ASC Report No. 17/2015

A new proof of convergence for radial perfectly matched layer discretizations of Helmholtz scattering and resonance problems

M. Halla

Institute for Analysis and Scientific Computing —
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-05-6

Most recent ASC Reports

16/2015	*M. Feischl, G. Gantner, A. Haberl, D. Praetorius* Optimal convergence for adaptive IGA boundary element methods for weakly-singular integral equations
15/2015	*M. Feischl, D. Praetorius, and K.G. van der Zee* An abstract analysis of optimal goal-oriented adaptivity
14/2015	*A. Spielauer and E.B. Weinmüller* Numerical solution of boundary value problems in ordinary differential equations with time and space singularities
13/2015	*E. Daus, A. Jüngel, C. Mouhot, and N. Zamponi* Hypocoercivity for a linearized multi-species Boltzmann system
12/2015	*M. Aurada, M. Feischl, T. Führer, M. Karkulik, J.M. Melenk, and D. Praetorius* Local inverse estimates for non-local boundary integral operators
11/2015	*M. Feischl, G. Gantner, A. Haberl, D. Praetorius* Adaptive 2D IGA boundary element methods
10/2015	*M. Halla* Convergence of Hardy space infinite elements for Helmholtz scattering and resonance problems
09/2015	*M. Kaltenbäck* Spectral Theorem for definitizable normal linear operators on Krein spaces
08/2015	*M. Faustmann, J.M. Melenk, D. Praetorius* Existence of $\mathcal{H}$-matrix approximants to the inverse of BEM matrices: the hyper-singular intergral operator
07/2015	*W. Auzinger, O. Koch, M.Quell* Splittingverfahren für die Gray-Scott-Gleichung

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: `admin@asc.tuwien.ac.at`
WWW: `http://www.asc.tuwien.ac.at`
FAX: `+43-1-58801-10196`

ISBN 978-3-902627-05-6

A new proof of convergence for radial perfectly matched layer discretizations of Helmholtz scattering and resonance problems*

Martin Halla†

May 27, 2015

Abstract

The analysis of discrete PML approximations is usually split into an analysis of the modeling error, i.e. the truncation of the PM layer, and an analysis of the discretization of the truncated problem. We point out how PML discretizations can be understood as conforming Galerkin discretizations, which allows to apply standard literature. Thus we present a new proof of convergence for radial PML discretizations of Helmholtz scattering and resonance problems. We also derive previously unknown error estimates for the approximation of resonances. In particular we achieve exponential convergence rates with respect to the width of the (perfectly matched) layer.

1 Introduction

The numerical treatment of wave-equations posed on unbounded domains is a challenging task. Since the introduction of perfectly matched layers by Berenger [1] to reduce unbounded domains to bounded ones, the technique became widely spread. The method was soon recognized as a complex scaling technique [3], which allowed to construct perfectly matched layers based on cylindrical and spherical coordinate systems [4]. An analysis of radial PML methods in the setting of time-harmonic acoustics was given in [2] for scattering and in [11] for resonance problems. The analysis therein follows the

*The author acknowledges support from the Austrian Science Fund (FWF) grant W1245-N25.

†Institute for Analysis and Scientific Computing, Vienna University of Technology, Austria (martin.halla@tuwien.ac.at)

widespread technique first to compare the truncated problem to the original problem and secondly to compare the discretization of the truncated problem to the continuous truncated problem. In this paper we present a different analysis for the above problems, which recognizes the discretized truncated problems as conforming Galerkin approximations of the original problem. Thus we achieve essentially the same results as in [2, 11] for the Helmholtz equation, but with a much easier analysis and a bit stronger result on the approximation of resonances.

The paper is structured as follows. In Section 2 we introduce the investigated problem and derive a variational formulation as in [2, 11]. In Section 3 we emphasize our point of view on the interpretation of PML discretizations. This allows us to exploit the structure "coercive+compact perturbation to directly obtain our results.

