mathrm{d}x + C_1\mathbb{E}\int_0^t\int_{\mathcal{O}}\sum_{i=1}^n\left|\nabla(u_i^{(N)})^{1-m}\right|^2\,\mathrm{d}x\,\mathrm{d}s\le C_2, \tag{5}$$

where C_1, $C_2>0$ are independent of N and R and $m<1$. Since the right-hand side does not depend on R, we may pass to the limit $R\to\infty$, thus obtaining global approximate solutions $u^{(N)}$.

Step 4: Tightness of the laws. The tightness of the laws of $(u^{(N)})$ in a sub-Polish space is shown by applying the tightness criterion of Brzeźniak and Motyl [9]. It involves the verification of some a priori estimates which can be deduced from (5).

Step 5: Convergence. The tightness of the laws of $(u^{(N)})$ and the Skorokhod–Jakubowski theorem allow us to perform the limit $N\to\infty$ in the sense that there exist random variables $\widetilde{u}^{(N)}$, with the same law as $u^{(N)}$, converging to a martingale solution to (1)–(2). Unfortunately, the property $u^{(N)}(x,t)\in\overline{\mathcal{D}}$ does not directly imply that $\widetilde{u}^{(N)}(x,t)\in\overline{\mathcal{D}}$ since only the laws of these random variables coincide. Our idea is to show, using the Kuratowski theorem, that $\sum_{i=1}^n\|\widetilde{u}_i^{(N)}\|_{L^\infty}\le 1$ $\widetilde{\mathbb{P}}$-a.s. and that $\widetilde{u}_i^{(N)}$ lies in the union of

the unit balls around zero and around one (with respect to the L^∞ norm) from which we conclude that $\widetilde{u}(x,t) \in \overline{\mathcal{D}}$ $\widetilde{\mathbb{P}}$-a.s.

These steps are detailed in Section 2. Two examples fulfilling Assumptions (A3)–(A7) are presented in Section 3 and some theorems from stochastic analysis are recalled in Appendix A.

2. Existence analysis

We prove Theorem 3 by approximating system (1) by a stochastic Galerkin method and later by a Wong–Zakai type approximation of the $\mathbb{R}^n$-valued Wiener process.

2.1. **Stochastic Galerkin approximation.** We prove the existence of a strong (in the probability sense) solution to an approximate system up to a stopping time by using the Banach fixed-point theorem.

The approximate system is obtained from projecting (1) onto the finite-dimensional Hilbert space $H_N = \operatorname{span}\{e_1, \dots, e_N\}$, where $N \in \mathbb{N}$ and $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathcal{O})$ such that $H_N \subset H^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$. We introduce the projection operator $\Pi_N : L^2(\mathcal{O}) \to H_N$ by

$$\Pi_N(v) = \sum_{i=1}^N (v, e_i)_{L^2(\mathcal{O})} e_i \quad \text{for } v \in L^2(\mathcal{O}).$$

We need the basis in $H^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$ for later purposes, i.e in the proof of Proposition 5.

The approximate problem is the following system of stochastic differential equations,

$$\begin{aligned} \mathrm{d}u_i^{(N)} &= \Pi_N \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u^{(N)})\nabla u_j^{(N)}\right)\mathrm{d}t + \sum_{j=1}^n \Pi_N\big(\sigma_{ij}(u^{(N)})\big)\,\mathrm{d}W_j(t) \\ &\quad + \frac{1}{2}\Pi_N\left(\sum_{k=1}^n\sum_{j=1}^n \sigma_{kj}(u^{(N)})\frac{\partial\sigma_{ij}}{\partial u_k}(u^{(N)})\right)\mathrm{d}t, \quad i = 1, \dots, N, \end{aligned} \tag{6}$$

with the initial conditions

$$u_i^{(N)}(0) = \Pi_N(u_i^0), \quad i = 1, \dots, N. \tag{7}$$

Since the solutions $u^{(N)}$ may not lie in the Gibbs simplex $\overline{\mathcal{D}}$, we need to extend the functions A_{ij} and σ_{ij} to the whole space $\mathbb{R}^n$. This is done in such a way that A_{ij} and σ_{ij} are Lipschitz continuous on $\mathbb{R}^n$ (we do not change the notation). This implies that A_{ij} and σ_{ij} grow at most linearly.

Given $T > 0$, we introduce the space $X_T = L^2(\Omega; L^\infty(0,T; H_N))$ with the norm $\|u\|_{X_T}^2 := \mathbb{E}(\sup_{0<t<T}\|u(t)\|_{H_N})^2$. For given $R > 0$ and $u \in X_T$, we define the stopping time

$$\tau_R := \inf\{t \in [0,T] : \|u(t)\|_{H^1(\mathcal{O})} > R\}.$$

Furthermore, we introduce the Itô correction operator $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_n) : L^2(\mathcal{O};\mathbb{R}^n) \to L^2(\mathcal{O};\mathbb{R}^n)$ by

$$\mathcal{T}_i(u) = \sum_{k=1}^n\sum_{j=1}^n \sigma_{kj}(u)\frac{\partial\sigma_{ij}}{\partial u_k}(u), \quad u \in L^2(\mathcal{O};\mathbb{R}^n).$$

Proposition 4. *Let $T > 0$, $R > 0$ be fixed, and let Assumptions (A1)–(A5) hold. Then there exists a unique strong (in the probabilistic sense) solution $u^{(N)} \in X_{T\wedge\tau_R}$ to (6)–(7) such that for any $t \in [0, T \wedge \tau_R]$,*

$$\begin{aligned}\langle u^{(N)}(t), \phi\rangle &= \langle u^0, \phi\rangle - \int_0^t \langle A(u^{(N)}(s))\nabla u^{(N)}(s), \nabla\phi\rangle\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \langle \mathcal{T}(u^{(N)}(s)), \phi\rangle\,\mathrm{d}s + \int_0^t \langle \sigma(u^{(N)}(s))\,\mathrm{d}W(s), \phi\rangle \end{aligned} \tag{8}$$

for any $\phi = (\phi_1, \ldots, \phi_N) \in L^2(0, T; C^\infty(\mathcal{O}) \cap H_N)^n$.

Proof. The idea of the proof is to apply the Banach fixed-point theorem to the mapping $S : X_T \to X_T$,

$$\begin{aligned}\langle S(u^{(N)})(t), \phi\rangle &= \langle u^0, \phi\rangle - \int_0^t \langle A(u^{(N)})(s)\nabla u^{(N)}(s), \nabla\phi\rangle\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \langle \mathcal{T}(u^{(N)}(s)), \phi\rangle\,\mathrm{d}s + \int_0^t \langle \sigma(u^{(N)}(s))\,\mathrm{d}W(s), \phi\rangle, \end{aligned} \tag{9}$$

where $u^{(N)} \in X_T$ and $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n) \cap H_N^n$. The linear growth of A and σ allows us to show that S indeed maps X_T into itself and that S is a contraction for some $T^* \in (0, T \wedge \tau_R]$. Although the arguments are rather standard, we provide a full proof for completeness.

We show first the self-mapping property. Let $u \in X_T$ and $\phi \in C_0^\infty(\mathcal{O}; \mathbb{R}^n) \cap H_N^n$. Then Definition (9) gives

$$\begin{aligned}&\|\langle S(u), \phi\rangle\|^2_{L^2(\Omega; L^\infty(0,T\wedge\tau_R))} = \mathbb{E}\bigg(\sup_{0<t<T\wedge\tau_R} |\langle S(u)(t), \phi\rangle|\bigg)^2 \\ &\quad \le \|\phi\|^2_{L^2(\mathcal{O})}\mathbb{E}\|u^0\|^2_{L^2(\mathcal{O})} + C\mathbb{E}\int_0^{T\wedge\tau_R} |\langle A(u(s))\nabla u(s), \nabla\phi\rangle|^2\,\mathrm{d}s \\ &\qquad + C\mathbb{E}\int_0^{T\wedge\tau_R} |\langle \mathcal{T}(u(s)), \phi\rangle|^2\,\mathrm{d}s + C\mathbb{E}\bigg(\sup_{0<t<T\wedge\tau_R}\bigg|\int_0^t \langle \sigma(u(s))\,\mathrm{d}W(s), \phi\rangle\bigg|\bigg)^2 \\ &\quad =: I_1 + \cdots + I_4.\end{aligned}$$

We estimate the terms I_2, I_3, and I_4. Because of the linear growth of A and the equivalence of the norms in H_N, we find that

$$\begin{aligned}I_2 &\le C\|\nabla\phi\|^2_{L^\infty(\mathcal{O})}\mathbb{E}\int_0^{T\wedge\tau_R} (1 + \|u(s)\|^2_{L^2(\mathcal{O})})\|\nabla u(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s \\ &\le C(T\wedge\tau_R)\|\nabla\phi\|^2_{L^\infty(\mathcal{O})}\mathbb{E}\bigg(1 + \sup_{0<t<T\wedge\tau_R}\|u(t)\|^2_{L^2(\mathcal{O})}\bigg)R^2 \\ &\le C(N, R)T\|\phi\|^2_{H_N}\big(1 + \|u\|^2_{X_{T\wedge\tau_R}}\big).\end{aligned}$$

Assumption (A4) implies that $\mathcal{T}(u)$ grows at most linearly, so

$$\begin{aligned} I_3 &\le C\|\phi\|_{L^2(\mathcal{O})}^2 \mathbb{E}\int_0^{T\wedge\tau_R} \|\mathcal{T}(u(s))\|_{L^2(\mathcal{O})}^2 \,\mathrm{d}s \\ &\le C\|\phi\|_{L^2(\mathcal{O})}^2 \mathbb{E}\int_0^{T\wedge\tau_R} \big(1+\|u(s)\|_{L^2(\mathcal{O})}^2\big)\,\mathrm{d}s \le C(N)T\|\phi\|_{H_N}^2\big(1+\|u\|_{X_{T\wedge\tau_R}}^2\big). \end{aligned}$$

We obtain from the Burkholder–Davis–Gundy inequality [37, Prop. 2.12]

$$\begin{aligned} I_4 &\le C\|\phi\|_{L^2(\mathcal{O})}^2 \mathbb{E}\int_0^{T\wedge\tau_R} \|\sigma(u(s))\|_{\mathcal{L}_2(\mathbb{R}^n;L^2(\mathcal{O}))}^2 \,\mathrm{d}s \\ &\le C\|\phi\|_{L^2(\mathcal{O})}^2 \mathbb{E}\int_0^{T\wedge\tau_R} \big(1+\|u(s)\|_{L^2(\mathcal{O})}^2\big)\,\mathrm{d}s \le C(N)T\|\phi\|_{L^2(\mathcal{O})}^2\big(1+\|u\|_{X_{T\wedge\tau_R}}^2\big). \end{aligned}$$

Summarizing these estimates, we find that

$$\|S(u)\|_{X_{T\wedge\tau_R}}^2 \le C\mathbb{E}\|u^0\|_{L^2(\mathcal{O})}^2 + C(N,R)T\big(1+\|u\|_{X_{T\wedge\tau_R}}^2\big),$$

which implies that S maps $X_{T\wedge\tau_R}$ to $X_{T\wedge\tau_R}$.

Next, we show that $S : X_T \to X_T$ is a contraction if $0 < T < \tau_R$ is sufficiently small. Let u, $v \in X_T$, $\phi \in C_0^\infty(\mathcal{O};\mathbb{R}^n) \cap H_N^n$, and $R > 0$ and set

$$\tau_R = \inf\big\{t\in[0,T] : \|u(t)\|_{H^1(\mathcal{O})} > R\big\} \wedge \inf\big\{t\in[0,T] : \|v(t)\|_{H^1(\mathcal{O})} > R\big\}.$$

Then

$$\begin{aligned} &\|\langle S(u)-S(v),\phi\rangle\|_{L^2(\Omega;L^\infty(0,T\wedge\tau_R))}^2 \\ &\quad\le C\mathbb{E}\bigg(\sup_{0<t<T\wedge\tau_R}\bigg|\int_0^t \langle A(u(s))\nabla u(s) - A(v(s))\nabla v(s), \nabla\phi(s)\rangle\,\mathrm{d}s\bigg|\bigg)^2 \\ &\qquad + \frac{C}{2}\mathbb{E}\bigg(\sup_{0<t<T\wedge\tau_R}\bigg|\int_0^t \langle \mathcal{T}(u(s)) - \mathcal{T}(v(s)),\phi\rangle\,\mathrm{d}s\bigg|\bigg)^2 \\ &\qquad + C\mathbb{E}\bigg(\sup_{0<t<T\wedge\tau_R}\bigg|\int_0^t \big\langle\big(\sigma(u(s)) - \sigma(v(s))\big)\,\mathrm{d}W(s),\phi\big\rangle\bigg|\bigg)^2 \\ &\quad =: I_5 + I_6 + I_7. \end{aligned}$$

Assumption (A3) shows that

$$\begin{aligned} I_5 &\le CT\mathbb{E}\int_0^{T\wedge\tau_R} \big|\big\langle (A(u)-A(v))\nabla u + A(v)\nabla(u-v), \nabla\phi\big\rangle\big|^2\,\mathrm{d}s \\ &\le CT\|\nabla\phi\|_{L^\infty(\mathcal{O})}^2\mathbb{E}\int_0^{T\wedge\tau_R} \Big(\|u(s)-v(s)\|_{L^2(\mathcal{O})}^2\|\nabla u(s)\|_{L^2(\mathcal{O})}^2 \\ &\qquad + \big(1+\|v(s)\|_{L^2(\mathcal{O})}^2\big)\|\nabla(u-v)(s)\|_{L^2(\mathcal{O})}^2\Big)\,\mathrm{d}s \\ &\le C(N)R^2T^2\|\nabla\phi\|_{L^\infty(\mathcal{O})}^2\|u-v\|_{X_{T\wedge\tau_R}}^2. \end{aligned}$$

Similarly, exploiting the linear growth of σ and $\mathcal{T}$,

$$I_6 \le CT\mathbb{E}\int_0^{T\wedge\tau_R} |\langle \mathcal{T}(u)-\mathcal{T}(v),\phi\rangle|^2\,\mathrm{d}s \le CT^2\|\phi\|_{L^2(\mathcal{O})}^2\|u-v\|_{X_{T\wedge\tau_R}}^2,$$

$$I_7 \le CT\|\phi\|_{L^2(\mathcal{O})}^2\mathbb{E}\int_0^{T\wedge\tau_R} \|\sigma(u)-\sigma(v)\|_{\mathcal{L}_2(\mathbb{R}^n;L^2(\mathcal{O}))}^2\,\mathrm{d}s \le CT^2\|\phi\|_{L^2(\mathcal{O})}^2\|u-v\|_{X_{T\wedge\tau_R}}^2.$$

Consequently,

$$\|S(u)-S(v)\|_{X_{T\wedge\tau_R}} \le C(N,R)T^2\|u-v\|_{X_{T\wedge\tau_R}},$$

which shows that $S : X_{T^*} \to X_{T^*}$ is a contraction for $0 < T^* < T\wedge\tau_R$ satisfying $C(N,R)(T^*)^2 < 1$.

