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Density of the spectrum of Jacobi matrices with power asymptotics

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Abstract: We consider Jacobi matrices J with off-diagonal $n^{\beta_1}\left(x_0 + \frac{x_1}{n} + \mathrm{O}(n^{-2})\right)$ and diagonal $n^{\beta_2}\left(y_0 + \frac{y_1}{n} + \mathrm{O}(n^{-2})\right)$. If $\beta_1 > \beta_2$, or $\beta_1 = \beta_2$ and $|y_0| \leq 2x_0$, J is of type C, and we study the upper density of its spectrum.

Keywords: Jacobi matrix, Spectral analysis, Difference equation, growth of entire function, canonical system, Berezanskiĭ's theorem

AMS MSC 2010: 47B36, 34L20, 30D15

1 Introduction

A Jacobi matrix J is a tridiagonal semi-infinite matrix

$$J = \begin{pmatrix} q_0 & \rho_0 & & \\ \rho_0 & q_1 & \rho_1 & \\ & \rho_1 & q_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

with real q_n and positive ρ_n. Each Jacobi matrix induces a closed symmetric operator T_J on $\ell^2(\mathbb{N})$, namely as the closure of the natural action of J on the subspace of finitely supported sequences, see, e.g., [1, Chapter 4.1]. There occurs an alternative: Either T_J is selfadjoint (one speaks of the limit point case, or, in the language of [1], type D), or T_J has defect index $(1,1)$ and is entire in the sense of M.G. Kreĭn (called the limit circle case, or, synonymously, type C). If J is of type D, the spectrum of T_J may be discrete, continuous, or be composed of different types. If J is of type C, the spectrum is discrete.

In general it is difficult to decide from the parameters ρ_n, q_n whether J is of type C or D. Two classical necessary conditions for type C are Carleman's condition which says that $\sum_{n=0}^{\infty} \rho_n^{-1} = \infty$ implies type D, cf. [7], and Wouk's theorem that a dominating diagonal in the sense that $\sup_{n\geq 0}(\rho_n + \rho_{n-1} - q_n) < \infty$ or $\sup_{n\geq 0}(\rho_n + \rho_{n-1} + q_n) < \infty$ implies type D, cf. [20]. A more subtle result, which gives a sufficient condition for type C, is due to Yu.M.Berezanskiĭ, cf. [2] or [3, VII,Theorem 1.5]: Assume that $\sum_{n=0}^{\infty} \rho_n^{-1} < \infty$, that $\sum_{n=0}^{\infty} \frac{|q_n|}{\rho_n} < \infty$, and that the off-diagonal parameters behave regular in the sense that $\rho_n^2 \geq \rho_{n+1}\rho_{n-1}$ (log-concavity). Then J is of type C.

There is a vast literature dealing with Jacobi matrices of type D, whose aim is to establish discreteness of the spectrum and investigate spectral asymptotics, e.g., [6, 8, 11, 12, 19]. Contrasting this, if J is of type C, not much is known about the asymptotic behaviour of the spectrum.

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The probably first result in this direction is due to M.Riesz [17] and states that ($\lambda_n^\pm$ denote the sequences of positive or negative spectral points arranged according to increasing modulus)

$$\lim_{n\to\infty} \frac{n}{\lambda_n^\pm} = 0.$$

A deeper result is contained in the already mentioned work of Yu.M.Berezanskiĭ (for an explicit formulation, see [4]). The upper density of the spectrum w.r.t. a power r^α is studied, and it is shown that under the mentioned assumptions (denoting by $n_\sigma(r)$ the number of spectral points in the interval $[-r, r]$)

$$\limsup_{r\to\infty} \frac{n_\sigma(r)}{r^\alpha} = \begin{cases} 0 & , \quad \alpha > \text{ convergence exponent of } (\rho_n)_{n=0}^\infty \\ \infty & , \quad \alpha < \text{ convergence exponent of } (\rho_n)_{n=0}^\infty \end{cases} \tag{1.1}$$

Studying the upper density of the spectrum is natural, since this quantity is accessible via the growth of the canonical product having the spectrum as its zero-set. Passing to a canonical product and applying the theory of entire functions is a common tool in the theory of operators with compact resolvents, e.g., [10]. It was applied in various instances to investigate the asymptotic behaviour of the spectrum, e.g., [9].