2 Variational Formulation

Let $K \subset \mathbb{R}^3$ be a compact set, such that $\Omega := \mathbb{R}^3 \setminus K$ is a Lipschitz domain. In this paper we are concerned with the scattering problem to find for given boundary datum $g \in H^{-1/2}(\partial\Omega)$ and given frequency $\omega > 0$ a function $u \in H^1_{\text{loc}}(\Omega)$ such that

$$-\Delta u - \omega^2 u = 0 \quad \text{in } \Omega, \tag{1a}$$
$$\partial_n u = g \quad \text{at } \partial\Omega, \tag{1b}$$
$$u \text{ is outgoing,} \tag{1c}$$

and the resonance problem to find $(\omega, u) \in \{z \in \mathbb{C}\colon \Re(z) > 0\} \times H^1_{\text{loc}}(\Omega) \setminus \{0\}$ such that

$$-\Delta u - \omega^2 u = 0 \quad \text{in } \Omega, \tag{2a}$$
$$\partial_n u = 0 \quad \text{at } \partial\Omega, \tag{2b}$$
$$u \text{ is outgoing.} \tag{2c}$$

With "u outgoing" we thereby man that outside a ball containing K, u admits a representation

$$u(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=0}^{M_l} \alpha_{l,m} h_l^{(1)}(\omega|\mathbf{x}|) Y_{l,m}\left(|\mathbf{x}|^{-1}\mathbf{x}\right) \tag{3}$$

where $h_l^{(1)}$ are spherical Hankel functions of the first kind, $Y_{l,m}$ are spherical harmonics and $\alpha_{l,m} \in \mathbb{C}$. We follow [2] to derive complex scaled formulations of the above equations. For $R > 0$ let $B_R := \{x \in \mathbb{R}^3 \colon |x| < R\}$. Let $R_1 > 0$ be such that $K \subset B_{R_1}$ and $R_2 > R_1$. Then we define the interior, transitional and exterior domains as

$$\Omega_{\text{int}} := \Omega \cap B_{R_1}, \quad \Omega_{\text{trans}} := B_{R_2} \setminus \overline{B_{R_1}}, \quad \Omega_{\text{ext}} := \mathbb{R}^3 \setminus \overline{B_{R_2}}. \tag{4}$$

We assume a profile function $r\tilde{\alpha}(r)$ to be given, such that $\alpha_0 > 0$ and $\tilde{\alpha} \in C^2(\mathbb{R}^+)$ satisfies

$$\tilde{\alpha}(r) = 0 \quad \text{for } 0 < r < R_1, \tag{5a}$$
$$\tilde{\alpha}'(r) > 0 \quad \text{for } R_1 < r < R_2, \tag{5b}$$
$$\tilde{\alpha}(r) = \alpha_0 \quad \text{for } R_2 < r. \tag{5c}$$

We define the complex scaled variable $\tilde{r}$ as

$$\tilde{r}(r) := r\tilde{d}(r), \quad \tilde{d}(r) := 1 + i\tilde{\alpha}(r), \tag{6}$$

and the quantities

$$\alpha(r) := \tilde{\alpha}(r) + r\tilde{\alpha}'(r), \quad d(r) := 1 + i\alpha(r), \quad d_0 := 1 + i\alpha_0, \tag{7}$$

such that $\tilde{r}'(r) = d(r)$. Since solutions to (1) or (2) admit a representation (3), using polar coordinates (r, θ, ϕ) we can define for such u the scaled function $\tilde{u}$ as

$$\tilde{u}(\mathbf{x}(r, \theta, \phi)) := u(\mathbf{x}(\tilde{r}(r), \theta, \phi)). \tag{8}$$