By the Banach fixed-point theorem, there exists a unique fixed point $u^{(N)} \in X_{T^*}$, which means that $u^{(N)}$ solves (8) for any $t \in (0,T^*)$. The local solution can be uniquely extended to a global one on the whole interval $[0,T\wedge\tau_R]$ since $T^* > 0$ is independent of the initial datum. Standard results [38, Lemma 3.23] show that the stopping time τ_R is $\mathbb{P}$-a.s. positive. □

2.2. **Wong–Zakai-type approximation.** We prove the existence of global-in-time solutions to another approximate system of (1), consisting of a system of ordinary differential equations (ODE). For this, we introduce two levels of approximations with the following parameters: the Galerkin dimension $N \in \mathbb{N}$ and a Wong–Zakai type approximation of the $\mathbb{R}^n$-valued Wiener process with time step $\eta > 0$. More precisely, we project (1) as in the previous subsection onto the finite-dimensional Galerkin space H_N and introduce a uniform partition of the time interval $[0,T]$ with time step $\eta = T/M$, where $M \in \mathbb{N}$ and $t_k = k\eta$ for $k = 0,\ldots,M$. The Wiener process is approximated by the process

$$W^{(\eta)}(t) = W(t_k) + \frac{t-t_k}{\eta}(W(t_{k+1})-W(t_k)), \quad t\in[t_k,t_{k+1}],\ k=0,\ldots,M. \tag{10}$$

Approximations like this or via convolution with a smooth kernel are generally referred to as Wong–Zakai approximations and were introduced in [48] in one dimension and in [45] for systems. Further generalizations can be found in [41], [46]–[47].

The approximate equations read as

$$\frac{\mathrm{d}u^{(N,\eta)}}{\mathrm{d}t} = \Pi_N \operatorname{div}\left(A(u^{(N,\eta)})\nabla u^{(N,\eta)}\right) + \Pi_N\big(\sigma(u^{(N,\eta)})\big)\frac{\mathrm{d}W^{(\eta)}}{\mathrm{d}t}, \tag{11}$$

with the initial conditions

$$u^{(N,\eta)}(0) = \Pi_N(u^0). \tag{12}$$

This is a finite-dimensional system of ODEs. The existence of global-in-time solutions is deduced from the boundedness-by-entropy technique of [33].

Proposition 5. *Let $T > 0$, $N \in \mathbb{N}$, $\eta > 0$, and let Assumptions (A1)–(A5) hold. Then for almost all $\omega \in \Omega$, there exists a global-in-time weak solution $u^{(N,\eta)} = (u_1^{(N,\eta)},\ldots,u_n^{(N,\eta)})$ to* (11)–(12) *satisfying*

$$u_i^{(N,\eta)}(\omega,\cdot,\cdot) \in L^2(0,T;H^1(\mathcal{O})), \quad \partial_t u_i^{(N,\eta)}(\omega,\cdot,\cdot) \in L^2(0,T;H^1(\mathcal{O})')$$

for $i = 1, \ldots, n$ *and a.e.* $\omega \in \Omega$,

$$u^{(N,\eta)}(x,t) \in \overline{\mathcal{D}} \quad \text{for } (x,t) \in \mathcal{O} \times (0,T) \ \mathbb{P}\text{-a.s.},$$

$u^{(N,\eta)}(0) = \Pi_N(u^0)$ *in the sense of* $H^1(\mathcal{O})'$, *and*

$$\begin{aligned}\langle u^{(N,\eta)}(t), \phi\rangle = \langle u^0, \phi\rangle &- \int_0^t \langle A(u^{(N,\eta)}(s))\nabla u^{(N,\eta)}(s), \nabla\phi\rangle \,\mathrm{d}s \\ &+ \int_0^t \left\langle \sigma(u^{(N,\eta)}(s))\frac{\mathrm{d}W^{(\eta)}}{\mathrm{d}t}(s), \phi \right\rangle \mathrm{d}s\end{aligned}$$

for any $\phi \in L^2(0,T; H^1(\mathcal{O}) \cap H_N)^n$.

Proof. In principle, the proof follows by applying the boundedness-by-entropy method [33, Theorem 2] to the cross-diffusion system (11) with the source term

$$f(u,t) := \Pi_N(\sigma(u^{(N,\eta)}(t)))\frac{\mathrm{d}W^{(\eta)}}{\mathrm{d}t}(t).$$

We drop the ω dependence to simplify the notation. For the convenience of those readers who are not familiar with this technique, we recall the main steps of the proof. Details can be found in [33, 34].

The idea is to formulate (11) as a finite-dimensional diffusion problem with variable $w = h'(u^{(N,\eta)})$. After solving this problem in w, we can then define $u^{(N,\eta)} := (h')^{-1}(w)$, and since the range of $(h')^{-1}$ is the bounded set $\mathcal{D}$, we find that $u^{(N,\eta)}(\omega, x, t) \in \mathcal{D}$ for a.e. $\omega \in \Omega$. The transformation causes two difficulties: First, the flux transforms to $A(u^{(N,\eta)})\nabla u^{(N,\eta)} = B(w)\nabla w$, but the new diffusion matrix $B(w) = A(u^{(N,\eta)})h''(u^{(N,\eta)})^{-1}$ is generally only positive semi-definite. Second, the time derivative becomes $\partial_t u^{(N,\eta)} = h''(u^{(N,\eta)})\partial_t w$, but $h''(u^{(N,\eta)})$ may be not invertible on $\partial\mathcal{D}$. Both issues can be solved by discretizing (11) in time and adding a regularization. In fact, for the fixed $T > 0$, $L \in \mathbb{N}$, we consider a time grid π_L (which is finer than the uniform time partition considered for the Wong–Zakai type approximation), and set $\tau = T/L > 0$. Let $\varepsilon > 0$ and $w^{k-1} \in L^\infty(\mathcal{O}; \mathbb{R}^n)$ be given. We wish to solve i.e. find $w^k \in H^1(\mathcal{O}; \mathbb{R}^n)$, such that

$$\begin{aligned}\frac{1}{\tau}\int_{\mathcal{O}} \big(u(w^k) - u(w^{k-1})\big) \cdot \phi \,\mathrm{d}x + \int_{\mathcal{O}} \sum_{i,j=1}^n B_{ij}(w^k)\nabla\phi_i \cdot \nabla w_j^k \,\mathrm{d}x \\ + \varepsilon \int_{\mathcal{O}} w^k \cdot \phi \,\mathrm{d}x = \int_{\mathcal{O}} f(u(w^k), t_k) \cdot \phi \,\mathrm{d}x,\end{aligned} \tag{13}$$

where $u(w) := (h')^{-1}(w)$ and $\phi \in H^1(\mathcal{O}; \mathbb{R}^n)$.

Step 1: Solution of the approximate problem. We prove the existence of a solution to (12) and (13) by applying the Leray–Schauder fixed-point theorem. Let the Galerkin space H_N be a subset of $H^1(\mathcal{O}; \mathbb{R}^n)$ such that $H_N \subset L^\infty(\mathcal{O}; \mathbb{R}^n)$. (This is possible by choosing appropriate basis functions.) Let $y \in L^\infty(\mathcal{O}; \mathbb{R}^n)$ and $\vartheta \in [0,1]$ be given. We consider the following linear problem: Find $w = w^k \in H_N$ such that

$$a(w, \phi) = F(\phi) \quad \text{for all } \phi \in H_N, \tag{14}$$

where

$$a(w,\phi) = \int_{\mathcal{O}} \sum_{i,j=1}^n B_{ij}(y)\nabla\phi_i \cdot \nabla w_j^k \,\mathrm{d}x + \varepsilon \int_{\mathcal{O}} w^k \cdot \phi \,\mathrm{d}x,$$
$$F(\phi) = -\frac{\vartheta}{\tau}\int_{\mathcal{O}} \big(u(y) - u(w^{k-1})\big)\cdot\phi\,\mathrm{d}x + \vartheta\int_{\mathcal{O}} f(u(y),t_k)\cdot\phi\,\mathrm{d}x.$$

The boundedness of y and the Cauchy–Schwarz inequality show that a and F are bounded on H_N. Since $B(y)$ is positive semi-definite and all norms are equivalent in finite dimensions,

$$a(w,w) \ge \varepsilon\|w\|_{L^2(\mathcal{O})}^2 \ge \varepsilon C(N)\|w\|_{H^1(\mathcal{O})}^2,$$

which means that a is coercive on H_N. By the Lax–Milgram lemma, there exists a unique solution $w \in H_N$ to (14) and it holds that $w \in L^\infty(\mathcal{O};\mathbb{R}^n)$. This defines the fixed-point operator $S : H_N \times [0,1] \to H_N$, $S(y,\vartheta) = w$, where w solves (14).

We verify the assumptions of the Leray–Schauder theorem. The only solution to (14) with $\vartheta = 0$ is $w = 0$; thus $S(y,0) = 0$. The continuity of S follows from standard arguments; see the proof of [33, Lemma 5] for details. Since H_N is finite-dimensional, S is compact. It remains to prove a uniform bound for all fixed points of $S(\cdot,\vartheta)$. Let $w \in H_N$ be such a fixed point. Then w solves (14) with y replaced by w. Choosing the test function $\phi = w$, we obtain $\mathbb{P}$-a.s.

$$\frac{\vartheta}{\tau}\int_{\mathcal{O}} \big(u(w) - u(w^{k-1})\big)\cdot w\,\mathrm{d}x + \int_{\mathcal{O}}\sum_{i,j=1}^n B_{ij}(w)\nabla w_i\cdot\nabla w_j\,\mathrm{d}x$$
$$+\varepsilon\int_{\mathcal{O}}|w|^2\,\mathrm{d}x = \vartheta\int_{\mathcal{O}} f(u(w),t_k)\cdot w\,\mathrm{d}x. \tag{15}$$

The convexity of h (see Assumption (A5i)) shows that

$$\frac{\vartheta}{\tau}\int_{\mathcal{O}} \big(u(w) - u(w^{k-1})\big)\cdot w\,\mathrm{d}x \ge \frac{\vartheta}{\tau}\int_{\mathcal{O}} \big(h(u(w)) - h(u(w^{k-1}))\big)\,\mathrm{d}x.$$

Since $B(w)$ is positive semi-definite, we have $\sum_{i,j=1}^n B_{ij}(w)\nabla w_i\cdot\nabla w_j \ge 0$. Finally, we use Assumption (A4), (10) along with Kolmogorov's continuity theorem to infer that for all $u \in [0,\infty)^n$,

$$f(u(w),t_k)\cdot h'(u(w)) = \frac{1}{\eta}\sum_{i,j=1}^n \sigma_{ij}(u(w))(W_j(t_{k+1}) - W_j(t_k))\frac{\partial h}{\partial u_i}(u(w))$$
$$\le \frac{1}{\eta}\sum_{j=1}^n |W_j(t_{k+1}) - W_j(t_k)| \max_{j=1,\dots,n}\sum_{i=1}^n \left|\sigma_{ij}(u(w))\frac{\partial h}{\partial u_i}(u(w))\right| \le C(\eta).$$

This shows that the right-hand side of (15) is bounded uniformly in ϑ and w. We infer that $\varepsilon\|w\|_{L^2(\Omega)}^2 \le C(\eta)$ and consequently $\|w\|_{H^1(\Omega)} \le C(\eta,\varepsilon,N)$ $\mathbb{P}$-a.s. This yields the desired uniform bound, and we can apply the Leray–Schauder fixed-point theorem to conclude the existence of a weak solution $w^k \in H_N$ to (13).