In this paper we study the upper density of the spectrum for Jacobi matrices J whose parameters have power asymptotics

$$\rho_n = n^{\beta_1}\left(x_0 + \frac{x_1}{n} + \mathrm{O}(n^{-2})\right), \quad q_n = n^{\beta_2}\left(y_0 + \frac{y_1}{n} + \mathrm{O}(n^{-2})\right), \tag{1.2}$$

with $x_0 > 0$, $y_0 \neq 0$. We assume $\beta_1 > 1$, $\delta := \beta_1 - \beta_2 \geq 0$, and $|y_0| \leq 2x_0$ if $\delta = 0$. These conditions are necessary for J being of type C by Carleman and Wouk.

Having (1.2) implies that $(\rho_n)_{n=0}^\infty$ is log-concave. Hence, if $\delta > 1$, Berezanskiĭ's theorem applies and yields (1.1). Observe that the convergence exponent of a sequence $(\rho_n)_{n=0}^\infty$ with (1.2) is $\frac{1}{\beta_1}$.

Our main result is the following theorem which, roughly speaking, says that (1.1) remains valid for $\delta \in (0, 1]$, and even in some cases where $\delta = 0$, i.e. where diagonal and off-diagonal parameters are comparable.

1.1 Theorem. *Let J be the Jacobi matrix with parameters ρ_n, q_n, let T be a selfadjoint extension of T_J in $\ell^2(\mathbb{N})$, and let n_σ be the counting function of the spectrum of T. Assume that ρ_n and q_n have the asymptotics* (1.2) *where $x_0 > 0$, $y_0 \neq 0$, $\beta_1 > 1$ and $\delta := \beta_1 - \beta_2 \in [0, 1]$. If $\delta = 0$, assume further that $|y_0| < 2x_0$.*

Then J is of type C, and for all $m \in \mathbb{N}$ we have

$$\limsup_{r\to\infty} \frac{n_\sigma(r)}{r^{\frac{1}{\beta_1}}\left(\ln^{[m]} r\right)^{1-\frac{1}{\beta_1}}} < +\infty, \qquad \limsup_{r\to\infty} \frac{n_\sigma(r)}{r^{\frac{1}{\beta_1}}} > 0, \tag{1.3}$$

where $\ln^{[0]} x := x$ and $\ln^{[n+1]} x := \ln(\ln^{[n]} x)$ for $n \in \mathbb{N}$.

In the situation of (1.1), it seems to be likely that $\limsup_{r\to\infty} n_\sigma(r)/r^{\frac{1}{\beta_1}} < \infty$, but I have not been able to achieve this result.

In the proof of this theorem we use the already mentioned fact that the growth of the counting function n_σ relates to the growth of the corresponding

canonical product, pass from the Jacobi operator to a unitarily equivalent model operator of a canonical system, and apply a recent theorem of R.Romanov [18] to estimate the growth of the monodromy matrix of this system. A crucial step is to establish that the power asymptotics (1.2) of the Jacobi parameters give rise to similar power asymptotics for the data determining the canonical system. To achieve this we use recent work of R.-J.Kooman [14] and a discrete Levinson type theorem.

2 Canonical systems

Let H be a 2×2-matrix valued integrable function on an interval $[0, L]$ whose values are almost everywhere real and positive semidefinite matrices. The *canonical system with Hamiltonian* H is the equation

$$y'(x) = zJH(x)y(x), \quad x \in [0, L],$$

where J is the symplectic matrix $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and z is a complex parameter. Making a suitable change of variable, one can always achieve that $\operatorname{tr} H(x) = 1$ for almost all x. The *fundamental solution* of the system is the solution of the initial value problem

$$\begin{cases} \frac{d}{dx} W(x, z)J = zW(x, z)H(x), & x \in [0, L), \\ W(0, z) = I. \end{cases}$$

In the limit circle case, i.e. if $L < \infty$, the fundamental solution exists up to L, and $W(L, z)$ is called the *monodromy matrix*.

