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LOCAL INVERSE ESTIMATES FOR NON-LOCAL BOUNDARY INTEGRAL OPERATORS

M. AURADA, M. FEISCHL, T. FÜHRER, M. KARKULIK, J. M. MELENK, AND D. PRAETORIUS

Abstract. We prove local inverse-type estimates for the four non-local boundary integral operators associated with the Laplace operator on a bounded Lipschitz domain Ω in $\mathbb{R}^d$ for $d \geq 2$ with piecewise smooth boundary. For piecewise polynomial ansatz spaces and $d \in \{2,3\}$, the inverse estimates are explicit in both the local mesh width and the approximation order. An application to efficiency estimates in *a posteriori* error estimation in boundary element methods is given.

1. Introduction

Inverse estimates are general tools for the numerical analysis of discretizations of partial differential equations (PDEs). They provide a means to bound a stronger (semi-) norm of a discrete function by a weaker norm up to some negative power of the mesh width. For example, in the context of finite element methods, it is textbook knowledge that

$$\|h\nabla V_h\|_{L^2(\Omega)} \leq C\,\|V_h\|_{L^2(\Omega)} \quad \text{for all continuous } \mathcal{T}_h\text{-piecewise polynomials } V_h. \tag{1.1}$$

The constant $C > 0$ depends only on the shape regularity of the underlying triangulation $\mathcal{T}_h$ of $\Omega \subset \mathbb{R}^d$ and the polynomial degree of V_h. Here, $h \in L^\infty(\Omega)$ is the local mesh width function defined by $h|_T := \mathrm{diam}(T)$ for $T \in \mathcal{T}_h$. Inverse estimates have also been derived for fractional-order Sobolev spaces [GHS05, DFG+04]. The usual proof of inverse estimates like (1.1) relies on scaling arguments, i.e., the powers of h arise by elementwise, i.e., *local* considerations and transformations to reference configurations.

In the present work we consider the four classical boundary integral operators (BIOs) associated with the Laplacian, e.g., the 3D simple-layer integral operator

$$\mathfrak{V}\phi(x) = \frac{1}{4\pi}\int_{\partial\Omega} \frac{1}{|x-y|}\,\phi(y)\,dy \quad \text{for } x \in \partial\Omega. \tag{1.2}$$

Here, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain with piecewise C^1-boundary $\partial\Omega$. Let $\Gamma \subseteq \partial\Omega$ be a relatively open subset of the boundary $\partial\Omega$. Our main result for $\mathfrak{V}$ and $d \in \{2,3\}$ reads, simplified,

$$\|h^{1/2}(p+1)^{-1}\nabla_\Gamma \mathfrak{V}\Phi_h\|_{L^2(\Gamma)} \leq C\,\|\Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \tag{1.3}$$

for all $\mathcal{T}_h$-piecewise polynomials Φ_h of degree $p \in \mathbb{N}_0$, where $\nabla_\Gamma(\cdot)$ denotes the surface gradient. The constant $C > 0$ depends only on the shape regularity of the underlying triangulation $\mathcal{T}_h$ of Γ. In typical settings, $\mathfrak{V}$ is an isomorphism between $\widetilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, so that we observe that (1.3) is in fact an inverse estimate for the finite dimensional space

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$\{\mathfrak{V}\Phi_h : \Phi_h \text{ a } \mathcal{T}_h\text{-piecewise polynomial of degree } p \in \mathbb{N}_0\}$ for the weighted H^1-seminorm and the natural $H^{1/2}$-norm. Inverse estimates of the form (1.3) will be shown for all four BIOs associated with the Laplacian and discrete spaces with spatially varying polynomial degree, cf. Corollary 3.2. In fact, in Theorem 3.1 we will show more general results of the type

$$\|w_h \nabla_\Gamma \mathfrak{V}\phi\|_{L^2(\Gamma)} \lesssim \Big\|\frac{w_h}{h^{1/2}}\Big\|_{L^\infty(\Gamma)} \|\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|w_h\phi\|_{L^2(\Gamma)} \qquad \text{for all } \phi \in L^2(\Gamma), \tag{1.4}$$

where w_h is a fairly general weight function. The correct choice of the weight function w_h and an inverse estimate from [GHS05, KMR14] for the weighted L^2-norm allows one to infer (1.3) from (1.4).