Step 2: Uniform estimates. Since we do not have any uniform estimates for w, we switch to the original variable $u(w^k)$. Let $w^{(\tau)}(\omega,x,t)=w^k(\omega,x)$ and $u^{(\tau)}(\omega,x,t)=u(w^k(\omega,x))$ for $\omega\in\Omega$, $x\in\mathcal{O}$, and $t\in((k-1)\tau,k\tau]$, $k=1,\ldots,L$. At time $t=0$, we set $w^{(\tau)}(\cdot,0)=h'(u^0)$ and $u^{(\tau)}(\cdot,0)=u^0$. We also need the shift operator $(\Gamma_\tau u^{(\tau)})(\omega,x,t)=u(w^{k-1}(\omega,x))$ for $\omega\in\Omega$, $x\in\mathcal{O}$, and $t\in((k-1)\tau,k\tau]$. In this notation, the weak formulation (13) can be written as

$$\begin{aligned}\frac{1}{\tau}\int_0^T\int_{\mathcal{O}}(u^{(\tau)}-\Gamma_\tau u^{(\tau)})\cdot\phi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathcal{O}}\sum_{i,j=1}^n A_{ij}(u^{(\tau)})\nabla\phi_i\cdot\nabla u_j^{(\tau)}\,\mathrm{d}x\,\mathrm{d}t\\ +\varepsilon\int_0^T\int_{\mathcal{O}}w^{(\tau)}\cdot\phi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_{\mathcal{O}}f(u^{(\tau)})\cdot\phi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{16}$$

for piecewise constant functions $\phi:(0,T)\to H_N$.

We derive now some uniform estimates, using the test function $\phi=w^{(\tau)}$ in (16). At this point, we need Assumption (A5ii):

$$\begin{aligned}\sum_{i,j=1}^n A_{ij}(u^{(\tau)})\nabla w_i^{(\tau)}\cdot\nabla u_j^{(\tau)}&=\sum_{i,j=1}^n\big(h''(u^{(\tau)})A(u^{(\tau)})\big)_{ij}\nabla u_i^{(\tau)}\cdot\nabla u_j^{(\tau)}\\ &\ge c_h\sum_{i=1}^n\frac{|\nabla u^{(\tau)}|^2}{(u^{(\tau)})^{2m}}=\frac{c_h}{(1-m)^2}\sum_{i=1}^n|\nabla(u^{(\tau)})^{1-m}|^2.\end{aligned}$$

Hence, summing (16) over $k=1,\ldots,j$ with $j\le L$, it follows similarly as in Step 1 that $\mathbb{P}$-a.s.

$$\int_{\mathcal{O}}h(u(w^j))\,\mathrm{d}x+\tau\sum_{k=1}^j\sum_{i=1}^n\int_{\mathcal{O}}|\nabla u_i(w^k)^{1-m}|^2\,\mathrm{d}x+\varepsilon\tau\sum_{k=1}^j\|w^k\|_{L^2(\mathcal{O})}^2\le C,$$

where $C>0$ depends on $h(u^0),\eta$ but not on ε or τ. Together with the uniform L^∞ bound for $u^{(\tau)}$, this yields

$$\|(u^{(\tau)})^{1-m}\|_{L^2(0,T;H^1(\mathcal{O}))}+\sqrt{\varepsilon}\|w^{(\tau)}\|_{L^2(0,T;L^2(\mathcal{O}))}\le C.$$

Moreover, $\nabla u^{(\tau)}=(1-m)^{-1}(u^{(\tau)})^m\nabla(u^{(\tau)})^{1-m}$ is uniformly bounded in $L^2(\mathcal{O}\times(0,T))$. (Here, we need that $0\le m<1$.) A straightforward computation shows that $\tau^{-1}(u^{(\tau)}-\Gamma_\tau u^{(\tau)})$ is uniformly bounded in $L^2(0,T;H^1(\mathcal{O})')$.

Step 3: Limit $\varepsilon\to0$ and $\tau\to0$. The uniform estimates from Step 2 allow us to apply the Aubin–Lions lemma in the version of [18], which provides the existence of a subsequence of $(u^{(\tau)})$, which is not relabeled, such that, as $(\varepsilon,\tau)\to0$,

$$u^{(\tau)}\to u\quad\text{strongly in } L^1(\mathcal{O}\times(0,T))\ \mathbb{P}\text{-a.s.}$$

In view of the uniform L^∞ bound, this convergence holds in any $L^p(\mathcal{O}\times(0,T))$ for $p<\infty$ and a.e. in $\mathcal{O}\times(0,T)$ $\mathbb{P}$-a.s. This allows us to identify the nonlinear weak limits. Moreover, by weak compactness, $\mathbb{P}$-a.s.

$$\nabla u^{(\tau)}\rightharpoonup\nabla u\quad\text{weakly in } L^2(0,T;L^2(\mathcal{O})),$$

$$\tau^{-1}(u^{(\tau)} - \Gamma_\tau u^{(\tau)}) \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0,T;H^1(\mathcal{O})'),$$
$$\varepsilon w^{(\tau)} \to 0 \quad \text{strongly in } L^2(0,T;L^2(\mathcal{O})).$$

Performing the limit $(\varepsilon,\tau) \to 0$ in (16) shows that $u^{(N,\eta)} := u$ solves (11) for all test functions $\phi \in L^2(0,T;H^1(\mathcal{O}))$ (by density). We verify as in [33] that u satisfies the initial condition (12). □

The proof of [33, Theorem 2] provides some a priori estimates through the entropy inequality, but they depend on η because of the dependence of the source term on η. We derive some uniform bounds in Section 2.3.

Next, we show that the Wong–Zakai approximations converge to the strong solution to (6)–(7). The key consequence is the L^∞ bound for the solution to (6)–(7).

Proposition 6. *Let $u^{(N,\eta)}$ be the solution to* (11)–(12)*, constructed in Proposition 5, and let $u^{(N)}$ be the unique strong (in the probabilistic sense) solution to* (6)–(7)*, proved in Proposition 4. Then $u^{(N,\eta)} \to u^{(N)}$ in probability up to a stopping time $\tau_R = \inf\{t \in [0,T] : \|u^{(N)}(t)\|_{H^1(\mathcal{O})} > R\}$ as $\eta \to 0$ ($M \to \infty$). Moreover, it holds that $u^{(N)}(x,t) \in \overline{\mathcal{D}}$ for $(x,t) \in \mathcal{O} \times (0,T)$ $\mathbb{P}$-a.s.*

Proof. The result is a consequence of Theorem 14 in the Appendix. We can apply this theorem since the right-hand side of (11) is Lipschitz continuous and has linear growth in $u^{(N,\eta)}$ (see the proof of Proposition 4). □

2.3. **Uniform estimates.** We prove some estimates uniform in the approximation parameter N. The starting point is a stochastic version of the entropy inequality.

Lemma 7. *The solution $u^{(N)}$ to* (6)–(7) *is global-in-time and satisfies the a priori estimate*

$$\mathbb{E}\int_\mathcal{O} h(u^{(N)}(t))\,\mathrm{d}x + C_1\mathbb{E}\int_0^t\int_\mathcal{O}\sum_{i=1}^n \left|\nabla(u_i^{(N)})^{1-m}\right|^2\,\mathrm{d}x\,\mathrm{d}s \le C_2,$$

where C_1, $C_2 > 0$ are independent of N and R.

Proof. Let $u^{(N)}$ be the solution to (6)–(7) up to the stopping time τ_R. Since the entropy density h, defined in Assumption (A5i), may be not a C^2 function on $\overline{\mathcal{D}}$, we cannot apply the Itô lemma to this function. Therefore, we need to regularize h. Let us recall the notation from Assumption (A7): Let $\delta > 0$ and define $[u]_\delta = ([u_1]_\delta,\ldots,[u_n]_\delta)$, where

$$[u_i]_\delta := \frac{u_i + \delta/n}{1+\delta},\ i = 1,\ldots,n, \quad [u_{n+1}]_\delta := \frac{u_{n+1}}{1+\delta},$$

and $u_{n+1} = 1 - \sum_{i=1}^n u_i$. Then $[u_{n+1}]_\delta = 1 - \sum_{i=1}^n [u_i]_\delta$ and $[u]_\delta \in \mathcal{D}$ for any $u \in \overline{\mathcal{D}}$. It follows that $h_\delta(u) := h([u]_\delta)$ satisfies $h_\delta \in C^2(\overline{\mathcal{D}};[0,\infty))$.

We can now apply the Itô lemma to h_δ. It holds for $t \in [0, T\wedge\tau_R]$ that

$$\int_\mathcal{O} h_\delta(u^{(N)}(t\wedge\tau_R))\,\mathrm{d}x = \int_\mathcal{O} h_\delta(u^{(N)}(0))\,\mathrm{d}x$$

$$
\begin{aligned}
&- \int_0^{t\wedge\tau_R} \int_{\mathcal{O}} \nabla u^{(N)}(s) : h_\delta''(u^{(N)}(s)) A(u^{(N)}(s)) \nabla u^{(N)}(s) \,\mathrm{d}x \,\mathrm{d}s \\
&+ \int_0^{t\wedge\tau_R} \int_{\mathcal{O}} (\sigma(u^{(N)}(s)) \,\mathrm{d}W(s)) \cdot h_\delta'(u^{(N)}(s)) \,\mathrm{d}x \\
&+ \frac{1}{2} \int_0^{t\wedge\tau_R} \int_{\mathcal{O}} h_\delta'(u^{(N)}(s)) \cdot \mathcal{T}(u^{(N)}(s)) \,\mathrm{d}x \,\mathrm{d}s \\
&+ \frac{1}{2} \int_0^{t\wedge\tau_R} \int_{\mathcal{O}} \mathrm{Tr}\left(\sigma(u^{(N)}(s)) h_\delta''(u^{(N)}(s)) \sigma(u^{(N)}(s))^*\right) \mathrm{d}x \,\mathrm{d}s.
\end{aligned}
$$

Taking the expectation on both sides and observing that the expectation of the Itô integral vanishes, we find that

$$
\begin{aligned}
\mathbb{E} \int_{\mathcal{O}} h_\delta(u^{(N)}(t\wedge\tau_R)) \,\mathrm{d}x &= \int_{\mathcal{O}} h_\delta(u^{(N)}(0)) \,\mathrm{d}x \\
&- \mathbb{E} \int_0^{t\wedge\tau_R} \int_{\mathcal{O}} \nabla u^{(N)}(s) : h_\delta''(u^{(N)}(s)) A(u^{(N)}(s)) \nabla u^{(N)}(s) \,\mathrm{d}x \,\mathrm{d}s \\
&+ \frac{1}{2} \mathbb{E} \int_0^{t\wedge\tau_R} \int_{\mathcal{O}} h_\delta'(u^{(N)}(s)) \cdot \mathcal{T}(u^{(N)}(s)) \,\mathrm{d}x \,\mathrm{d}s \\
&+ \frac{1}{2} \mathbb{E} \int_0^{t\wedge\tau_R} \int_{\mathcal{O}} \mathrm{Tr}\left(\sigma(u^{(N)}(s)) h_\delta''(u^{(N)}(s)) \sigma(u^{(N)}(s))^*\right) \mathrm{d}x \,\mathrm{d}s \\
&=: J_1^{(\delta)} + \cdots + J_4^{(\delta)}. \qquad (17)
\end{aligned}
$$

Our aim is to perform the limit $\delta \to 0$ in (17). We know that the function h is continuous on $\overline{\mathcal{D}}$ and that, as $\delta \to 0$,

$$
[u^{(N)}(\omega, x, t\wedge\tau_R)]_\delta \to u^{(N)}(\omega, x, t\wedge\tau_R) \quad \text{for a.e. } (\omega, x, t) \in \Omega \times \mathcal{O} \times (0, T).
$$

This implies that

$$
h_\delta(u^{(N)}(\omega, x, t\wedge\tau_R)) = h([u^{(N)}(\omega, x, t\wedge\tau_R)]_\delta) \to h(u^{(N)}(\omega, x, t\wedge\tau_R))
$$

for a.e. $(\omega, x, t) \in \Omega \times \mathcal{O} \times (0, T)$. Moreover, the integral $\mathbb{E} \int_{\mathcal{O}} h_\delta(u^{(N)}(t\wedge\tau_R)) \,\mathrm{d}x$ is uniformly bounded in δ (since h is bounded on $\overline{\mathcal{D}}$ by assumption). We conclude from the dominated convergence theorem that, as $\delta \to 0$,

$$
\begin{aligned}
\mathbb{E} \int_{\mathcal{O}} h_\delta(u^{(N)}(t\wedge\tau_R)) \,\mathrm{d}x &\to \mathbb{E} \int_{\mathcal{O}} h(u^{(N)}(t\wedge\tau_R)) \,\mathrm{d}x, \\
J_1^{(\delta)} = \mathbb{E} \int_{\mathcal{O}} h_\delta(u^{(N)}(0)) \,\mathrm{d}x &\to \mathbb{E} \int_{\mathcal{O}} h(\Pi_N(u^0)) \,\mathrm{d}x.
\end{aligned}
$$