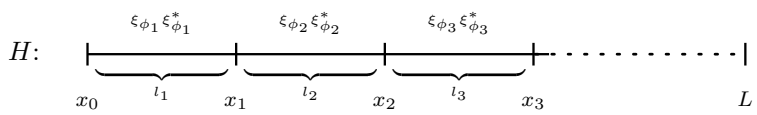
The following class of Hamiltonians plays an important role: For sequences $(l_n)_{n=1}^{\infty}$ and $(\phi_n)_{n=1}^{\infty}$ with $l_n > 0$ and $\phi_{n+1} \not\equiv \phi_n \bmod \pi$, $n \in \mathbb{N}$, we set

$$x_0 := 0, \qquad x_n := \sum_{k=1}^{n} l_k, \ n \in \mathbb{N}, \qquad L := \sum_{k=1}^{\infty} l_k \in (0, \infty],$$

and call the function $H : [0, L) \to \mathbb{R}^{2\times 2}$ defined as ($\xi_\phi := (\cos\phi, \sin\phi)^T$)

$$H(x) := \xi_{\phi_n} \xi_{\phi_n}^*, \quad x \in [x_{n-1}, x_n), \ n \in \mathbb{N},$$

the *Hamburger Hamiltonian* with *lengths* $(l_n)_{n=1}^{\infty}$ and *angles* $(\phi_n)_{n=1}^{\infty}$.



After a suitable normalisation, Hamburger Hamiltonians are in a one-to-one correspondence with Jacobi matrices, cf. [13]. Hereby, the monodromy matrix of the Hamiltonian coincides with the Nevanlinna matrix which describes all solutions of the corresponding moment problem. The lengths and angles of a Hamburger Hamiltonian are related to the parameters of the associated Jacobi matrix via

$$l_n = P_n(0)^2 + Q_n(0)^2, \ \left|\sin(\phi_{n+1} - \phi_n)\right| = \left(\rho_n \sqrt{l_n l_{n+1}}\right)^{-1}, \tag{2.1}$$

$$l_n q_n = -\cot(\phi_{n+1} - \phi_n) - \cot(\phi_n - \phi_{n-1}),$$

where $P_n(z)$ and $Q_n(z)$ denote the orthogonal polynomials of the first and second kind, respectively.

We are going to employ [18, Theorem 1] which provides an upper bound for the growth of the monodromy matrix. This theorem is based on finding appropriate approximations of a given Hamiltonian by simple ones.

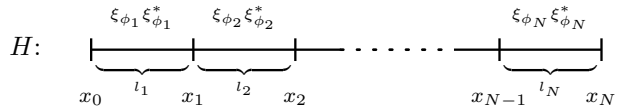
2.1 Definition. Let $N \in \mathbb{N}$, let $(l_n)_{n=1}^N$ be a finite sequence of positive numbers, and let $(\phi_n)_{n=1}^N$ be a finite sequence of real numbers with $\phi_{n+1} \not\equiv \phi_n \mod \pi$, $n = 1, \dots, N-1$. Set

$$x_0 := 0, \qquad x_n := \sum_{k=1}^{n} l_k, \ n = 1, \dots, N. \tag{2.2}$$

Then we speak of the function $H : [0, x_N) \to \mathbb{R}^{2\times 2}$ which is defined by

$$H(x) := \xi_{\phi_n} \xi_{\phi_n}^*, \quad x \in [x_{n-1}, x_n), \ n = 1, \dots, N,$$

as the *finite rank Hamiltonian with parameters* $\langle N, (l_n)_{n=1}^N, (\phi_n)_{n=1}^N \rangle$.



$\Diamond$

The next theorem is a formulation of Romanov's theorem for growth functions $\lambda(r)$ instead of powers r^a. For the definition of growth functions in general see [15, Section I.6] or [16, Section I.12]. Note here that growth functions are exponentials of proximate orders. Classical examples are functions of the form $\lambda(r) = r^a \ln^b(r)$ or, more general, $\lambda(r) = r^a (\ln^{[n]} r)^b$, and we will only use such growth functions.