Applications. The inverse-type estimate (1.3) arises naturally in adaptive BEM (boundary element method) when one tries to transfer the convergence and quasi-optimality analysis from adaptive FEM [CKNS08, Ste07] to adaptive BEM [FKMP13, Gan13]. Indeed, the present results allow us to prove quasi-optimality of adaptive BEM for piecewise smooth geometries and higher (fixed) order discretizations; we refer to [FFK+14] and [FFK+15], where this is worked out in detail for weakly singular and hypersingular integral equations, respectively. While the inverse estimate (1.3) features prominently in the analysis of quasi-optimality of adaptive BEM for symmetric problems, it is also a key ingredient for plain convergence in non-symmetric problems such as FEM-BEM couplings. We refer to [AFF+13a] and the earlier preprint [AFF+12] of the present work for a convergence proof of the adaptive coupling of FEM and BEM.

A further application of estimate (1.4) concerns the efficiency of weighted residual error estimators for both weakly singular and hypersingular integral equations [Car97, CMS01, CMPS04]. To fix ideas, consider the weakly singular case and suppose that $\phi \in L^2(\Gamma)$ solves $\mathfrak{V}\phi = f$ for some given $f \in H^1(\Gamma)$. Let Φ_h be the Galerkin approximation of ϕ, where the ansatz space consists of $\mathcal{T}_h$-piecewise polynomial of fixed degree $p \in \mathbb{N}_0$. While reliability

$$C_{\mathrm{rel}}^{-1} \|\phi - \Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \le \eta_{h,\mathfrak{V}} := \|h^{1/2}\nabla_\Gamma(f - \mathfrak{V}\Phi_h)\|_{L^2(\Gamma)} \tag{1.5}$$

is well-known (at least for polyhedral domains Ω), the converse efficiency estimate remained open. As a consequence of (1.4), we will see in Corollary 3.4 that

$$C_{\mathrm{eff}}^{-1}\eta_{h,\mathfrak{V}} \le \|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)}, \tag{1.6}$$

which expresses efficiency of the weighted residual error estimator with respect to the slightly stronger norm $\|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)} \gtrsim \|\phi - \Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)}$. We refer to Corollary 3.7 for the case of the hypersingular operator.

These efficiency bounds are specific instances of new stability estimates for the BIOs in locally weighted L^2-norms detailed in Corollaries 3.3 and 3.6.

Novelty. The discrete inequality (1.3) was first shown independently in [FKMP13] and [Gan13], however, under some restrictions. The work [FKMP13] considers only lowest-order polynomials, i.e., $\mathcal{T}_h$-piecewise constants, but works for polyhedral boundaries Γ. The work [Gan13] proves (1.3) for arbitrary $\mathcal{T}_h$-piecewise polynomials, but its wavelet-based analysis is restricted to $C^{1,1}$-boundaries Γ and the constant $C > 0$ depends on the polynomial degree. Our proof of (1.4) generalizes the works [FKMP13, Gan13] in the following ways: 1) we generalize the analysis of [FKMP13] for the simple-layer operator $\mathfrak{V}$ to all four BIOs associated with the Laplacian (i.e., the double-layer operator $\mathfrak{K}$, its adjoint $\mathfrak{K}'$, and the hypersingular operator $\mathfrak{W}$); 2) we extend our previous analysis from polyhedral domains to piecewise smooth geometries; 3) we lift the restriction to fixed-order polynomial ansatz space and permit very general ansatz spaces; 4) for ansatz spaces of piecewise polynomials of arbitrary order, we make the dependence on the polynomial degree in the inverse estimates explicit.