If u is a solution to (1) or (2) and $\Im(d_0\omega) > 0$, then $\tilde{u}$ solves

$$-\tilde{\Delta}\tilde{u} - \omega^2\tilde{u} = 0 \quad \text{in } \Omega, \tag{9a}$$
$$\partial_n \tilde{u} = g \quad \text{at } \partial\Omega, \tag{9b}$$
$$\tilde{u} \in H^1(\Omega), \tag{9c}$$

where

$$\tilde{\Delta} v := \frac{1}{\tilde{d}^2 d r^2} \partial_r \left(\frac{\tilde{d}^2 r^2}{d} \partial_r v \right) + \frac{1}{\tilde{d}^2 r^2 \sin\theta} \partial_\theta (\sin\theta \partial_\theta v) + \frac{1}{\tilde{d}^2 r^2 \sin^2\theta} \partial_\phi \partial_\phi v, \tag{10}$$

and $g = 0$ in the case of the resonance problem. Vice-versa, if $\tilde{u}$ solves (9), then $u(\mathbf{x}(r, \theta, \phi)) = \tilde{u}(\mathbf{x}(\tilde{r}^{-1}(r), \theta, \phi))$ solves (1) resp. (2), see [11][Thm. 2.3]. Multiplying (9a) with $\tilde{d}^2$ and testing with $v \in H^1(\Omega)$ yields

$$A(\omega; \tilde{u}, v) = \int_{\partial\Omega} \overline{v} g \, \mathrm{d}s \tag{11}$$

for all $v \in H^1(\Omega)$, where the sesquilinear form A is defined as

$$\begin{aligned} A(\omega; u, v) := &\left\langle \frac{\tilde{d}^2}{d} \partial_r u, \partial_r \left(\frac{v}{\overline{\overline{d}}} \right) \right\rangle_{L^2(\Omega)} + \left\langle \frac{1}{r^2} \partial_\theta u, \partial_\theta v \right\rangle_{L^2(\Omega)} \\ &+ \left\langle \frac{1}{r^2 \sin^2 \theta} \partial_\phi u, \partial_\phi v \right\rangle_{L^2(\Omega)} - \omega^2 \left\langle \tilde{d}^2 u, v \right\rangle_{L^2(\Omega)}. \end{aligned} \tag{12}$$

It is straight forward to see that $A(\omega; \bullet, \bullet)$ is bounded on $H^1(\Omega)^2$. Moreover, as done in [2] A can be decomposed as $A = B + K$ with bounded sesquilinear forms

$$\begin{aligned} B(\omega; u, v) := &\left\langle \frac{\tilde{d}^2}{d^2} \partial_r u, \partial_r v \right\rangle_{L^2(\Omega)} + \left\langle \frac{1}{r^2} \partial_\theta u, \partial_\theta v \right\rangle_{L^2(\Omega)} \\ &+ \left\langle \frac{1}{r^2 \sin^2 \theta} \partial_\phi u, \partial_\phi v \right\rangle_{L^2(\Omega)} - \omega^2 d_0^2 \left\langle u, v \right\rangle_{L^2(\Omega)} \end{aligned} \tag{13}$$

and

$$K(\omega; u, v) := -\left\langle \frac{\tilde{d}^2 d'}{d^3} \partial_r u, v \right\rangle_{L^2(\Omega)} - \omega^2 \left\langle (\tilde{d}^2 - d_0^2) u, v \right\rangle_{L^2(\Omega)}. \tag{14}$$

Let $\mathbf{A}(\omega), \mathbf{B}(\omega), \mathbf{K}(\omega)$ and f be the $H^1(\Omega)$-Riesz representations of $A(\omega; \bullet, \bullet)$, $B(\omega; \bullet, \bullet)$, $K(\omega; \bullet, \bullet)$ and the semi-linear form which involves g. Let $\arg z$ be the argument of a complex number $z \in \mathbb{C} \setminus \{0\}$ taking values in $(-\pi, \pi)$. Since $\arg \frac{\tilde{d}^2}{d^2} \in [c, 0]$ with a $c \in (-\pi, 0)$, $\mathbf{B}(\omega)$ is coercive, for all $\omega \in \mathbb{C} \setminus \{0\}$ satisfying $\arg \omega^2 d_0^2 \in (-\pi, -c)$, i.e. $|\langle \mathbf{B}(\omega) u, u \rangle_{H^1(\Omega)}| \geq c \|u\|^2_{H^1(\Omega)}$ for some $c > 0$ and all $u \in H^1(\Omega)$. In particular, this holds for $\arg \omega \in (\arg d_0^{-1}, \arg i d_0^{-1})$. As d' and $\tilde{d}^2 - d_0^2$ vanish in Ω_{ext}, $\mathbf{K}(\omega)$ is compact due to the compact Sobolev embedding $H^1(\Omega \setminus \overline{\Omega_{\text{ext}}}) \hookrightarrow L^2(\Omega \setminus \overline{\Omega_{\text{ext}}})$ for arbitrary ω. Variational Equation (11) can thus be written in operator form