2.2 Theorem ([18])**.** *Let $L \in (0, \infty)$, and let $H : [0, L) \to \mathbb{R}^{2\times 2}$ be a Hamiltonian with $\operatorname{tr} H = 1$ a.e. Let λ be a growth function, and assume that there exists a family of finite rank Hamiltonians*

$$H^\star(R), \ R > 1 \quad \Big(\textit{parameters } \big\langle N^\star(R), (l_n^\star(R))_{n=1}^{N^\star(R)}, (\phi_n^\star(R))_{n=1}^{N^\star(R)} \big\rangle\Big)$$

and a family of sequences of weights

$$\big(a_n(R)\big)_{n=1}^{N^\star(R)}, \ R > 1 \quad \textit{with} \quad a_n(R) \in (0, 1],$$

such that ($\|.\|$ denotes any matrix norm, and $x_n^\star(R)$ is defined as in (2.2) from $l_n^\star(R)$)

$$(i) \quad \sum_{n=1}^{N^\star(R)} \frac{1}{a_n^2(R)} \int_{x_{n-1}^\star(R)}^{x_n^\star(R)} \big\| H(x) - [H^\star(R)](x) \big\| \, dx = \mathrm{O}\left(\frac{\lambda(R)}{R}\right),$$

$$(ii) \quad \sum_{n=1}^{N^\star(R)} a_n^2(R) l_n^\star(R) = \mathrm{O}\left(\frac{\lambda(R)}{R}\right),$$

$$(iii)\quad \sum_{n=1}^{N^\star(R)-1} \ln\left(1+\frac{\left|\sin\left(\phi^\star_{n+1}(R)-\phi^\star_n(R)\right)\right|}{a_{n+1}(R)a_n(R)}\right) = \mathrm{O}\big(\lambda(R)\big),$$

$$(iv)\quad \big|\ln a_1(R)\big| + \big|\ln a_{N^\star(R)}(R)\big| + \sum_{n=1}^{N^\star(R)-1} \left|\ln\frac{a_{n+1}(R)}{a_n(R)}\right| = \mathrm{O}\big(\lambda(R)\big).$$

Then the entries of the monodromy matrix of H have finite λ-type.

The proof is verbatim the same as [18, Theorem 1], we skip details.

3 Proof of Theorem 1

Let a Jacobi matrix J whose parameters ρ_n and q_n have an asymptotic expansion (1.2) be given, and assume that $x_0 > 0$, $y_0 \neq 0$, $\beta_1 > 1$, $\delta \in [0,1]$, and that $|y_0| < 2x_0$ if $\delta = 0$.

In order to apply Theorem 2.2, we need knowledge about the lengths and angles of the Hamburger Hamiltonian associated with J. Since $P_n(0)$ and $Q_n(0)$ form a fundamental system of solutions of the difference equation

$$\rho_{n+1}u_{n+2} + q_{n+1}u_{n+1} + \rho_n u_n = 0, \tag{3.1}$$

we start with studying asymptotics of solutions of this equation.

Step 1: Growth of solutions, $\delta \in [0,1)$.

In the case $\delta \in [0,1)$, we begin with rewriting (3.1). Setting $r_i := \frac{-q_i}{2\rho_i}$ and dividing by $\rho_{n+1}\prod_{i=1}^{n+1} r_i$ gives

$$\frac{u_{n+2}}{\prod_{i=1}^{n+1} r_i} - 2\frac{u_{n+1}}{\prod_{i=1}^{n} r_i} + \frac{\rho_n}{\rho_{n+1}r_n r_{n+1}}\frac{u_n}{\prod_{i=1}^{n-1} r_i} = 0.$$

Introducing the new variable $v_n := u_n\left(\prod_{i=1}^{n-1} r_i\right)^{-1}$ and setting $C_n := 1 - \frac{\rho_n}{\rho_{n+1}r_n r_{n+1}}$ gives

$$v_{n+2} - 2v_{n+1} + (1-C_n)v_n = 0. \tag{3.2}$$

A computation shows

$$C_n = 1 - 4n^{2\delta}\left(z_0 + \frac{z_1}{n} + \mathrm{O}(n^{-2})\right)$$

with

$$z_0 := \left(\frac{x_0}{y_0}\right)^2, \quad z_1 := \frac{x_0}{y_0^3}\big(2(x_1y_0 - x_0y_1) - \beta_2 x_0 y_0\big).$$

Obviously, we have

$$\lim_{n\to\infty} n^{-2\delta}C_n = \begin{cases} -4z_0 & \delta \in (0,1) \\ 1-4z_0 & \delta = 0 \end{cases}.$$