The technical difficulty in the proof of (1.4) and (1.3) lies in the non-locality of the boundary integral operator $\mathfrak{V}$, which precludes simple elementwise considerations. We cope with the non-locality of the BIOs by splitting them into near-field and far-field contributions, each requiring different tools. The analysis of the near-field part relies on local arguments and stability properties of the BIOs. For the far-field part, the key observation is that the BIOs are derived from two volume potentials, namely, the simple-layer potential $\widetilde{\mathfrak{V}}$ and the double-layer potential $\widetilde{\mathfrak{K}}$ by taking appropriate traces. Since these potentials solve elliptic equations, "interior regularity" estimates are available for them and trace inequalities imply corresponding estimates for the BIOs. Section 4 proves the relevant estimates for the simple-layer potential $\widetilde{\mathfrak{V}}$, whereas Section 5 is concerned with the double-layer potential $\widetilde{\mathfrak{K}}$. The final Section 6 then combines these results to give the proof of Theorem 3.1.

Although the present paper considers only the four BIOs associated with the Laplacian, the scope is wider. As just mentioned, the key tool are interior estimates for potentials; such estimates are available for many elliptic equations, for example, the Lamé system, so that we expect that corresponding results can be proved as well for BIOs associated with these problems.

General notation. We close the introduction by stating that $|\cdot|$ denotes, depending on the context, the absolute value of a real number, the Euclidean norm of a vector in $\mathbb{R}^d$, the Lebesgue measure of a subset of $\mathbb{R}^{d-1}$ or $\mathbb{R}^d$ or the $(d-1)$-dimensional surface measure of a subset of $\partial\Omega$. The notation $a \lesssim b$ abbreviates $a \le C \cdot b$ for some constant $C > 0$ which will be clear from the context, and we write $a \simeq b$ to abbreviate $a \lesssim b \lesssim a$.

We write $B_r(x) := \{z \in \mathbb{R}^d : |x - z| \le r\}$ for the closed ball with radius r and center x.

2. Spaces, Operators, and Meshes

2.1. Sobolev spaces. Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \ge 2$, with piecewise C^1-boundary $\partial\Omega$ and corresponding exterior domain $\Omega^{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}$. The exterior unit normal vector field on $\partial\Omega$ is denoted by $\boldsymbol{\nu}$. Throughout, we will assume that $\Gamma \subseteq \partial\Omega$ is a non-empty, relatively open set that stems from a Lipschitz dissection $\partial\Omega = \Gamma \cup \partial\Gamma \cup (\partial\Omega \setminus \Gamma)$ as described in [McL00, pp. 99]. Note that $\Gamma = \partial\Omega$ is valid.

The non-negative order Sobolev spaces $H^{1/2+s}(\partial\Omega)$ for $s \in \{-1/2, 0, 1/2\}$ are defined as in [McL00, pp. 99] by use of Bessel potentials on $\mathbb{R}^{d-1}$ and lifting via the bi-Lipschitz maps that describe $\partial\Omega$. We also need the spaces $H^{1/2+s}(\Gamma)$ and $\widetilde{H}^{1/2+s}(\Gamma)$. In accordance with [McL00], these are defined as follows:

$$H^{1/2+s}(\Gamma) := \{v|_\Gamma \colon v \in H^{1/2+s}(\partial\Omega)\}, \tag{2.1}$$

$$\widetilde{H}^{1/2+s}(\Gamma) := \{v \colon E_{0,\Gamma} v \in H^{1/2+s}(\partial\Omega)\}, \tag{2.2}$$

where $E_{0,\Gamma}$ denotes the operator that extends a function defined on Γ to a function on $\partial\Omega$ by zero. These spaces are endowed with their natural norms, i.e., the quotient norm