By Assumption (A7), we have

$$
J_2^{(\delta)} = -\frac{1}{(1+\delta)^2} \mathbb{E} \int_0^{t\wedge\tau_R} \int_{\mathcal{O}} \nabla u^{(N)}(s) : h''([u^{(N)}(s)]_\delta) A(u^{(N)}(s)) \nabla u^{(N)}(s) \,\mathrm{d}x \,\mathrm{d}s
$$

$$
\begin{aligned}
&\le -\frac{c_h}{(1+\delta)^2}\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i=1}^n\frac{|\nabla u_i^{(N)}(s)|^2}{[u_i^{(N)}(s)]_\delta^{2m}}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad -\frac{1}{(1+\delta)^2}\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\nabla u^{(N)}(s):R_\delta(u^{(N)}(s))\nabla u^{(N)}(s)\,\mathrm{d}x\,\mathrm{d}s. \qquad (18)
\end{aligned}
$$

Since $R_\delta(u^{(N)}(s))\to 0$ as $\delta\to 0$ uniformly in $u^{(N)}(s)$ and $\nabla u^{(N)}(s)$ is bounded in $L^2(\mathcal{O})$, the last integral tends to zero as $\delta\to 0$. Because of

$$
\frac{1}{(1+\delta)^2}\frac{|\nabla u_i^{(N)}|^2}{[u_i^{(N)}]_\delta^{2m}}\nearrow\frac{|\nabla u_i^{(N)}|^2}{(u_i^{(N)})^{2m}}=\frac{|\nabla(u_i^{(N)})^{1-m}|^2}{(1-m)^2}
$$

as $\delta\to 0$, the monotone convergence theorem implies that

$$
\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i=1}^n\frac{|\nabla u_i^{(N)}(s)|^2}{[u_i^{(N)}]_\delta^{2m}}\,\mathrm{d}x\,\mathrm{d}s\to\frac{1}{(1-m)^2}\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i=1}^n|\nabla(u_i^{(N)})^{1-m}|^2\,\mathrm{d}x\,\mathrm{d}s
$$

and we infer from (18) that

$$
\lim_{\delta\to 0}J_2^{(\delta)}\le -\frac{c_h}{(1-m)^2}\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i=1}^n|\nabla(u_i^{(N)})^{1-m}|^2\,\mathrm{d}x\,\mathrm{d}s.
$$

The following a.e. pointwise limits hold:

$$
\begin{aligned}
h_\delta'(u^{(N)}(s))\cdot\mathcal{T}(u^{(N)}(s)) &= \frac{1}{1+\delta}\sum_{i,j,k=1}^n\sigma_{kj}(u^{(N)}(s))\frac{\partial\sigma_{ij}}{\partial u_k}(u^{(N)}(s))\frac{\partial h}{\partial u_i}([u^{(N)}(s)]_\delta) \\
&\to h'(u^{(N)})\cdot\mathcal{T}(u^{(N)}(s))
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Tr}\Big(\sigma(u^{(N)}(s))h_\delta''(u^{(N)}(s))\sigma(u^{(N)}(s))^*\Big) \\
&\quad=\frac{1}{(1+\delta)^2}\sum_{i,j,k=1}^n\sigma_{ik}(u^{(N)}(s))\frac{\partial^2 h}{\partial u_i\partial u_j}([u^{(N)}(s)]_\delta)\sigma_{jk}(u^{(N)}(s)) \\
&\quad\to\mathrm{Tr}\left(\sigma(u^{(N)}(s))h''(u^{(N)}(s))\sigma(u^{(N)}(s))^*\right)
\end{aligned}
$$

for a.e. $\Omega\times\mathcal{O}\times[0,T\wedge\tau_R]$. Then the bounds imposed in Assumption (A6) imply by dominated convergence that these expressions converge in $L^1(\Omega\times\mathcal{O}\times[0,T\wedge\tau_R])$, which means that

$$
\begin{aligned}
J_3^{(\delta)} &\to \frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}h'(u^{(N)}(s))\cdot\mathcal{T}(u^{(N)}(s))\,\mathrm{d}x\,\mathrm{d}s, \\
J_4^{(\delta)} &\to \frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\mathrm{Tr}\left(\sigma(u^{(N)}(s))h''(u^{(N)}(s))\sigma(u^{(N)}(s))^*\right)\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Using Assumption (A6) again, we see that the limits of $J_3^{(\delta)}$ and $J_4^{(\delta)}$ are bounded with respect to N and R, and Assumption (A2) implies that the limit of $J_1^{(\delta)}$ is uniformly bounded in N. Then the limit $\delta \to 0$ in (17) yields the entropy inequality

$$\begin{aligned}\mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(t\wedge\tau_R))\,\mathrm{d}x + C_1\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i=1}^n |\nabla(u_i^{(N)}(s))^{1-m}|^2\,\mathrm{d}x\,\mathrm{d}s \\ \le \mathbb{E}\int_{\mathcal{O}} h(\Pi_N(u^0))\,\mathrm{d}x + C_2,\end{aligned}$$

where the constants $C_1 > 0$ and $C_2 > 0$ are independent of N and R. Consequently, the right-hand side of this inequality does not depend on the chosen sequence of stopping times τ_R, and we can pass to the limit $R \to \infty$. Hence, the previous inequality holds for any $t \in [0,T]$. The uniform L^∞ estimate implies that

$$\sup_{N\in\mathbb{N}} \mathbb{E}\bigg(\sup_{0<t<T} \|u^{(N)}(t)\|_{L^2(\mathcal{O})}^p\bigg) \le C(T,u^0) \tag{19}$$

for all $1 \le p < \infty$, and the entropy inequality shows that

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|u^{(N)}\|_{L^2(0,T;H^1(\mathcal{O}))}^2 \le C(T,u^0), \tag{20}$$

where $C(T,u^0) > 0$ is independent of N, since

$$\begin{aligned}\mathbb{E}\|\nabla u_i^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))}^2 &= \int_\Omega\int_0^T\int_{\mathcal{O}}\bigg|\frac{1}{1-m}(u_i^{(N)})^m\nabla(u_i^{(N)})^{1-m}\bigg|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mathbb{P}(\omega) \\ &= \frac{1}{(1-m)^2}\int_\Omega\int_0^T\int_{\mathcal{O}}|u_i^{(N)}|^{2m}|\nabla(u_i^{(N)})^{1-m}|^2\mathrm{d}x\,\mathrm{d}t\mathrm{d}\mathbb{P}(\omega) \\ &\le \frac{1}{(1-m)^2}\int_\Omega\int_0^T\int_{\mathcal{O}}|\nabla(u_i^{(N)})^{1-m}|^2\mathrm{d}x\,\mathrm{d}t\mathrm{d}\mathbb{P}(\omega) \\ &= \frac{1}{(1-m)^2}\mathbb{E}\|\nabla(u_i^{(N)})^{1-m}\|_{L^2(0,T;L^2(\mathcal{O}))}^2 \le C.\end{aligned}$$

Here, we used $|u^{(N)}(\omega,x,t)| \le 1$ for almost all $(\omega,x,t) \in \Omega\times\mathcal{O}\times[0,T]$ and that $m < 1$. Since $T > 0$ was arbitrary, the solution $u^{(N)}$ to (6)–(7) is global-in-time. □

2.4. **Tightness of the laws of** $(u^{(N)})$**.** Let $u^{(N)}$ be a solution to (6)–(7), constructed in Lemma 7. We show that the laws of $u^{(N)}$ are tight in a certain sub-Polish space. (This is a topological space in which there exists a countable family of continuous functions that separate points [6, Definition 2.1.3].) For this, we proceed similarly as in [17] and introduce the following spaces:

- $C^0([0,T];H^3(\mathcal{O})')$ is the space of continuous functions $u : [0,T] \to H^3(\mathcal{O})'$ with the topology $\mathbb{T}_1$ induced by the norm $\|u\|_{C^0([0,T];H^3(\mathcal{O})')} = \sup_{t\in(0,T)}\|u(t)\|_{H^3(\mathcal{O})'}$;
- $L^2_w(0,T;H^1(\mathcal{O}))$ is the space $L^2(0,T;H^1(\mathcal{O}))$ with the weak topology $\mathbb{T}_2$;
- $L^2(0,T;L^2(\mathcal{O}))$ is the space of square integrable functions $u : (0,T) \to L^2(\mathcal{O})$ with the topology $\mathbb{T}_3$ induced by the norm $\|\cdot\|_{L^2(0,T;L^2(\mathcal{O}))}$;

- $C^0([0,T];L^2_w(\mathcal{O}))$ is the space of weakly continuous functions $u:[0,T]\to L^2(\mathcal{O})$ endowed with the weakest topology $\mathbb{T}_4$ such that for all $\psi\in L^2(\mathcal{O})$, the mappings
$$C^0([0,T];L^2_w(\mathcal{O}))\to C^0([0,T];\mathbb{R}),\quad u\mapsto (u(\cdot),\psi)_{L^2(\mathcal{O})},$$
are continuous.

We define the space
$$Z_T := C^0([0,T];H^3(\mathcal{O})')\cap L^2_w(0,T;H^1(\mathcal{O}))\cap L^2(0,T;L^2(\mathcal{O}))\cap C^0([0,T];L^2_w(\mathcal{O})),$$
endowed with the topology $\mathbb{T}$ that is the maximum of the topologies $\mathbb{T}_i$, $i=1,2,3,4$, of the corresponding spaces. It is shown in [17, Lemma 12] that Z_T is a sub-Polish space.

Lemma 8. *The set of laws $(\mathcal{L}(u^{(N)}))_{N\in\mathbb{N}}$ is tight in Z_T.*

Proof. The idea is to apply the tightness criterion of Brzeźniak and Motyl [9, Corollary 2.6] with the spaces $U=H^3(\mathcal{O})$, $V=H^1(\mathcal{O})$, and $H=L^2(\mathcal{O})$ (also see the proof of Lemma 11 in [17]). Estimates (19) and (20) are exactly conditions (a) and (b) in [9]. It remains to show that $(u^{(N)})_{N\in\mathbb{N}}$ satisfies the Aldous condition in $H^3(\mathcal{O})'$. We need to show that for any $\varepsilon>0$ and $\kappa>0$, there exists $\theta_0>0$ such that for any sequence $(\tau_N)_{N\in\mathbb{N}}$ of $\mathbb{F}$-stopping times, it holds that
$$\sup_{N\in\mathbb{N}}\sup_{0<\theta<\theta_0}\mathbb{P}\big\{\|u^{(N)}(\tau_N+\theta)-u^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\ge\kappa\big\}\le\varepsilon.$$
We proceed similarly as in [17, Lemma 11]. Let $(\tau_N)_{N\in\mathbb{N}}$ be a sequence of $\mathbb{F}$-stopping times such that $0\le\tau_N\le T$ and let $t\in[0,T]$ and $\phi\in H^3(\mathcal{O})$. The solution $u^{(N)}$ to (6)–(7) solves
$$\begin{aligned}\langle u_i^{(N)}(t),\phi\rangle &= \langle\Pi_N(u_i^0),\phi\rangle-\int_0^t\sum_{j=1}^n\big\langle A_{ij}(u^{(N)})\nabla u_j^{(N)},\nabla\Pi_N\phi\big\rangle\,\mathrm{d}s\\ &\quad+\frac12\int_0^t\langle\Pi_N(\mathcal{T}_i(u^{(N)}),\phi\rangle\,\mathrm{d}s+\left\langle\int_0^t\sum_{j=1}^n\Pi_N(\sigma_{ij}(u^{(N)}))\,\mathrm{d}W_j,\phi\right\rangle\\ &=: J_1^{(N)}+J_2^{(N)}(t)+J_3^{(N)}(t)+J_4^{(N)}(t).\end{aligned}\tag{21}$$

Consider first the term involving the diffusion coefficients. Let $\theta>0$. We use assumption (A3), the continuous embedding $H^3(\mathcal{O})\hookrightarrow W^{1,\infty}(\mathcal{O})$ (for $d\le3$), and estimates (19)–(20) to find that
$$\begin{aligned}&\mathbb{E}\left|\int_{\tau_N}^{\tau_N+\theta}\big\langle A_{ij}(u^{(N)})\nabla u_j^{(N)},\nabla\Pi_N\phi\big\rangle\,\mathrm{d}s\right|\\ &\le C\mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\big(1+\|u^{(N)}\|_{L^2(\mathcal{O})}\big)\|\nabla u^{(N)}\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^\infty(\mathcal{O})}\,\mathrm{d}s\\ &\le C\theta^{1/2}\mathbb{E}\big((1+\|u^{(N)}\|_{L^\infty(0,T;L^2(\mathcal{O}))})\|\nabla u^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))}\big)\|\phi\|_{H^3(\mathcal{O})}\\ &\le C\theta^{1/2}\left\{1+\mathbb{E}\left(\sup_{0<t<T}\|u^{(N)}(t)\|^2_{L^2(\mathcal{O})}\right)\right\}^{1/2}\left\{\mathbb{E}\int_0^T\|\nabla u^{(N)}\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s\right\}^{1/2}\|\phi\|_{H^3(\mathcal{O})}\end{aligned}$$

$$\leq C\theta^{1/2}\|\phi\|_{H^3(\mathcal{O})},$$

where we applied first the Cauchy–Schwarz inequality with respect to time and then with respect to the random variable. For the Itô correction term, we use the boundedness of $u^{(N)}$ and the Cauchy–Schwarz inequality:

$$\begin{aligned}\mathbb{E}\bigg|\int_{\tau_N}^{\tau_N+\theta}\langle\Pi_N(\mathcal{T}_i(u^{(N)}),\phi\rangle\,\mathrm{d}s\bigg| &\leq \mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\|\mathcal{T}_i(u^{(N)})\|_{L^2(\mathcal{O})}\|\phi\|_{L^2(\mathcal{O})}\\ &\leq C\theta^{1/2}\|\phi\|_{H^3(\mathcal{O})}.\end{aligned}$$