Our assumptions ensure that this limit is always negative. We can apply [14, Theorem 1 (i)], since we are in the situation described in [14, p.1039, Remark]. We get two linearly independent solutions $(v_n^{(j)})_{n=1}^{\infty}$, $j = 1, 2$, of (3.2) with

$$v_n^{(1)} = \overline{v_n^{(2)}} = (1 + \mathrm{o}(1))n^{-\frac{\delta}{2}} \prod_{k=1}^{n-1} \big(1 + i\sqrt{-C_k}\big).$$

We calculate the asymptotic behaviour

$$\begin{aligned}\Big|1 + i\sqrt{-C_k}\Big| = \sqrt{1 - C_k} &= \sqrt{4k^{2\delta}\Big(z_0 + \frac{z_1}{k} + \mathrm{O}(k^{-2})\Big)} \\ &= 2\sqrt{z_0}k^{\delta}\sqrt{1 + \frac{z_1}{z_0 k} + \mathrm{O}(k^{-2})},\end{aligned}$$

and get

$$\begin{aligned}\left|\prod_{k=1}^{n-1} \big(1 + i\sqrt{-C_k}\big)\right| &= (2\sqrt{z_0})^{n-1}\big[(n-1)!\big]^{\delta}\sqrt{\prod_{k=1}^{n-1}\big(1 + \frac{z_1}{z_0 k} + \mathrm{O}(k^{-2})\big)} \\ &= \Big(\frac{2x_0}{|y_0|}\Big)^{n-1}\big[(n-1)!\big]^{\delta}(d_1 + o(1))n^{\frac{z_1}{2z_0}}\end{aligned}$$

for some $d_1 > 0$, due to [14, Lemma 4] adding a summable perturbation. Thus

$$\begin{aligned}\big|v_n^{(1)}\big| = \big|v_n^{(2)}\big| &= (d_1 + o(1))\Big(\frac{2x_0}{|y_0|}\Big)^{n-1}\big[(n-1)!\big]^{\delta} n^{\frac{z_1}{2z_0} - \frac{\delta}{2}} \\ &= (d_1 + o(1))\Big(\frac{2x_0}{|y_0|}\Big)^{n-1}\big[(n-1)!\big]^{\delta} n^{\frac{x_1}{x_0} - \frac{y_1}{y_0} - \frac{\beta_1}{2}}. \qquad (3.3)\end{aligned}$$

Substituting back via $u_n = v_n \prod_{i=1}^{n-1} r_i$ produces two solutions $(u_n^{(j)})_{n=1}^{\infty}$, $j = 1, 2$, of (3.1). At first note

$$\begin{aligned}\prod_{k=1}^{n-1} r_k = \prod_{k=1}^{n-1} \frac{-q_k}{2\rho_k} &= \prod_{k=1}^{n-1} \frac{-1}{2} k^{-\delta} \frac{y_0 + \frac{y_1}{k} + \mathrm{O}(k^{-2})}{x_0 + \frac{x_1}{k} + \mathrm{O}(k^{-2})} \\ &= \prod_{k=1}^{n-1} \frac{-1}{2} k^{-\delta}\left(\frac{y_0}{x_0} + \frac{1}{k}\frac{y_0}{x_0}\Big(\frac{y_1}{y_0} - \frac{x_1}{x_0}\Big) + \mathrm{O}(k^{-2})\right) \\ &= \Big(\frac{-y_0}{2x_0}\Big)^{n-1}\big[(n-1)!\big]^{-\delta}\prod_{k=1}^{n-1}\left(1 + \frac{1}{k}\Big(\frac{y_1}{y_0} - \frac{x_1}{x_0}\Big) + \mathrm{O}(k^{-2})\right) \\ &= \Big(\frac{-y_0}{2x_0}\Big)^{n-1}\big[(n-1)!\big]^{-\delta}(d_2 + o(1))n^{\frac{y_1}{y_0} - \frac{x_1}{x_0}}, \qquad (3.4)\end{aligned}$$

for some $d_2 \neq 0$, again by [14, Lemma 4]. Combining (3.3) and (3.4) gives the asymptotic behaviour

$$\big|u_n^{(1)}\big| = \big|u_n^{(2)}\big| = \big|v_n^{(1)}\big|\left|\prod_{k=1}^{n-1} r_k\right| = (d_3 + o(1))n^{-\frac{\beta_1}{2}}, \qquad (3.5)$$

where $d_3 = d_1\,|d_2| > 0$. In particular, J is of type C.