$$\mathbf{A}(\omega) u = f. \tag{15}$$

with $\mathbf{A}(\omega) = \mathbf{B}(\omega) + \mathbf{K}(\omega)$, coercive $\mathbf{B}(\omega)$ and compact $\mathbf{K}(\omega)$.

3 Convergence Analysis

Since $\tilde{u}$ is exponentially decaying in r Equation (9)/(11)/(15) is usually approximated in the following way. A bounded Lipschitz domain Ω_{trunc} with $\Omega_{\text{int}} \cup \overline{\Omega_{\text{trans}}} \subset \Omega_{\text{trunc}} \subset \Omega$ is chosen. This domain is supplemented with either a homogeneous Dirichlet or a homogeneous Neumann boundary condition at $\partial\Omega_{\text{trunc}} \setminus \partial\Omega$. This gives the problem to find $\tilde{u_{\text{t}}} \in H^1(\Omega_{\text{trunc}})$ such that

$$-\tilde{\Delta} u_{\text{t}} - \omega^2 u_{\text{t}} = 0 \quad \text{in } \Omega_{\text{trunc}}, \tag{16a}$$

$$\partial_n u_{\text{t}} = g \quad \text{at } \partial\Omega_{\text{trunc}} \cap \partial\Omega, \tag{16b}$$

$$u_{\text{t}} \text{ (resp. } \partial_n u_{\text{t}}) = 0 \quad \text{at } \partial\Omega_{\text{trunc}} \setminus \partial\Omega. \tag{16c}$$

The weak formulation of Equation (16) is then discretized with a finite element method, which gives a discrete solution u_{t}^h. In the analysis, the error $\|u - u_{\text{t}}^h\|_{H^1(\Omega)}$ is usually estimated by $\|u - u_{\text{t}}\|_{H^1(\Omega)} + \|u_{\text{t}} - u_{\text{t}}^h\|_{H^1(\Omega)}$ and the two summands are further analyzed individually.

Contrary to this, we now present a different kind of analysis. To this end we restrict ourselves to homogeneous Dirichlet boundary conditions at $\partial\Omega_{\text{trunc}} \setminus \partial\Omega$. Let $H^1_D(\Omega_{\text{trunc}}) := \{v \in H^1(\Omega_{\text{trunc}})\colon \operatorname{tr} v = 0 \text{ at } \partial\Omega_{\text{trunc}} \setminus \partial\Omega\}$ and $V_{h,\Omega_{\text{trunc}}}$ be a finite dimensional subspace of $H^1_D(\Omega_{\text{trunc}})$. Then, as explained above, we are looking for $u_{\text{t}}^h \in V_{h,\Omega_{\text{trunc}}}$ such that

$$\begin{aligned}\int_{\partial\Omega} \overline{v_h} g \, \mathrm{d}s = {} & \left\langle \frac{\tilde{d}^2}{d} \partial_r u_{\text{t}}^h, \partial_r \left(\frac{v_h}{\overline{d}} \right) \right\rangle_{L^2(\Omega_{\text{trunc}})} + \left\langle \frac{1}{r^2} \partial_\theta u_{\text{t}}^h, \partial_\theta v_h \right\rangle_{L^2(\Omega_{\text{trunc}})} \\ & + \left\langle \frac{1}{r^2 \sin^2\theta} \partial_\phi u_{\text{t}}^h, \partial_\phi v_h \right\rangle_{L^2(\Omega_{\text{trunc}})} - \omega^2 \left\langle \tilde{d}^2 u_{\text{t}}^h, v_h \right\rangle_{L^2(\Omega_{\text{trunc}})},\end{aligned} \tag{17}$$