For the stochastic term, we take into account Assumption (A4), the Itô isometry, and again the Cauchy–Schwarz inequality:

$$\begin{aligned}&\mathbb{E}\bigg|\bigg\langle\int_{\tau_N}^{\tau_N+\theta}\Pi_N(\sigma_{ij}(u^{(N)}))\,\mathrm{d}W_j,\phi\bigg\rangle\,\mathrm{d}s\bigg|^2\\ &\quad\leq \mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\|\sigma(u^{(N)}(s))\|^2_{\mathcal{L}_2(\mathbb{R}^n;L^2(\mathcal{O}))}\,\mathrm{d}s\|\phi\|^2_{L^2(\mathcal{O})}\\ &\quad\leq \mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\big(1+\|u^{(N)}(s)\|^2_{L^2(\mathcal{O})}\big)\,\mathrm{d}s\|\phi\|^2_{L^2(\mathcal{O})}\\ &\quad\leq C\bigg\{\theta+\theta^{1/3}\bigg(E\int_0^T\|u^{(N)}(s)\|^3_{L^2(\mathcal{O})}\,\mathrm{d}s\bigg)^{2/3}\bigg\}\|\phi\|^2_{L^2(\mathcal{O})}\leq C\theta^{1/3}\|\phi\|^2_{H^3(\mathcal{O})}.\end{aligned}$$

Note that the previous estimates could be simplified since $u^{(N)}$ is uniformly bounded. Our estimates hold under minimal requirements and may be used for generalizations.

Let $\kappa>0$ and $\varepsilon>0$. In view of the previous estimates and using the Chebyshev inequality, it follows for $i=2,3$ that

$$\begin{aligned}&\mathbb{P}\big\{\|J_i^{(N)}(\tau_N+\theta)-J_i^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\geq\kappa\big\}\leq\frac{1}{\kappa}\mathbb{E}\big\|J_i^{(N)}(\tau_N+\theta)-J_i^{(N)}(\tau_N)\big\|_{H^3(\mathcal{O})'}\\ &\quad=\frac{1}{\kappa}\sup_{\|\phi\|_{H^3(\mathcal{O})}=1}\mathbb{E}\big|\big\langle J_i^{(N)}(\tau_N+\theta)-J_i^{(N)}(\tau_N),\phi\big\rangle\big|\leq\frac{C\theta^{1/2}}{\kappa},\end{aligned}$$

while for $i=4$, we have

$$\begin{aligned}&\mathbb{P}\big\{\|J_4^{(N)}(\tau_N+\theta)-J_4^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\geq\kappa\big\}\\ &\quad\leq\frac{1}{\kappa}\sup_{\|\phi\|_{H^3(\mathcal{O})}=1}\mathbb{E}\big|\big\langle J_i^{(N)}(\tau_N+\theta)-J_i^{(N)}(\tau_N),\phi\big\rangle\big|\\ &\quad\leq\frac{C}{\kappa}\sup_{\|\phi\|_{H^3(\mathcal{O})}=1}\Big(\mathbb{E}\big|\big\langle J_i^{(N)}(\tau_N+\theta)-J_i^{(N)}(\tau_N),\phi\big\rangle\big|^2\Big)^{1/2}\leq\frac{C\theta^{1/6}}{\kappa}.\end{aligned}$$

Thus, choosing $\theta_0=\min\{1,(\kappa\varepsilon/C)^6\}$, we infer that for $i=2,3,4$,

$$\sup_{N\in\mathbb{N}}\sup_{0<\theta<\theta_0}\mathbb{P}\big\{\|J_i^{(N)}(\tau_N+\theta)-J_i^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\geq\kappa\big\}\leq\varepsilon.$$

This shows that the Aldous condition holds for all three terms $J_i^{(N)}$ $(i = 2, 3, 4)$ and consequently, in view of (21), also for $(u_i^{(N)})_{N\in\mathbb{N}}$. Thus, by [9, Corollary 2.6], the set of laws of $(u^{(N)})_{N\in\mathbb{N}}$ is tight in Z_T. □

2.5. **Convergence of** $(u^{(N)})_{N\in\mathbb{N}}$**.** Since $Z_T \times C^0([0,T];\mathbb{R}^n)$ satisfies the assumptions of the Skorokhod–Jakubowski theorem [10, Theorem C1] and the sequence of laws of $(u^{(N)})_{N\in\mathbb{N}}$ is tight on $(Z_T, \mathbb{T})$ by Lemma 8, this theorem implies the existence of a subsequence of $(u^{(N)})_{N\in\mathbb{N}}$, which is not relabeled, a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and, on this space, $(Z_T \times C^0([0,T];\mathbb{R}^n))$-valued random variables $(\widetilde{u}, \widetilde{W})$ and $(\widetilde{u}^{(N)}, \widetilde{W}^{(N)})$ for $N \in \mathbb{N}$ such that $(\widetilde{u}^{(N)}, \widetilde{W}^{(N)})$ has the same law as $(u^{(N)}, W)$ on $\mathcal{B}(Z_T \times C^0([0,T];\mathbb{R}^n))$ and, as $N \to \infty$,

$$(\widetilde{u}^{(N)}, \widetilde{W}^{(N)}) \to (\widetilde{u}, \widetilde{W}) \quad \text{in } Z_T \times C^0([0,T];\mathbb{R}^n)\ \widetilde{\mathbb{P}}\text{-a.s.}$$

Because of the definition of the space Z_T, this convergence means $\widetilde{\mathbb{P}}$-a.s.,

$$\begin{aligned}
\widetilde{u}^{(N)} &\to \widetilde{u} \quad \text{in } C^0([0,T];H^3(\mathcal{O})'),\\
\widetilde{u}^{(N)} &\rightharpoonup \widetilde{u} \quad \text{weakly in } L^2(0,T;H^1(\mathcal{O})),\\
\widetilde{u}^{(N)} &\to \widetilde{u} \quad \text{in } L^2(0,T;L^2(\mathcal{O})),\\
\widetilde{u}^{(N)} &\to \widetilde{u} \quad \text{in } C^0([0,T];L^2_w(\mathcal{O})),\\
\widetilde{W}^{(N)} &\to \widetilde{W} \quad \text{in } C^0([0,T];\mathbb{R}^n).
\end{aligned} \tag{22}$$

We wish to derive some regularity properties for the limit $\widetilde{u}$. To this end, we proceed as in [17, Section 2.5]. Since $u^{(N)}$ is an element of $C^0([0,T];H_N)$ $\mathbb{P}$-a.s., $C^0([0,T];H_N)$ is a Borel set of $C^0([0,T];H^3(\mathcal{O})') \cap L^2(0,T;L^2(\mathcal{O}))$, and $u^{(N)}$ and $\widetilde{u}^{(N)}$ have the same law on $\mathcal{B}(Z_T)$, we infer that

$$\mathcal{L}(\widetilde{u}^{(N)})\big(C^0([0,T];H_N)\big) = 1 \quad \text{for all } N \in \mathbb{N}.$$

Observe that $\widetilde{u}$ is a Z_T-Borel random variable since $\mathcal{B}(Z_T \times C^0([0,T];\mathbb{R}^n))$ is a subset of $\mathcal{B}(Z_T) \times \mathcal{B}(C^0([0,T];\mathbb{R}^n))$. Furthermore, estimates (19)–(20) and the equivalence of the laws of $\widetilde{u}(N)$ and $u(N)$ on $\mathcal{B}(Z_T)$ yield for any $p \geq 1$ the following uniform estimates:

$$\begin{aligned}
\sup_{N\in\mathbb{N}} \widetilde{\mathbb{E}}\bigg(\sup_{0<t<T} \|\widetilde{u}^{(N)}(t)\|^p_{L^\infty(\mathcal{O})}\bigg) &\leq C,\\
\sup_{N\in\mathbb{N}} \widetilde{\mathbb{E}}\bigg(\int_0^T \|\widetilde{u}^{(N)}(t)\|^2_{H^1(\mathcal{O})}\,\mathrm{d}t\bigg) &\leq C.
\end{aligned}$$

We deduce the existence of a subsequence of $(\widetilde{u}^{(N)})_{N\in\mathbb{N}}$ (not relabeled) which is weakly* converging in $L^2(\widetilde{\Omega}; L^p(0,T;L^\infty(\mathcal{O})))$ and weakly converging in $L^p(\widetilde{\Omega}; L^\infty(0,T;H^1(\mathcal{O})))$ as $N \to \infty$. Since $\widetilde{u}^{(N)} \to \widetilde{u}$ in Z_T $\widetilde{\mathbb{P}}$-a.s., we conclude that $\widetilde{u} \in L^p(\widetilde{\Omega}; L^\infty(0,T;L^\infty(\mathcal{O})))$ for any $p \geq 1$ and $\widetilde{u} \in L^2(\widetilde{\Omega}; L^2(0,T;H^1(\mathcal{O})))$, i.e.

$$\widetilde{\mathbb{E}}\bigg(\sup_{0<t<T} \|\widetilde{u}(t)\|^p_{L^\infty(\mathcal{O})}\bigg) < \infty, \quad \widetilde{\mathbb{E}}\int_0^T \|\widetilde{u}(t)\|^2_{H^1(\mathcal{O})}\,\mathrm{d}t < \infty.$$

We claim that $\widetilde{u}$ is even bounded in $\overline{\mathcal{D}}$ $\widetilde{\mathbb{P}}$-a.s.

Lemma 9. *The limit $\widetilde{u}$ satisfies $\widetilde{u}(x,t) \in \overline{\mathcal{D}}$ for a.e. $(x,t) \in \mathcal{O} \times (0,T)$ $\widetilde{\mathbb{P}}$-a.s.*

Proof. By Proposition 6, $u^{(N)}(x,t) \in \overline{\mathcal{D}}$ for a.e. $(x,t) \in \mathcal{O} \times (0,T)$ $\mathbb{P}$-a.s. In particular,

$$\|u^{(N)}\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} := \sum_{i=1}^n \|u_i^{(N)}\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1. \tag{23}$$

The set $L^\infty(0,T;L^\infty(\mathcal{O}))$ is continuously embedded in $L^\infty(0,T;H^3(\mathcal{O})') \cap L^2(0,T;L^2(\mathcal{O}))$. Thus, by the Kuratowski theorem (see Theorem 13 in the Appendix), $L^\infty(0,T;L^\infty(\mathcal{O}))$ is a Borel set of $L^\infty(0,T;H^3(\mathcal{O})') \cap L^2(0,T;L^2(\mathcal{O}))$ and, in fact, also of $C^0([0,T];H^3(\mathcal{O})') \cap L^2(0,T;L^2(\mathcal{O}))$ (since the norms are the same). By [8, Lemma B.1], the set $L^\infty(0,T;L^\infty(\mathcal{O})) \cap Z_T$ is a Borel subset of Z_T. The equivalence of the laws of $\widetilde{u}^{(N)}$ and $u^{(N)}$ on $\mathcal{B}(Z_T)$ as well as (23) then show that

$$\widetilde{\mathbb{P}}\big\{\|\widetilde{u}^{(N)}\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1\big\} = \mathbb{P}\big\{\|u^{(N)}\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1\big\} = 1.$$

By the definition of the norm in (23), this means that

$$\sum_{i=1}^n |\widetilde{u}_i^{(N)}(x,t)| \le 1 \quad \text{for a.e. } (x,t) \in \mathcal{O} \times (0,T) \ \widetilde{\mathbb{P}}\text{-a.s.} \tag{24}$$

Next, we show that $\widetilde{u}_i^{(N)}(x,t) \ge 0$ for a.e. $(x,t) \in \mathcal{O} \times (0,T)$ $\widetilde{\mathbb{P}}$-a.s. Let $v \in L^\infty(0,T;L^\infty(\mathcal{O}))$ and define the closed unit ball

$$B(v) = \big\{u \in L^\infty(0,T;L^\infty(\mathcal{O})) : \|u - v\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1\big\}.$$

We deduce from (24) that

$$\widetilde{\mathbb{P}}\big(\widetilde{u}_i^{(N)} \in B(0)\big) = 1, \quad i = 1, \dots, n.$$

Since $0 \le u_i^{(N)}(x,t) \le 1$ a.e. in $\mathcal{O} \times (0,T)$ $\mathbb{P}$-a.s., we have $\|u_i^{(N)} - 1\|_{L^\infty(0,T;L^\infty(\mathcal{O}))} \le 1$ for all $i = 1, \dots, n$ and consequently, by the equivalence of the laws,

$$\widetilde{\mathbb{P}}\big(\widetilde{u}_i^{(N)} \in B(1)\big) = \mathbb{P}\big(u_i^{(N)} \in B(1)\big) = 1.$$

We infer that

$$\widetilde{\mathbb{P}}\big(\widetilde{u}_i^{(N)} \in B(1) \cap B(0)\big) = 1,$$

and this implies that $0 \le \widetilde{u}_i^{(N)}(x,t) \le 1$ $\widetilde{\mathbb{P}}$-a.s. and, taking into account (24), $\sum_{i=1}^n \widetilde{u}_i^{(N)}(x,t) \le 1$, i.e. $\widetilde{u}^{(N)}(x,t) \in \overline{\mathcal{D}}$ $\widetilde{\mathbb{P}}$-a.s. Moreover, from (22) we know that $\widetilde{u}^{(N)}$ converges to $\widetilde{u}$ strongly in $L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. and thus we conclude that $\widetilde{u}(x,t) \in \overline{\mathcal{D}}$ for a.e. $(x,t) \in \mathcal{O} \times (0,T)$ $\widetilde{\mathbb{P}}$-a.s. □

We denote by $\widetilde{\mathbb{F}}$ and $\widetilde{\mathbb{F}}^{(N)}$ the filtrations generated by $(\widetilde{u}, \widetilde{W})$ and $(\widetilde{u}^{(N)}, \widetilde{W}^{(N)})$, respectively. Lemmas 14–15 in [17] imply that $\widetilde{u}$ is progressively measurable with respect to $\widetilde{\mathbb{F}}$ and that $\widetilde{u}^{(N)}$ is progressively measurable with respect to $\widetilde{\mathbb{F}}^{(N)}$.