Step 2: Growth of solutions, $\delta = 1$.

The case $\delta = 1$ is not covered by [14, Theorem 1], but can be handled with direct computations reduced to Levinson's theorem.

Dividing (3.1) by ρ_{n+1} and shifting the index by 1 gives

$$u_{n+1} + \underbrace{\frac{q_n}{\rho_n}}_{=:a_n} u_n + \underbrace{\frac{\rho_{n-1}}{\rho_n}}_{=:b_n} u_{n-1} = 0. \tag{3.6}$$

Clearly $a_n = \frac{y_0}{x_0} n^{-1} + \mathrm{O}(n^{-2})$ and $b_n = 1 - \beta_1 n^{-1} + \mathrm{O}(n^{-2})$. By setting $\vec{u}_n = (u_n, u_{n+1})^T$ and

$$A_n := \begin{pmatrix} 0 & 1 \\ -b_n & -a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1}{n} \begin{pmatrix} 0 & 0 \\ \beta_1 & -\frac{y_0}{x_0} \end{pmatrix} + \mathrm{O}\left(n^{-2}\right),$$

we write (3.6) as the difference system

$$\vec{u}_n = A_n \vec{u}_{n-1}. \tag{3.7}$$

The idea is to diagonalise A_n modulo summable terms. To this end, we set

$$R_n := \begin{pmatrix} i^n & (-i)^n \\ i^{n+1} & (-i)^{n+1} \end{pmatrix},$$

and introduce the variable $\vec{v}_n := R_n^{-1} \vec{u}_n$. This leads to the difference system

$$\vec{v}_n = R_n^{-1} A_n R_{n-1} \vec{v}_{n-1}.$$

Set $z := \beta_1 + i \frac{y_0}{x_0}$. A calculation shows,

$$R_n^{-1} A_n R_{n-1} = \begin{pmatrix} 1 - \frac{1}{2n}\bar{z} & 0 \\ 0 & 1 - \frac{1}{2n} z \end{pmatrix} + \frac{(-1)^n}{2n} \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} + \mathrm{O}\left(n^{-2}\right).$$

In order to get rid of off-diagonal terms which are not summable we perform one more transformation:

$$S_n := I - \frac{1}{2} \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k} \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}$$

Note that S_n is invertible for sufficiently large n, say $n \geq n_0$. Then $\vec{w}_n := S_n^{-1} \vec{v}_n$ satisfies

$$\vec{w}_n = S_n^{-1} R_n^{-1} A_n R_{n-1} S_{n-1} \vec{w}_{n-1}, \tag{3.8}$$

with

$$(R_n S_n)^{-1} A_n R_{n-1} S_{n-1} = \begin{pmatrix} 1 - \frac{1}{2n}\bar{z} & 0 \\ 0 & 1 - \frac{1}{2n} z \end{pmatrix} + \mathrm{O}\left(n^{-2}\right).$$

Note that the diagonal entries are complex conjugated numbers which converge to 1. The discrete version of Levinsons's Fundamental Theorem [5, Theorem 3.4] gives a fundamental solution of (3.8) with

$$\begin{aligned} W_n &= (I + \mathrm{o}(1)) \prod_{k=n_0}^{n} \begin{pmatrix} 1 - \frac{1}{2k}\bar{z} & 0 \\ 0 & 1 - \frac{1}{2k} z \end{pmatrix} \\ &= (I + \mathrm{o}(1)) \begin{pmatrix} \prod_{k=n_0}^{n} (1 - \frac{1}{2k}\bar{z}) & 0 \\ 0 & \prod_{k=n_0}^{n} (1 - \frac{1}{2k} z) \end{pmatrix}. \end{aligned}$$