$\|v\|_{H^{1/2+s}(\Gamma)} := \inf\{\|V\|_{H^{1/2+s}(\partial\Omega)} \colon V|_\Gamma = v\}$ and $\|v\|_{\widetilde{H}^{1/2+s}(\Gamma)} := \|E_{0,\Gamma}v\|_{H^{1/2+s}(\partial\Omega)}$. Owing to the assumption that $\partial\Omega = \Gamma \cup \partial\Gamma \cup (\partial\Omega \setminus \Gamma)$ is a Lipschitz dissection, we have the following facts, stated here without proof:

Facts 2.1. *(i) For $s = 1/2$, we have the norm equivalences $\|u\|^2_{H^1(\partial\Omega)} \simeq \|u\|^2_{L^2(\partial\Omega)} + \|\nabla_\Gamma u\|^2_{L^2(\partial\Omega)}$ and $\|u\|^2_{\widetilde{H}^1(\Gamma)} \simeq \|u\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma u\|^2_{L^2(\Gamma)}$, where ∇_Γ is the (weak) surface gradient.*

(ii) For $s = 0$, the norms $\|u\|_{H^{1/2}(\partial\Omega)}$ and $\|u\|_{\widetilde{H}^{1/2}(\Gamma)}$ can equivalently be described by the Aronstein-Slobodeckii norms of u and $E_{0,\Gamma}u$ (cf. [McL00, (3.18)] *for the definition of the Aronstein-Slobodeckii norm).*

(iii) For $s = 0$, the spaces $H^{1/2}(\partial\Omega)$ and $\widetilde{H}^{1/2}(\Gamma)$ are obtained from interpolating between the cases $s = -1/2$ (i.e., $L^2(\partial\Omega)$ or $L^2(\Gamma)$) and $s = 1/2$ (i.e., $H^1(\partial\Omega)$ or $\widetilde{H}^1(\Gamma)$) using the K-method (cf., e.g., [McL00, Thm. B.11] *for the case of $H^s(\partial\Omega)$).*

Negative order Sobolev spaces are defined by duality, namely, for $s \in \{-1/2, 0, 1/2\}$,

$$\begin{aligned} H^{-1/2}(\partial\Omega) &:= H^{1/2}(\partial\Omega)', \\ \widetilde{H}^{-(1/2+s)}(\Gamma) &:= H^{1/2+s}(\Gamma)', \quad \text{and} \\ H^{-(1/2+s)}(\Gamma) &:= \widetilde{H}^{1/2+s}(\Gamma)', \end{aligned} \tag{2.3}$$

where duality pairings $\langle\cdot,\cdot\rangle$ are understood to extend the standard L^2-scalar product on $\partial\Omega$ or Γ. We observe the continuous inclusions

$$\widetilde{H}^{\pm(1/2+s)}(\Gamma) \subseteq H^{\pm(1/2+s)}(\Gamma) \quad \text{as well as} \quad \widetilde{H}^{\pm(1/2+s)}(\partial\Omega) = H^{\pm(1/2+s)}(\partial\Omega).$$

We also note that for $\psi \in L^2(\Gamma)$ the zero extension $E_{0,\Gamma}\psi$ satisfies $E_{0,\Gamma}\psi \in H^{-1/2}(\partial\Omega)$ with

$$\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} = \|E_{0,\Gamma}\psi\|_{H^{-1/2}(\partial\Omega)}. \tag{2.4}$$

We denote by $\gamma_0^{\mathrm{int}}(\cdot)$ the interior trace operator, i.e., $\gamma_0^{\mathrm{int}}u$ is the restriction of a function $u \in H^1(\Omega)$ to the boundary $\partial\Omega$. With $H^1_\Delta(\Omega) := \{u \in H^1(\Omega) \,:\, -\Delta u \in L^2(\Omega)\}$, the interior conormal derivative operator $\gamma_1^{\mathrm{int}} : H^1_\Delta(\Omega) \to H^{-1/2}(\partial\Omega)$ is defined by the first Green's formula, viz.,

$$\langle \gamma_1^{\mathrm{int}}u \,,\, v\rangle_{\partial\Omega} = \langle \nabla u \,,\, \nabla v\rangle_\Omega - \langle -\Delta u \,,\, v\rangle_\Omega \quad \text{for all } v \in H^1(\Omega). \tag{2.5}$$