The following lemma is needed to prove that $(\widetilde{u}, \widetilde{W})$ is a martingale solution to (1)–(2).

Lemma 10. *It holds for all s, $t \in [0,T]$ with $s \le t$ and all $\phi_1 \in L^2(\mathcal{O})$ and $\phi_2 \in H^3(\mathcal{O})$ satisfying $\nabla\phi_2 \cdot \nu = 0$ on $\partial\mathcal{O}$ that*

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \big\langle \widetilde{u}_i^{(N)}(t) - \widetilde{u}_i(t), \phi_1 \big\rangle^2 \,\mathrm{d}t = 0, \tag{25}$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \big\langle \widetilde{u}_i^{(N)}(0) - \widetilde{u}_i(0), \phi_1 \big\rangle^2 = 0, \tag{26}$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \bigg| \sum_{j=1}^n \int_0^t \Big\langle A_{ij}(\widetilde{u}^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s) - A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s), \nabla\phi_2 \Big\rangle \,\mathrm{d}s \bigg| \,\mathrm{d}t = 0, \tag{27}$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \bigg| \int_0^t \Big\langle \mathcal{T}_i(\widetilde{u}^{(N)}(s)) - \mathcal{T}_i(\widetilde{u}(s)), \phi_1 \Big\rangle \,\mathrm{d}s \bigg| \,\mathrm{d}t = 0, \tag{28}$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \bigg| \sum_{j=1}^n \int_0^t \Big\langle \sigma_{ij}(\widetilde{u}^{(N)}(s))\mathrm{d}\widetilde{W}_j^{(N)}(s) - \sigma_{ij}(\widetilde{u}(s))\mathrm{d}\widetilde{W}_j(s), \phi_1 \Big\rangle \bigg|^2 \,\mathrm{d}t = 0. \tag{29}$$

Proof. The convergences (25) and (26) can be shown as in the proof of [17, Lemma 16]. The convergence (27) follows from the Lipschitz continuity of A_{ij} in the bounded domain $\overline{\mathcal{D}}$:

$$\begin{aligned}
&\bigg| \int_0^t \Big\langle A_{ij}(\widetilde{u}^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s) - A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s), \nabla\phi_2 \Big\rangle \,\mathrm{d}s \bigg| \\
&\quad \le \int_0^t \|A_{ij}(\widetilde{u}^{(N)}(s)) - A_{ij}(\widetilde{u}(s))\|_{L^2(\mathcal{O})} \|\nabla\widetilde{u}_j^{(N)}(s)\|_{L^2(\mathcal{O})} \|\nabla\phi_2\|_{L^\infty(\mathcal{O})} \,\mathrm{d}s \\
&\qquad + \bigg| \int_0^t A_{ij}(\widetilde{u}(s))\nabla(\widetilde{u}^{(N)}(s) - \widetilde{u}(s)) \cdot \nabla\phi_2 \,\mathrm{d}s \bigg|.
\end{aligned}$$

Since $(\widetilde{u}^{(N)})$ is bounded in $L^\infty(0,T;L^\infty(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. and the function $u \mapsto A_{ij}(u)$ is Lipschitz continuous on bounded sets, the strong L^2 convergence of $(\widetilde{u}^{(N)})$ implies that $A_{ij}(\widetilde{u}^{(N)}) \to A_{ij}(\widetilde{u})$ strongly in $L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. Therefore, the first term on the right-hand side converges to zero. We deduce from the weak convergence $\nabla\widetilde{u}^{(N)} \to \nabla\widetilde{u}$ weakly in $L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. that also the second term on the right-hand side converges to zero. This shows that

$$\lim_{N\to\infty} \int_0^t \big\langle A_{ij}(\widetilde{u}^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s), \nabla\phi_2 \big\rangle \,\mathrm{d}s = \int_0^t \big\langle A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s), \nabla\phi_2 \big\rangle \,\mathrm{d}s \quad \widetilde{\mathbb{P}}\text{-a.s.} \tag{30}$$

for all $\phi_2 \in H^3(\mathcal{O})$ satisfying $\nabla\phi_2 \cdot \nu = 0$ on $\partial\Omega$. We compute

$$\begin{aligned}
&\widetilde{\mathbb{E}} \bigg| \int_0^t \big\langle A_{ij}(\widetilde{u}^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s), \nabla\phi_2 \big\rangle \,\mathrm{d}s \bigg|^{3/2} \\
&\quad \le \|\nabla\phi_2\|_{L^\infty(\mathcal{O})}^{3/2} \widetilde{\mathbb{E}} \bigg| \int_0^t \big(1 + \|\widetilde{u}^{(N)}(s)\|_{L^2(\mathcal{O})}\big) \|\nabla\widetilde{u}^{(N)}(s)\|_{L^2(\mathcal{O})} \,\mathrm{d}s \bigg|^{3/2}
\end{aligned}$$

$$\leq C\|\phi_2\|_{H^3(\mathcal{O})}^{3/2}T^{3/4}\widetilde{\mathbb{E}}\bigg\{\big(1+\|\widetilde{u}^{(N)}(s)\|_{L^\infty(0,T;L^2(\mathcal{O}))}\big)^{3/2}\bigg(\int_0^T\|\nabla\widetilde{u}^{(N)}(s)\|_{L^2(\mathcal{O})}^2\,\mathrm{d}s\bigg)^{3/4}\bigg\}$$
$$\leq C\|\phi_2\|_{H^3(\mathcal{O})}^{3/2}T^{3/4}\Big(\widetilde{\mathbb{E}}\big(1+\|\widetilde{u}^{(N)}\|_{L^\infty(0,T;L^2(\mathcal{O}))}^6\big)\Big)^{1/4}\Big(\widetilde{\mathbb{E}}\|\widetilde{u}^{(N)}\|_{L^2(0,T;H^1(\mathcal{O}))}^2\Big)^{3/4}\leq C.$$

This bound and the $\widetilde{\mathbb{P}}$-a.s. convergence (30) allow us to apply the Vitali convergence theorem to infer that (27) holds.

Analogous arguments lead to the convergence $\mathcal{T}_i(\widetilde{u}^{(N)})\to\mathcal{T}_i(\widetilde{u})$ strongly in $L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. (since $\partial\sigma/\partial u_k$ is bounded). Moreover, for $\phi_1\in L^2(\mathcal{O})$,

$$\widetilde{\mathbb{E}}\bigg|\int_0^t\big\langle\mathcal{T}_i(\widetilde{u}^{(N)}(s)),\phi_1\big\rangle\,\mathrm{d}s\bigg|^2\leq\|\phi_1\|_{L^2(\mathcal{O})}^2\widetilde{\mathbb{E}}\bigg|\int_0^t\|\mathcal{T}_i(\widetilde{u}^{(N)}(s))\|_{L^2(\mathcal{O})}\,\mathrm{d}s\bigg|^2$$
$$\leq C\|\phi_1\|_{L^2(\mathcal{O})}^2T\widetilde{\mathbb{E}}\big(1+\|\widetilde{u}^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))}^2\big)\leq C,$$

and Vitali's convergence theorem implies that (28) holds.

It remains to prove convergence (29). Since $\widetilde{W}^{(N)}\to\widetilde{W}$ in $C^0([0,T];\mathbb{R}^n)$, it is sufficient to show that $\sigma_{ij}(\widetilde{u}^{(N)})\to\sigma_{ij}(\widetilde{u})$ in $L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. We estimate for $\phi_1\in L^2(\mathcal{O})$,

$$\int_0^t\big|\big\langle\sigma_{ij}(\widetilde{u}^{(N)}(s))-\sigma_{ij}(\widetilde{u}(s)),\phi_1\big\rangle\big|^2\,\mathrm{d}s$$
$$\leq\int_0^t\big\|\sigma_{ij}(\widetilde{u}^{(N)}(s))-\sigma_{ij}(\widetilde{u}(s))\big\|_{L^2(\mathcal{O})}^2\|\phi_1\|_{L^2(\mathcal{O})}^2\,\mathrm{d}s$$
$$\leq C\|\widetilde{u}^{(N)}(s)-\widetilde{u}(s)\|_{L^2(0,T;L^2(\mathcal{O}))}^2\|\phi_1\|_{L^2(\mathcal{O})}^2.$$

Then, by the strong L^2 convergence $\widetilde{\mathbb{P}}$-a.s. of $(\widetilde{u}^{(N)})$,

$$\lim_{N\to\infty}\int_0^t\big|\big\langle\sigma_{ij}(\widetilde{u}^{(N)}(s))-\sigma_{ij}(\widetilde{u}(s)),\phi_1\big\rangle\big|^2\,\mathrm{d}s=0.$$

Furthermore,

$$\widetilde{\mathbb{E}}\bigg|\int_0^t\big|\big\langle\sigma_{ij}(\widetilde{u}^{(N)}(s))-\sigma_{ij}(\widetilde{u}(s)),\phi_1\big\rangle\big|^2\,\mathrm{d}s\bigg|^2$$
$$\leq C\|\phi_1\|_{L^2(\mathcal{O})}^4\widetilde{\mathbb{E}}\int_0^t\big(\|\sigma_{ij}(\widetilde{u}^{(N)}(s))\|_{L^2(\mathcal{O})}^4+\|\sigma_{ij}(\widetilde{u}(s)\|_{L^2(\mathcal{O})}^4\big)\,\mathrm{d}s$$
$$\leq CT\|\phi_1\|_{L^2(\mathcal{O})}^4\widetilde{E}\bigg(\sup_{0<s<T}\|\widetilde{u}^{(N)}(s)\|_{L^2(\mathcal{O})}^4+\sup_{0<s<T}\|\widetilde{u}(s)\|_{L^2(\mathcal{O})}^4\bigg)\leq C.$$

In view of Vitali's convergence theorem, we deduce from this bound and the previous convergence that

$$\lim_{N\to\infty}\widetilde{\mathbb{E}}\int_0^t\big|\big\langle\sigma_{ij}(\widetilde{u}^{(N)}(s))-\sigma_{ij}(\widetilde{u}(s)),\phi_1\big\rangle\big|^2\,\mathrm{d}s=0.$$

We deduce from the Itô isometry that

$$\lim_{N\to\infty}\widetilde{\mathbb{E}}\left|\left\langle\int_0^t\big(\sigma_{ij}(\widetilde{u}^{(N)}(s)-\sigma_{ij}(\widetilde{u}(s))\big)\mathrm{d}\widetilde{W}_j(s),\phi_1\right\rangle\right|^2=0, \tag{31}$$

and we can estimate as

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left|\left\langle\int_0^t\big(\sigma_{ij}(\widetilde{u}^{(N)}(s))-\sigma_{ij}(\widetilde{u}(s))\big)\mathrm{d}\widetilde{W}_j(s),\phi_1\right\rangle\right|^2\\
&\quad=\widetilde{\mathbb{E}}\int_0^t\left|\langle\sigma_{ij}(\widetilde{u}^{(N)}(s))-\sigma_{ij}(\widetilde{u}(s)),\phi_1\rangle\right|^2\mathrm{d}s\\
&\quad\le\|\phi_1\|_{L^2(\mathcal{O})}^2\widetilde{\mathbb{E}}\int_0^t\|\sigma_{ij}(\widetilde{u}^{(N)}(s))-\sigma_{ij}(\widetilde{u}(s))\|_{L^2(\mathcal{O})}^2\,\mathrm{d}s\\
&\quad\le CT\|\phi_1\|_{L^2(\mathcal{O})}^2\widetilde{\mathbb{E}}\left(\sup_{0<s<T}\|\widetilde{u}^{(N)}(s)\|_{L^2(\mathcal{O})}^2+\sup_{0<s<T}\|\widetilde{u}(s)\|_{L^2(\mathcal{O})}^2\right)\le C.
\end{aligned}$$