Substituting back yields that

$$R_n S_n W_n = (I + \mathrm{o}(1))\, R_n \begin{pmatrix} \prod_{k=n_0}^{n}(1-\frac{1}{2k}\bar{z}) & 0 \\ 0 & \prod_{k=n_0}^{n}(1-\frac{1}{2k}z) \end{pmatrix}$$

$$= (I + \mathrm{o}(1)) \begin{pmatrix} i^n \prod_{k=n_0}^{n}(1-\frac{1}{2k}\bar{z}) & (-i)^n \prod_{k=n_0}^{n}(1-\frac{1}{2k}z) \\ i^{n+1} \prod_{k=n_0}^{n}(1-\frac{1}{2k}\bar{z}) & (-i)^{n+1} \prod_{k=n_0}^{n}(1-\frac{1}{2k}z) \end{pmatrix}$$

is a fundamental solution of (3.7). By inspecting the first row we get two solutions $u_n^{(1)}, u_n^{(2)}$ of (3.6), or equivalently (3.1), with

$$|u_n^{(1)}|, |u_n^{(2)}| \asymp {}^{1}\left| \prod_{k=n_0}^{n} (1 - \frac{1}{2k}\bar{z}) \right| \asymp n^{-\frac{\operatorname{Re} z}{2}} = n^{-\frac{\beta_1}{2}}. \tag{3.9}$$

In particular, J is of type C.

Step 3: Conclusions concerning the spectrum

We have seen in the first and second step, cf. (3.5) and (3.9) respectively, that the difference equation (3.1) has a fundamental system of solutions $u_n^{(1)}, u_n^{(2)}$ with $|u_n^{(j)}| \asymp n^{-\frac{\beta_1}{2}}$, $j \in \{1, 2\}$.

Recall that also $P_n(0)$ and $Q_n(0)$ are linearly independent solutions of (3.1). The quotient $\big(|P_n(0)|+|Q_n(0)|\big)/n^{-\frac{\beta_1}{2}}$ is bounded above since $P_n(0)$ and $Q_n(0)$ can be written as linear combinations of $u_n^{(1)}$ and $u_n^{(2)}$. It is also bounded away from zero, since $u_n^{(1)}$ is a linear combination of $P_n(0)$ and $Q_n(0)$ and $\big|u_n^{(1)}\big|/n^{-\frac{\beta_1}{2}}$ is bounded away from zero. Thus, we obtain $P_n(0)^2 + Q_n(0)^2 \asymp n^{-\beta_1}$.

Consider the canonical system related to J. By (2.1), the lengths and angles of the corresponding Hamburger Hamiltonian H satisfy

$$l_n = P_n(0)^2 + Q_n(0)^2 \asymp n^{-\beta_1},$$
$$\big|\sin(\phi_{n+1} - \phi_n)\big| = \big(\rho_n \sqrt{l_n l_{n+1}}\big)^{-1} \asymp n^{\beta_1 - \beta_1} = 1.$$

Let $m \in \mathbb{N}$ be arbitrary. Our aim is to employ Theorem 2.2 on H with the growth function $\lambda(r) = r^{\frac{1}{\beta_1}} (\ln^{[m]} r)^{1-\frac{1}{\beta_1}}$. To this end, we take the following family of finite rank Hamiltonians $H^\star(R)$:

$$N^\star(R) := \lfloor R^{\frac{1}{\beta_1}} (\ln^{[m]} R)^{-\frac{1}{\beta_1}} \rfloor$$
$$l_n^\star(R) := l_n,\ \phi_n^\star(R) := \phi_n \qquad n = 1, \dots, N^\star(R) - 1$$

On the last interval, we take $l_{N^\star(R)}^\star := \sum_{n \geq N^\star(R)} l_n$ and $\phi_{N^\star(R)}^\star := 0$. For $k = 1, \dots, m-1$ set $N_k(R) := \lfloor R^{\frac{1}{\beta_1}} (\ln^{[k]} R)^{-1} \rfloor$ as well as $N_0(R) := 1$, $N_m(R) := N^\star(R)$, and define the weights via

$$a_n(R) := \begin{cases} (\ln^{[k]} R)^{\frac{1-\beta_1}{2}} & n = N_k, \dots, N_{k+1} - 1;\ k = 0, \dots, m-1 \\ 1 & n = N^\star(R). \end{cases}$$

[1] We write $f_n \asymp g_n$ if there are constants $c, d > 0$ s. t. $c g_n \leq f_n \leq d g_n$ for n large enough.