Remark 2.2. *The operator $\gamma_1^{\mathrm{int}}(\cdot)$ generalizes the classical normal derivative operator: if $u \in H^1_\Delta(\Omega)$ is sufficiently smooth near a boundary point x_0, then $\gamma_1^{\mathrm{int}}u$ can be represented near x_0 by a function given by the pointwise defined normal derivative $\partial_{\boldsymbol{\nu}} u$.* ∎

The exterior trace γ_0^{ext} and the exterior conormal derivative operator γ_1^{ext} are defined analogously to their interior counterparts. To that end, we fix a bounded Lipschitz domain $U \subset \mathbb{R}^d$ with $\overline{\Omega} \subset U$. The exterior trace operator $\gamma_0^{\mathrm{ext}} : H^1(U \setminus \overline{\Omega}) \to H^{1/2}(\partial\Omega)$ is defined by restricting to $\partial\Omega$, and the exterior conormal derivative γ_1^{ext} is characterized by $\langle \gamma_1^{\mathrm{ext}}u \,,\, v\rangle_{\partial\Omega} = \langle \nabla u \,,\, \nabla v\rangle_{U\setminus\overline{\Omega}} - \langle -\Delta u \,,\, v\rangle_{U\setminus\overline{\Omega}}$ for all $v \in H^1(U \setminus \overline{\Omega})$ with $\gamma_0^{\mathrm{ext}}v = 0$ on ∂U.

For a function u that admits both conormal derivatives or both traces, we define the jumps $[\gamma_1 u] := \gamma_1^{\mathrm{ext}}u - \gamma_1^{\mathrm{int}}u$ and $[u] = \gamma_0^{\mathrm{ext}}u - \gamma_0^{\mathrm{int}}u$, respectively.

2.2. Boundary integral operators. We briefly introduce the pertinent boundary integral operators and refer to the monographs [McL00, HW08, SS11] for further details and proofs. Green's function for the Laplace operator is given by

$$G(x,y) = \begin{cases} -\frac{1}{|\mathbb{S}^1|} \log|x-y|, & \text{for } d=2, \\ +\frac{1}{|\mathbb{S}^{d-1}|} |x-y|^{-(d-2)}, & \text{for } d \geq 3, \end{cases} \tag{2.6}$$

where $|\mathbb{S}^{d-1}|$ denotes the surface measure of the Euclidean sphere in $\mathbb{R}^d$, e.g., $|\mathbb{S}^1| = 2\pi$ and $|\mathbb{S}^2| = 4\pi$. The classical simple-layer potential $\widetilde{\mathfrak{V}}$ and the double-layer potential $\widetilde{\mathfrak{K}}$ are formally defined by

$$(\widetilde{\mathfrak{V}}\psi)(x) := \int_{\partial\Omega} G(x,y)\psi(y)\,dy, \qquad (\widetilde{\mathfrak{K}}v)(x) := \int_{\partial\Omega} \partial_{\boldsymbol{\nu}(y)} G(x,y) v(y)\,dy, \qquad x \in \mathbb{R}^d \setminus \partial\Omega;$$

here, $\partial_{\boldsymbol{\nu}(y)}$ denotes the (outer) normal derivative with respect to the variable y. These pointwise defined operators can be extended to bounded linear operators

$$\widetilde{\mathfrak{V}} \in L\big(H^{-1/2}(\partial\Omega); H^1(U)\big) \quad \text{and} \quad \widetilde{\mathfrak{K}} \in L\big(H^{1/2}(\partial\Omega); H^1(U \setminus \partial\Omega)\big). \tag{2.7}$$