This bound and convergence (31) allow us to apply the dominated convergence theorem to conclude that for any $\phi_1\in L^2(\mathcal{O})$,

$$\lim_{N\to\infty}\widetilde{\mathbb{E}}\int_0^T\left|\left\langle\int_0^t\big(\sigma_{ij}(\widetilde{u}^{(N)}(s))-\sigma_{ij}(\widetilde{u}(s))\big)\mathrm{d}\widetilde{W}_j(s),\phi_1\right\rangle\right|^2\mathrm{d}t=0.$$

This shows (29) and finishes the proof. □

We define

$$\begin{aligned}
\Lambda_i^{(N)}\big(\widetilde{u}^{(N)},\widetilde{W}^{(N)},\phi\big)(t)&:=\big\langle\Pi_N(\widetilde{u}_i(0)),\phi\big\rangle\\
&\quad-\sum_{j=1}^n\int_0^t\big\langle A_{ij}(\widetilde{u}^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s),\nabla\phi\big\rangle\,\mathrm{d}s\\
&\quad+\frac{1}{2}\int_0^t\big\langle\Pi_N\mathcal{T}_i(\widetilde{u}^{(N)}(s)),\phi\big\rangle\,\mathrm{d}s+\sum_{j=1}^n\left\langle\int_0^t\Pi_N\sigma_{ij}(\widetilde{u}^{(N)}(s))\mathrm{d}\widetilde{W}_j^{(N)}(s),\phi\right\rangle,\\
\Lambda_i\big(\widetilde{u},\widetilde{W},\phi\big)(t)&:=\langle\widetilde{u}_i(0),\phi\rangle-\sum_{j=1}^n\int_0^t\big\langle A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s),\nabla\phi\big\rangle\,\mathrm{d}s\\
&\quad+\frac{1}{2}\int_0^t\big\langle\mathcal{T}_i(\widetilde{u}(s)),\phi\big\rangle\,\mathrm{d}s+\sum_{j=1}^n\left\langle\int_0^t\sigma_{ij}(\widetilde{u}(s))\mathrm{d}\widetilde{W}_j(s),\phi\right\rangle,
\end{aligned}$$

for $t\in[0,T]$ and $i=1,\dots,n$. The following corollary is essentially a consequence of Lemma 10; see [17, Corollary 17] for a proof.

Corollary 11. *It holds for any $\phi_1\in L^2(\mathcal{O})$ and any $\phi_2\in H^3(\mathcal{O})$ satisfying $\nabla\phi_2\cdot\nu=0$ on $\partial\mathcal{O}$ that*

$$\lim_{N\to\infty}\big\|\langle\widetilde{u}_i^{(N)},\phi_1\rangle-\langle\widetilde{u}_i,\phi_1\rangle\big\|_{L^2(\tilde{\Omega}\times(0,T))}=0,$$

$$\lim_{N\to\infty} \big\|\Lambda_i^{(N)}\big(\widetilde{u}^{(N)},\widetilde{W}^{(N)},\phi_2\big)-\Lambda_i\big(\widetilde{u},\widetilde{W},\phi_2\big)\big\|_{L^1(\widetilde{\Omega}\times(0,T))}=0.$$

With these preparations, we can finish the proof of Theorem 3. Indeed, since $u^{(N)}$ is a strong solution to (6)–(7), it satisfies the identity

$$\langle u_i^{(N)}(t),\phi\rangle=\Lambda_i^{(N)}(u^{(N)},W,\phi)(t)\quad \mathbb{P}\text{-a.s.}$$

for a.e. $t\in[0,T]$, $i=1,\dots,n$, and $\phi\in H^1(\mathcal{O})$. In particular, it follows that

$$\int_0^T \mathbb{E}\big|\langle u_i^{(N)}(t),\phi\rangle-\Lambda_i^{(N)}(u^{(N)},W,\phi)(t)\big|\,\mathrm{d}t=0,\quad i=1,\dots,n.$$

Moreover, since the laws $\mathcal{L}(u^{(N)},W)$ and $\mathcal{L}(\widetilde{u}^{(N)},\widetilde{W}^{(N)})$ coincide,

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle \widetilde{u}_i^{(N)}(t),\phi\rangle-\Lambda_i^{(N)}(\widetilde{u}^{(N)},\widetilde{W}^{(N)},\phi)(t)\big|\,\mathrm{d}t=0,\quad i=1,\dots,n.$$

We deduce from Corollary 11 that in the limit $N\to\infty$, this equation becomes

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle \widetilde{u}_i(t),\phi\rangle-\Lambda_i(\widetilde{u},\widetilde{W},\phi)(t)\big|\,\mathrm{d}t=0,\quad i=1,\dots,n.$$

This identity holds for all $\phi\in H^3(\mathcal{O})$ satisfying $\nabla\phi\cdot\nu=0$ on $\partial\mathcal{O}$ and, by density, also for all $\phi\in H^1(\mathcal{O})$. Hence, for a.e. $t\in[0,T]$ and $\widetilde{\mathbb{P}}$-a.s.,

$$\big|\langle \widetilde{u}_i(t),\phi\rangle-\Lambda_i(\widetilde{u},\widetilde{W},\phi)(t)\big|=0,\quad i=1,\dots,n.$$

The definition of Λ_i implies that for a.e. $t\in[0,T]$ $\widetilde{\mathbb{P}}$-a.s. and for all $\phi\in H^1(\mathcal{O})$,

$$\begin{aligned}\langle \widetilde{u}_i(t),\phi\rangle=\langle \widetilde{u}_i(0),\phi\rangle&-\sum_{j=1}^n\int_0^t\big\langle A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s),\nabla\phi\big\rangle\,\mathrm{d}s\\&+\frac12\int_0^t\langle\mathcal{T}_i(\widetilde{u}(s)),\phi\rangle\,\mathrm{d}s+\sum_{j=1}^n\Big\langle\int_0^t\sigma_{ij}(\widetilde{u}(s))\mathrm{d}\widetilde{W}_j(s),\phi\Big\rangle.\end{aligned}$$

Setting $\widetilde{U}=(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{F}},\widetilde{\mathbb{P}})$, we deduce that $(\widetilde{U},\widetilde{u},\widetilde{W})$ is a martingale solution to (1)–(2), and the stochastic process $\widetilde{u}$ satisfies the estimates

$$\widetilde{\mathbb{E}}\int_0^T\|\widetilde{u}(t)\|_{H^1(\mathcal{O})}^2\,\mathrm{d}t<\infty,\quad \widetilde{\mathbb{E}}\bigg(\sup_{0<t<T}\|\widetilde{u}(t)\|_{L^\infty(\mathcal{O})}^p\bigg)<\infty\quad\text{for } p<\infty.$$

3. Examples

We present two examples that fulfill Assumptions (A3)–(A7).

3.1. **Maxwell–Stefan systems.** Maxwell–Stefan equations describe the dynamics of fluid mixtures in the diffusion regime. Applications include membrane electrolysis processes [29], ion transport through nanopores [4], and dynamics of lithium-ion batteries [42]. Here, we consider an uncharged three-species mixture with the concentrations u_1, u_2 and the solvent concentration $u_3 = 1 - u_1 - u_2$. The diffusion matrix is given by

$$A(u) = \frac{1}{a(u)}\begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix},$$
$$\text{where } a(u) = d_0 d_1 u_1 + d_0 d_2 u_2 + d_1 d_2 u_3,$$

and $d_i > 0$ for $i = 0, 1, 2$ are diffusion coefficients [34, Section 4.1]. The matrix $A(u)$ is Lipschitz continuous on $\overline{\mathcal{D}}$ since $a(u)$ is strictly positive and bounded from above (Assumption (A3)). The entropy density is given by

$$h(u) = \sum_{i=1}^{3} \big(u_i(\log u_i - 1) + 1\big).$$

Its derivative $w = h'(u) = (\log(u_1/u_3), \log(u_2/u_3))^\top$ can be explicitly inverted on $\mathcal{D}$:

$$u_i = \frac{e^{w_i}}{1 + e^{w_1} + e^{w_2}}, \quad i = 1, 2.$$

Moreover, there exists $c > 0$ such that for $z \in \mathbb{R}^n$,

$$z^\top h''(u)A(u)z = \frac{d_2 z_1^2}{u_1 a(u)} + \frac{d_1 z_2^2}{u_2 a(u)} + \frac{d_0(z_1 + z_2)^2}{u_3 a(u)} \geq c\left(\frac{z_1^2}{u_1} + \frac{z_2^2}{u_2}\right).$$

Thus, Assumption (A5) is satisfied with $m = 1/2$.

We choose the multiplicative noise

$$\sigma(u) = \begin{pmatrix} u_1 u_3 & 0 \\ 0 & u_2 u_3 \end{pmatrix},$$

where we recall that $u_3 = 1 - u_1 - u_2$. This noise term guarantees that the solutions stay in the Gibbs simplex a.s. Similar terms are well-known in stochastic reaction-diffusion equations; see, e.g. [36, (8)]. Then the expressions

$$\left|\frac{\partial h}{\partial u_i}(u)\sigma_{ii}(u)\right| = \left|u_i u_3 \log\frac{u_i}{u_3}\right|,$$
$$\left|\sigma_{ii}(u)\frac{\partial \sigma_{ii}}{\partial u_i}(u)\frac{\partial h}{\partial u_i}(u)\right| = \left|u_i u_3 (u_3 - u_i)\log\frac{u_i}{u_3}\right|,$$
$$\left|\sigma_{ii}(u)\frac{\partial^2 h}{\partial u_i^2}(u)\sigma_{ii}(u)\right| = u_i u_3 (u_i + u_3), \quad i = 1, 2,$$

are bounded for $u \in \overline{\mathcal{D}}$, proving Assumption (A6). It remains to verify Assumption (A7). To simplify the notation, we set $u^\delta = (u_1^\delta, u_2^\delta)$ with $u_i^\delta := [u_i]_\delta$. and $u_3^\delta = 1 - u_1^\delta - u_2^\delta$. We

compute the elements M_{ij}^δ of the matrix $h''(u^\delta)A(u)$:

$$\begin{aligned}
M_{11}^\delta &= \frac{1}{a(u)}\bigg(\frac{d_2}{u_1^\delta} + (d_0-d_2)\bigg(\frac{u_1}{u_1^\delta} - \frac{u_3}{u_3^\delta}\bigg) + \frac{d_0}{u_3^\delta}\bigg),\\
M_{12}^\delta &= \frac{1}{a(u)}\bigg((d_0-d_1)\bigg(\frac{u_1}{u_1^\delta} - \frac{u_3}{u_3^\delta}\bigg) + \frac{d_0}{u_3^\delta}\bigg),\\
M_{21}^\delta &= \frac{1}{a(u)}\bigg((d_0-d_2)\bigg(\frac{u_2}{u_2^\delta} - \frac{u_3}{u_3^\delta}\bigg) + \frac{d_0}{u_3^\delta}\bigg),\\
M_{22}^\delta &= \frac{1}{a(u)}\bigg(\frac{d_2}{u_1^\delta} + (d_0-d_1)\bigg(\frac{u_2}{u_2^\delta} - \frac{u_3}{u_3^\delta}\bigg) + \frac{d_0}{u_3^\delta}\bigg).
\end{aligned}$$

It holds for $z \in \mathbb{R}^n$ that

$$z^\top h''(u^\delta)A(u)z - c_h\sum_{i=1}^2 \frac{z_i^2}{u_i^\delta} \ge z^\top R^\delta(u)z,$$

where $c_h = \min\{d_0d_1, d_0d_2, d_1d_2\} > 0$ and

$$\begin{aligned}
z^\top R^\delta(u)z &= \frac{d_0-d_2}{a(u)(1+\delta)^2}\bigg(\frac{u_1}{u_1^\delta} - \frac{u_3}{u_3^\delta}\bigg)z_1^2 + \frac{d_0-d_1}{a(u)(1+\delta)^2}\bigg(\frac{u_1}{u_1^\delta} - \frac{u_3}{u_3^\delta}\bigg)z_1z_2\\
&\quad + \frac{d_0-d_2}{a(u)(1+\delta)^2}\bigg(\frac{u_2}{u_2^\delta} - \frac{u_3}{u_3^\delta}\bigg)z_1z_2 + \frac{d_0-d_1}{a(u)(1+\delta)^2}\bigg(\frac{u_2}{u_2^\delta} - \frac{u_3}{u_3^\delta}\bigg)z_2^2.
\end{aligned}$$

Since u_i/u_i^δ is bounded for $u \in \overline{\mathcal{D}}$ and $i = 1,2,3$, it follows that $R_\delta(u) \to 0$ as $\delta \to 0$ uniformly in $u \in \overline{\mathcal{D}}$. We infer that Assumption (A7) is fulfilled.