We need to check conditions $(i)-(iv)$. Considering the last length,

$$l^\star_{N^\star(R)} = \sum_{n=N^\star(R)}^{\infty} l_n \le C \sum_{n=N^\star(R)}^{\infty} n^{-\beta_1} = \mathrm{O}\left(N^\star(R)^{1-\beta_1}\right)$$
$$= \mathrm{O}\left(R^{\frac{1}{\beta_1}-1}\left(\ln^{[m]} R\right)^{1-\frac{1}{\beta_1}}\right) = \mathrm{O}\left(\frac{\lambda(R)}{R}\right),$$

shows (i). For $k = 1, \dots, m-1$ note

$$\sum_{n=N_k}^{N_{k+1}-1} l_n \le \sum_{n=N_k}^{\infty} n^{-\beta_1} = \mathrm{O}\big(N_k^{1-\beta_1}\big) = \mathrm{O}\big(R^{\frac{1}{\beta_1}-1}(\ln^{[k]} R)^{\beta_1-1}\big),$$

as well as $(\ln^{[0]} R)^{1-\beta_1} = R^{1-\beta_1} \le R^{\frac{1}{\beta_1}-1}$, since $2 < \beta_1 + \beta_1^{-1}$. Therefore, condition (ii) follows:

$$l^\star_{N^\star(R)} + \sum_{k=0}^{m-1}\left(\sum_{n=N_k}^{N_{k+1}-1} l_n\right)(\ln^{[k]} R)^{1-\beta_1} = \mathrm{O}\left(\frac{\lambda(R)}{R}\right).$$

Since $a_n(R)$ is increasing in n, we have

$$1 + \frac{\left|\sin\left(\phi^\star_{n+1}(R) - \phi^\star_n(R)\right)\right|}{a_{n+1}(R)a_n(R)} \le 1 + \frac{1}{a_n^2(R)} \le \frac{2}{a_n^2(R)},$$

and the left-hand side of condition (iii) can be bounded above by

$$\sum_{k=0}^{m-1}(N_{k+1} - N_k)\left(\ln(2) + \ln\left((\ln^{[k]} R)^{\beta_1-1}\right)\right) \le$$
$$\le \ln(2)N^\star(R) + (\beta_1 - 1)\sum_{k=0}^{m-1} N_{k+1}(\ln^{[k+1]} R) = \mathrm{O}\big(\lambda(R)\big).$$

Condition (iv) is fulfilled as well, for trivial reasons. Theorem 2.2 gives that all entries of the monodromy matrix have finite λ-type. By [16, Theorem 17], the upper density of the zeros of the entry B is finite, i.e.

$$\limsup_{r\to\infty} \frac{n_B(r)}{\lambda(r)} < +\infty,$$

where

$$n_B(r) = \#\{z \in \mathbb{C} : B(z) = 0, |z| < r\}.$$

The first assertion in (1.3) follows, since the spectrum of T interlaces with the zeros of B.

For the second assertion in (1.3), note that due to [4, Proposition 7.1 (ii),(iii)] the order and type of any entry of the monodromy matrix is bounded below by the order and type of the entire function $H(z)$,

$$H(z) = \sum_{n=0}^{\infty} b_{n,n} z^n,$$

where $b_{n,n} = (\rho_1\rho_2\dots\rho_{n-1})^{-1}$ denotes the leading coefficient of $P_n(z)$. The asymptotics of ρ_n yields

$$b_{n,n} = (d + \mathrm{o}(1))\big[n!\big]^{-\beta_1} n^{\beta_1 - \frac{x_1}{x_0}} x_0^{-n+1},$$

for a constant $d > 0$. Using the standard formula for the order and type of a power series, [16, Theorem 2], we get that the order of $H(z)$ is $\frac{1}{\beta_1}$, and the type w.r.t. this order is equal to $\beta_1 x_0^{-1/\beta_1}$. In particular, the type is not zero.

Due to [16, Theorem 14], the upper density of the zeros of B is greater than zero,

$$\limsup_{r\to\infty} \frac{n_B(r)}{r^{\frac{1}{\beta_1}}} > 0.$$

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