for all $v_h \in V_{h,\Omega_{\text{trunc}}}$. This is indeed nothing else then looking for $u_{\text{t}}^h \in V_{h,\Omega_{\text{trunc}}}$ such that

$$A(\omega; u_{\text{t}}^h, v_h) = \int_{\partial\Omega} \overline{v_h} g \, \mathrm{d}s, \tag{18}$$

for all $v_h \in V_{h,\Omega_{\text{trunc}}}$. Now the crucial observation is that $V_{h,\Omega_{\text{trunc}}} \subset H^1(\Omega)$, which shows that (17)/(18) is nothing else than a conforming Galerkin discretization of (15). This observation is not new and was already exploited in [8] for Helmholtz problems on cylindrical wave-guide domains. Since $\mathbf{A}(\omega) = \mathbf{B}(\omega) + \mathbf{K}(\omega)$ with $\mathbf{B}$ being coercive, $\mathbf{K}$ being compact and both $\mathbf{B}, \mathbf{K}$ being holomorphic in ω, as in [7] we immediately obtain the following results.

Theorem 3.1. *Let $V_n \subset H^1(\Omega), n \in \mathbb{N}$ be a sequence of finite dimensional subspaces, such that the orthogonal projection $P_n\colon H^1(\Omega) \to V_n$ onto V_n converges to the identity $I\colon H^1(\Omega) \to H^1(\Omega)$ in the strong operator norm. For $\omega > 0$, $g \in H^{-1/2}(\partial K)$ and given $\tilde{\alpha}$ as in Sec. 2 consider the problem to find*

$u \in H^1(\Omega)$, such that

$$\langle \mathbf{A}(\omega)u, v\rangle_{H^1(\Omega)} = \int_{\partial\Omega} g\overline{v}\,\mathrm{d}s, \tag{19}$$

for all $v \in H^1(\Omega)$ and the discrete problem to find $u_n \in V_n$, such that

$$\langle \mathbf{A}(\omega)u_n, v_n\rangle_{H^1(\Omega)} = \int_{\partial\Omega} g\overline{v_n}\,\mathrm{d}s, \tag{20}$$

for all $v_n \in V_n$. Then there exists a unique solution u to (19), an index $N > 0$ and a constant $C > 0$, such that (20) admits a unique solution u_n for all $n > N$ and u_n converges to u in $H^1(\Omega)$, with the estimate

$$\|u - u_n\|_{H^1(\Omega)} \leq C \inf_{v_n \in V_n} \|u - v_n\|_{H^1(\Omega)}. \tag{21}$$

Proof. From [5] it is known that (1) admits a unique solution and hence does (19). The Lemma of Lax-Milgram allows us to apply [12, Thm. 13.6, Thm. 13.7], which yield the claim. □

Theorem 3.2. *Let $V_n \subset H^1(\Omega), n \in \mathbb{N}$ be a sequence of finite dimensional subspaces, such that the orthogonal projection $P_n\colon H^1(\Omega) \to V_n$ onto V_n converges to the identity $I\colon H^1(\Omega) \to H^1(\Omega)$ in the strong operator norm. For given α as in Sec. 2 let $\mathbb{K}(\alpha_0) := \{z \in \mathbb{C} \setminus \{0\}\colon \arg z \in (\arg d_0^{-1}, \arg id_0^{-1})\}$. Consider the eigenvalue problem to find $(\omega, u) \in \mathbb{K}(\alpha_0) \times H^1(\Omega) \setminus \{0\}$ such that*

$$\mathbf{A}(\omega)u = 0. \tag{22}$$

and its discretization to find $(\omega, u_n) \in \mathbb{K}(\alpha_0) \times V_n \setminus \{0\}$ such that

$$\mathbf{A}_n(\omega)u_n = 0, \tag{23}$$

with $\mathbf{A}_n(\omega)\colon V_n \to V_n\colon v_n \mapsto P_n\mathbf{A}(\omega)v_n$. Then there hold the spectral properties of $\mathbf{A}$:

1. *Let $\rho(\mathbf{A}) := \{z \in \mathbb{K}(\alpha_0)\colon \mathbf{A}(z)$ is invertible$\}$ be the resolvent set and $\sigma(\mathbf{A}) := \mathbb{K}(\alpha_0) \setminus \rho(\mathbf{A})$ the spectrum of $\mathbf{A}(\bullet)$. Then $\sigma(\mathbf{A})$ has no cluster points in $\mathbb{K}(\alpha_0)$,*

2. *every $\omega \in \sigma(\mathbf{A})$ is an eigenvalue of $\mathbf{A}(\bullet)$,*

3. *the operator function $\mathbf{A}^{-1}(\bullet)$ defined on $\rho(\mathbf{A})$ by $\mathbf{A}^{-1}(\omega) = \mathbf{A}(\omega)^{-1}$ is analytic on $\rho(\mathbf{A})$ and has poles of finite order at every point $\omega \in \sigma(\mathbf{A})$.*

For $\Lambda \subset \mathbb{K}(\alpha_0)$ let $\mu(\Lambda, \mathbf{A})$ be the sum of the algebraic multiplicities of all eigenvalues of $\mathbf{A}$ in Λ. Then there hold the spectral convergence properties:

4. *For every eigenvalue ω of $\mathbf{A}(\bullet)$ exists a sequence $\omega_n, n \in \mathbb{N}$ converging to ω with ω_n being eigenvalues of $\mathbf{A}_n$ for almost all $n \in \mathbb{N}$,*

5. *if $\omega_n, n \in \mathbb{N}$ and $u_n, n \in \mathbb{N}$ are some sequences of eigenvalues ω_n of $\mathbf{A}_n$ and normalized eigenelements u_n of $\mathbf{A}_n(\omega_n)$ so that $\omega_n \to \omega \in \mathbb{K}(\alpha_0)$, then*

   (a) ω is an eigenvalue of $\mathbf{A}(\bullet)$,

   (b) $u_n, n \in \mathbb{N}$ is a compact sequence and its cluster points are normalized eigenelements of $\mathbf{A}(\omega)$,

6. *for every compact $\Lambda \subset \mathbb{K}(\alpha_0)$ with boundary $\partial\Lambda \subset \rho(\mathbf{A})$ exists an index $n(\Lambda)$, so that $n \geq n(\Lambda) \Rightarrow \mu(\Lambda, \mathbf{A}_n) = \mu(\Lambda, \mathbf{A})$.*

Denote $\eta(\omega, \mathbf{A})$ the order of the pole ω of the operator function $\mathbf{A}^{-1}$ and $G(\omega, \mathbf{A})$ the generalized eigenspace of $\mathbf{A}(\omega)$. Let $\operatorname{dist}(u, V) := \inf_{v \in V} \|u - v\|_{H^1(\Omega)}$ *and*

$$d_n := \max_{u \in G(\mathbf{A},\omega), \|u\|_{H^1(\Omega)}=1} \operatorname{dist}(u, V_n), \quad d_n^* := \max_{u \in G(\mathbf{A}^*,\omega), \|u\|_{H^1(\Omega)}=1} \operatorname{dist}(u, V_n),$$

where $\mathbf{A}^*$ denotes the adjoint operator of $\mathbf{A}$. Let $\Lambda \subset \mathbb{K}(\alpha_0)$ be compact with boundary $\partial\Lambda \subset \rho(\mathbf{A})$, such that $\Lambda \cap \sigma(\mathbf{A}) = \{\omega\}$. Then the following convergence estimates hold: There exists $c > 0$ such that

7. *$|\omega_n - \omega| \leq c(d_n d_n^*)^{1/\eta(\omega,\mathbf{A})}$ for all $\omega_n \in \sigma(\mathbf{A}_n) \cap \Lambda$,*

8. *$|\overline{\omega}_n - \omega| \leq c d_n d_n^*$ where $\overline{\omega}_n$ is the weighted mean* $\overline{\omega}_n := \sum_{\tilde{\omega} \in \sigma(\mathbf{A}_n) \cap \Lambda} \tilde{\omega} \frac{\mu(\tilde{\omega}, \mathbf{A}_n)}{\mu(\omega, \mathbf{A})}$.