3.2. **Biofilm model.** Consider a fluid mixture consisting of n concentrations $u_1, \dots, u_n$ and the solvent concentration u_{n+1} such that $\sum_{i=1}^{n+1} u_i = 1$. We suppose that the concentrations are driven by the partial pressures $p_i = u_i$ $(i = 1, \dots, n)$, while the solvent has the constant partial pressure p_{n+1}. Allowing for the presence of an interphase force and neglecting inertia effects, a volume-filling cross-diffusion model with diffusion matrix $A(u)$, defined by

$$A_{ii}(u) = 1 - u_i, \quad A_{ij}(u) = -u_i \quad \text{for } i \ne j,$$

was formally derived in [34, Example 4.3] from an Euler system with linear friction force. This model can be also used to describe the dynamics of a bacterial biofilm with subpopulations $u_1, \dots, u_n$ and the volume fraction u_{n+1} of "free space", in which the biofilm can expand [14]. As in the previous example, we choose the entropy density and the noise term

$$h(u) = \sum_{i=1}^{n+1}\big(u_i(\log u_i - 1) + 1\big), \quad \sigma_{ii}(u) = u_iu_{n+1}, \quad \sigma_{ij}(u) = 0 \quad \text{for } i \ne j.$$

The previous example has shown that Assumption (A6) is satisfied. Assumption (A5) is fulfilled with $m = 1/2$ since for all $u \in \mathcal{D}$ and $z \in \mathbb{R}^n$,

$$z^\top h''(u)A(u)z = \sum_{i=1}^n \frac{z_i^2}{u_i}.$$

It remains to check Assumption (A7). For this, we compute

$$z^\top h''(u^\delta)A(u)z - \sum_{i=1}^n \frac{z_i^2}{u_i^\delta} = z^\top R_\delta(u)z, \quad \text{where}$$

$$R_\delta(u) = \sum_{i,j=1}^n \left(\frac{u_{n+1}}{u_{n+1}^\delta} - \frac{u_i}{u_i^\delta}\right) z_i z_j.$$

It holds that $R_\delta(u) \to 0$ as $\delta \to 0$ uniformly in $u \in \overline{\mathcal{D}}$.

Appendix A. Technical results

For the convenience of the reader, we recall some technical results used in this paper. Since we are working on the non-metric space Z_T, we need Jakubowski's generalization of the Skorokhod theorem in the form given in [10, Theorem C.1] (see [32] for the original theorem).

Theorem 12 (Skorokhod–Jakubowski)**.** *Let Z be a topological space such that there exists a sequence $(f_m)_{m\in\mathbb{N}}$ of continuous functions $f_m : Z \to \mathbb{R}$ that separate points of Z. Let S be the σ-algebra generated by $(f_m)_{m\in\mathbb{N}}$. Then*

(1) Every compact subset of Z is metrizable.
(2) If $(\mu_m)_{m\in\mathbb{N}}$ is a tight sequence of probability measures on (Z, S), then there exists a subsequence $(\mu_{m_k})_{k\in\mathbb{N}}$, a probability space $(\widetilde{\Omega}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}})$, and Z-valued Borel measurable random variables ξ_k and ξ such that (i) *μ_{m_k} is the law of ξ_k and* (ii) *$\xi_k \to \xi$ almost surely on $\widetilde{\Omega}$.*

The following result is proved in [39] (also see [8, Theorem B2]).

Theorem 13 (Kuratowski)**.** *Let X be a separable complete metric space, Y a Borel set of X, and $f : Y \to X$ a one-to-one Borel measurable mapping. Then for any Borel set $B \subset Y$, the image $f(B)$ is a Borel set.*

The Wong–Zakai approximations converge to the Wiener process. This was proved in [48] in the one-dimensional case, extended in [45] to higher dimensions, and unified in [31, Chapter 6, Theorem 7.2].

Theorem 14 (Convergence of Wong–Zakai approximations)**.** *Let $X^{(\eta)}$ be the solutions to the family of ODEs, indexed by the random variable $\omega \in \Omega$, on a finite-dimensional vector space H,*

$$\mathrm{d}X^{(\eta)}(t) = a(X^{(\eta)}(t), t)\,\mathrm{d}t + b(X^{(\eta)}(t), t)\,\mathrm{d}W^{(\eta)}(t),\ t \in [0, T], \quad X^{(\eta)}(0) = X^0,$$

where $W^{(\eta)}$ are the Wong–Zakai approximations (10) *of a Wiener process with time step $\eta > 0$; $a(X,\cdot)$, $b(X,\cdot)$, $(\partial b/\partial t)(X,\cdot)$, and $(\partial b/\partial X)(X,\cdot)$ are continuous; and $a(\cdot,t)$, $b(\cdot,t)$, and $(\partial b/\partial X)(\cdot,t)$ are Lipschitz continuous (and consequently grow at most linearly). Furthermore, let X be a solution to the Stratonovich stochastic differential equation*

$$\mathrm{d}X(t) = a(X(t),t)\,\mathrm{d}t + b(X(t),t) \circ \mathrm{d}W(t),\ t \in [0,T], \quad X(0) = X^0.$$

Then

$$\lim_{\eta\to 0} \mathbb{E}\bigg(\sup_{0<t<T} \|X^{(\eta)}(t) - X(t)\|_H^2 \bigg) = 0.$$

References

[1] H. Amann. Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems. *Diff. Integral Eqs.* 3 (1990), 13–75.

[2] I. Bailleul, A. Debussche, and M. Hofmanová. Quasilinear generalized parabolic Anderson model equation. *Stoch. Partial Diff. Eqs. Anal. Comput.* 7 (2019), 40–63.

[3] V. Barbu, G. Da Prato and M. Röckner. Existence of strong solutions for stochastic porous media equation under general monotonicity conditions. *Ann. Probab.* 37 (2009), 428–452.

[4] S. Bhatia, M. Bonilla, and D. Nicholson. Molecular transport in nanopores: a theoretical perspective. *Phys. Chem. Chem. Phys.* 13 (2011), 15350–15383.

[5] D. Bothe. On the Maxwell–Stefan equations to multicomponent diffusion. In: J. Escher et al. (eds). *Parabolic Problems. Progr. Nonlinear Diff. Eqs. Appl.* 80, pp. 81–93. Springer, Basel, 2011.

[6] D. Breit, E. Feireisl, and M. Hofmanová. *Stochastically Forced Compressible Fluid Flows.* De Gruyter, Berlin, 2018.

[7] M. Bruna and J. Chapman. Diffusion of finite-size particles in confined geometries. *Bull. Math. Biol.* 76 (2014), 947–982.

[8] Z. Brzeźniak and G. Dhariwal. Stochastic tamed Navier–Stokes equations on $\mathbb{R}^3$: existence, uniqueness of solution and existence of an invariant measure. Submitted for publication, 2019. arXiv:1904.13295.

[9] Z. Brzeźniak and E. Motyl. The existence of martingale solutions to the stochastic Boussinesq equations. *Global Stoch. Anal.* 1 (2014), 175–216.

[10] Z. Brzeźniak and M. Ondreját. Stochastic wave equations with values in Riemannian manifolds. *Stochastic partial differential equations and applications, Quad. Mat.* 25 (2010), 65–97.

[11] M. Burger, M. Di Francesco, J.-F. Pietschmann, and B. Schlake. Nonlinear cross-diffusion with size exclusion. *SIAM J. Math. Anal.* 42 (2010), 2842–2871.

[12] K. Dareiotis, M. Gerencsér, and B. Gess. Entropy solutions for stochastic porous media equations. *J. Diff. Eqs.* 266 (2019), 3732–3763.

[13] E. Daus, J.-P. Milišić, and N. Zamponi. Analysis of a degenerate and singular volume-filling cross-diffusion system modeling biofilm growth. *SIAM J. Math. Anal.* 51 (2019), 3569–3605.

[14] F. Davidson. Personal communication, 2018.

[15] A. Debussche, S. de Moor, and M. Hofmanová. A regularity result for quasilinear stochastic partial differential equations of parabolic type. *SIAM J. Math. Anal.* 47 (2015), 1590–1614.

[16] A. Debussche, M. Hofmanová, and J. Vovelle. Degenerate parabolic stochastic partial differential equations: Quasilinear case. *Ann. Probab.* 44 (2016), 1916–1955.

[17] G. Dhariwal, A. Jüngel, and N. Zamponi. Global martingale solutions for a stochastic population cross-diffusion system. *Stoch. Process. Appl.* 129 (2019), 3792–3820.

[18] M. Dreher and A. Jüngel. Compact families of piecewise constant functions in $L^p(0,T;B)$. *Nonlinear Anal.* 75 (2012), 3072–3077.

[19] W. Dreyer, C. Guhlke, and R. Müller. Overcoming the shortcomings of the Nernst–Planck–Poisson model. *Phys. Chem. Chem. Phys.* 15 (2013), 7075–7086.

[20] L. Evans. *An Introduction to Stochastic Differential Equations*. American Mathematical Society, Providence, 2013.
[21] B. Fehrmann and B. Gess. Well-posedness of nonlinear diffusion equations with nonlinear, conservative noise. *Arch. Ration. Mech. Anal.* 233 (2019), 249–322.
[22] M. Furlan and M. Gubinelli. Paracontrolled quasilinear SPDEs. *Ann. Probab.* 47 (2019), 1096–1135.
[23] M. Gerencsér and M. Hairer. A solution theory for quasilinear singular SPDEs. *Comm. Pure Appl. Math.* 72 (2019), 1983–2005.
[24] A. Gerstenmayer and A. Jüngel. Analysis of a degenerate parabolic cross-diffusion system for ion transport. *J. Math. Anal. Appl.* 461 (2018), 523–543.
[25] B. Gess. Strong solutions for stochastic partial differential equations of gradient type. *J. Funct. Anal.* 263 (2012), 2355–2383.
[26] B. Gess and M. Hofmanová. Well-posedness and regularity for quasilinear degenerate parabolic-hyperbolic SPDE. *Ann. Probab.* 46 (2018), 2495–2544.
[27] V. Giovangigli and M. Massot. The local Cauchy problem for multicomponent flows in full vibrational non-equilibrium. *Math. Meth. Appl. Sci.* 21 (1998), 1415–1439.
[28] M. Hofmanová and T. Zhang. Quasilinear parabolic stochastic partial differential equations: existence, uniqueness. *Stoch. Process. Appl.* 127 (2017), 3354–3371.
[29] J. Hogendoorn, A. van der Veen, J. van der Stegen, J. Kuipers, and G. Versteeg. Application of the Maxwell–Stefan theory to the membrane electrolysis process. *Computers Chem. Engin.* 25 (2001), 1251–1265.
[30] L. Hornung. Semilinear and quasilinear stochastic evolution equations in Banach spaces. PhD Thesis, Karlsruhe University of Technology, 2017.
[31] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. Second edition. North-Holland, Amsterdam, 1989.
[32] A. Jakubowski. The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Teor. Veroyatnost. i Primenen.* 42 (1997), 209–216. English translation in *Theory Probab. Appl.* 42 (1998), 167–174.
[33] A. Jüngel. The boundedness-by-entropy method for cross-diffusion systems. *Nonlinearity* 28 (2015), 1963–2001.
[34] A. Jüngel. *Entropy Methods for Diffusive Partial Differential Equations*. SpringerBriefs in Mathematics, Springer, 2016.
[35] A. Jüngel and I. V. Stelzer. Existence analysis of Maxwell–Stefan systems for multicomponent mixtures. *SIAM J. Math. Anal.* 45 (2013), 2421–2440.
[36] S. Kliem. Convergence of rescaled competing species processes to a class of SPDEs. *Electron. J. Probab.* 16 (2011), 618–657.
[37] R. Kruse. *Strong and Weak Approximation of Semilinear Stochastic Evolution Equations*. Lecture Notes Math. 2093. Springer, Cham, 2014.
[38] C. Kuehn and A. Neamţu. Pathwise mild solutions for quasilinear stochastic partial differential equations. Submitted for publication, 2018. arXiv:1802.10016.
[39] K. Kuratowski. *Topology I*. Academic Press, New York, 1966.
[40] W. Liu and M. Röckner. SPDE in Hilbert space with locally monotone coefficients. *J. Funct. Anal.* 259 (2010), 2902–2922.
[41] T. Ma and R. Zhu. Wong–Zakai approximation and support theorem for SPDEs with locally monotone coefficients. *J. Math. Anal. Appl.* 469 (2019), 623–660.
[42] A. Nyman, M. Behm, and G. Lindbergh. Electrochemical characterisation and modelling of the mass transport phenomena in LiPF6-EC-EMC electrolyte. *Electrochim. Acta* 53 (2008), 6356–6365.
[43] F. Otto and H. Weber. Quasilinear SPDEs via rough paths. *Arch. Ration. Mech. Anal.* 232 (2019), 873–950.
[44] C. Prévôt and M. Röckner. *A Concise Course on Stochastic Partial Differential Equations*. Lecture Notes Math. 1905. Springer, Berlin, 2007.

[45] D. Stroock and S. Varadhan. On the support of diffusion processes with applications to the strong maximum principle. In: *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, vol. 3, pp. 333–359. University of California Press, Berkeley, 1972.

[46] K. Twardowska. An extension of Wong–Zakai theorem for stochastic evolution equations in Hilbert spaces. *Stoch. Anal. Appl.* 10 (1992), 471–500.

[47] K. Twardwowska. Wong–Zakai approximations for stochastic differential equations. *Acta Appl. Math.* 43 (1996), 317–359.

[48] E. Wong and M. Zakai. On the convergence of ordinary integrals to stochastic integrals. *Ann. Math. Statist.* 36 (1965), 1560–1564.

[49] N. Zamponi and A. Jüngel. Analysis of degenerate cross-diffusion population models with volume filling. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 34 (2017), 1–29. (Erratum: 34 (2017), 789–792.)

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