Proof. The first three claims are standard properties of holomorphic Fredholm operator functions, see e.g. [9]. [6, (32)] enables us to apply [9, Thm. 2] for the second three claims and [10, Thm. 2, Thm. 3] for the remaining ones. □

Thm. 3.1 and 3.2 show that for large enough Ω_{trunc} and fine enough finite element space $V_h(\Omega_{\text{trunc}}) \subset H^1(\Omega_{\text{trunc}})$ the finite element solution u_t^h converges to the solution $\tilde{u}$ of (9) in $H^1(\Omega)$ with the best approximation rate up to a constant. The best approximation error can thereby be splitted as

$$\inf_{v_h \in V_h(\Omega_{\text{trunc}})} \|\tilde{u} - v_h\|_{H^1(\Omega)}^2 = \inf_{v_h \in V_h(\Omega_{\text{trunc}})} \|\tilde{u} - v_h\|_{H^1(\Omega_{\text{trunc}})}^2 + \|\tilde{u}\|_{H^1(\Omega_{\text{ext}})}^2, \quad (24)$$

The first term is an approximation error, which can be analyzed with finite element analysis. Since

$$\tilde{u}(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=0}^{M_l} \alpha_{l,m} h_l^{(1)}(\omega d_0 |\mathbf{x}|) Y_{l,m} \left(|\mathbf{x}|^{-1} \mathbf{x} \right) \tag{25}$$

in Ω_{ext}, the second term converges exponentially to zero as Ω_{trunc} tends to Ω and can be interpreted as a truncation error.

Acknowledgments

This paper was written during a visit of the author to the POems laboratory (Palaiseau, France).

References

[1] J.-P. Berenger, *A perfectly matched layer for the absorption of electromagnetic waves*, J. Comput. Phys., 114 (1994), pp. 185–200.

[2] J. H. Bramble and J. E. Pasciak, *Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell and acoustic scattering problems*, Math. Comp., 76 (2007), pp. 597–614 (electronic).

[3] W. C. Chew and W. H. Weedon, *A 3d perfectly matched medium from modified Maxwell's equations with stretched coordinates*, Microwave Optical Tech. Letters, 7 (1994), pp. 590–604.

[4] F. Collino and P. Monk, *The perfectly matched layer in curvilinear coordinates*, SIAM J. Sci. Comput., 19 (1998), pp. 2061–2090 (electronic).

[5] D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*, vol. 93 of Applied Mathematical Sciences, Springer-Verlag, Berlin, second ed., 1998.

[6] R. Grigorieff and H. Jeggle, *Approximation von Eigenwertproblemen bei nichtlinearer Parameterabhängigkeit*, manuscripta mathematica, 10 (1973), pp. 245–271.

[7] M. Halla, *Convergence of hardy space infinite elements for Helmholtz scattering and resonance problems*, Preprint 10/2015, Institute for Analysis and Scientific Computing; Vienna University of Technology, Wien, 2015.

[8] T. Hohage and L. Nannen, *Convergence of infinite element methods for scalar waveguide problems*, BIT Numerical Mathematics, 55 (2015), pp. 215–254.

[9] O. Karma, *Approximation in eigenvalue problems for holomorphic Fredholm operator functions. I*, Numer. Funct. Anal. Optim., 17 (1996), pp. 365–387.

[10] ——, *Approximation in eigenvalue problems for holomorphic Fredholm operator functions. II. (Convergence rate)*, Numer. Funct. Anal. Optim., 17 (1996), pp. 389–408.

[11] S. Kim and J. E. Pasciak, *The computation of resonances in open systems using a perfectly matched layer*, Math. Comp., 78 (2009), pp. 1375–1398.

[12] R. Kress, *Linear integral equations*, vol. 82 of Applied Mathematical Sciences, Springer-Verlag, New York, second ed., 1999.