It is well-known that $\Delta\widetilde{\mathfrak{V}}\psi = 0 = \Delta\widetilde{\mathfrak{K}}v$ in $U \setminus \partial\Omega$ for all $\psi \in H^{-1/2}(\partial\Omega)$ and $v \in H^{1/2}(\partial\Omega)$. The simple-layer, double-layer, adjoint double-layer, and the hypersingular integral operator are defined as follows:

$$\mathfrak{V} = \gamma_0^{\text{int}}\widetilde{\mathfrak{V}}, \quad \mathfrak{K} = \frac{1}{2} + \gamma_0^{\text{int}}\widetilde{\mathfrak{K}}, \quad \mathfrak{K}' = -\frac{1}{2} + \gamma_1^{\text{int}}\widetilde{\mathfrak{V}}, \text{ and } \quad \mathfrak{W} = -\gamma_1^{\text{int}}\widetilde{\mathfrak{K}}. \tag{2.8}$$

These linear operators are bounded linear operators for $s \in \{-1/2, 0, 1/2\}$ as follows:

$$\mathfrak{V} \in L(H^{-1/2+s}(\partial\Omega); H^{1/2+s}(\partial\Omega)), \tag{2.9}$$

$$\mathfrak{K} \in L(H^{1/2+s}(\partial\Omega); H^{1/2+s}(\partial\Omega)), \tag{2.10}$$

$$\mathfrak{K}' \in L(H^{-1/2+s}(\partial\Omega); H^{-1/2+s}(\partial\Omega)), \tag{2.11}$$

$$\mathfrak{W} \in L(H^{1/2+s}(\partial\Omega); H^{-1/2+s}(\partial\Omega)), \tag{2.12}$$

The operators $\widetilde{\mathfrak{V}}$, $\mathfrak{V}$, $\mathfrak{K}'$ will often be applied to functions in $L^2(\Gamma)$. Throughout the paper, we employ the convention that for $\psi \in L^2(\Gamma)$ we implicitly extend by zero, e.g.,

$$\widetilde{\mathfrak{V}}\psi \text{ means } \widetilde{\mathfrak{V}}(E_{0,\Gamma}\psi), \quad \mathfrak{V}\psi \text{ means } \mathfrak{V}(E_{0,\Gamma}\psi), \quad \text{and } \mathfrak{K}'\psi \text{ means } \mathfrak{K}'(E_{0,\Gamma}\psi). \tag{2.13}$$

An analogous extension is obviously used when $\widetilde{\mathfrak{K}}$, $\mathfrak{K}$, $\mathfrak{W}$ are applied to an $v \in \widetilde{H}^{1/2}(\Gamma)$.

Remark 2.3. *Ellipticity of $\mathfrak{V}$ and $\mathfrak{W}$ is not used in our analysis of Theorem 3.1 and Corollary 3.2. In particular, there is no need to scale Ω to ensure* $\operatorname{diam}(\Omega) < 1$ *in 2D or to assume that Γ is connected.* ∎

2.3. Surface simplices and admissible triangulations. Fix the reference simplex $T_{\text{ref}} := \{x \in \mathbb{R}^{d-1},\ 0 < x_1, \dots, x_{d-1}, \sum_{j=1}^{d-1} x_j < 1\}$, which is the convex hull of the d vertices $\{0, e_1, \dots, e_{d-1}\}$ ("0-faces"). The convex hull of any $j+1$ of these vertices is called a "j-face" of T_{ref}. We call the $(d-2)$-faces "facets" of T_{ref}.

We require the concept of regular, shape-regular triangulations $\mathcal{T}_h$ of Γ.

Definition 2.4 (regular and shape-regular triangulations). *A set $\mathcal{T}_h$ of subsets of Γ is called a* regular *triangulation of Γ if the following is true:*

(i) The elements *$T \in \mathcal{T}_h$ are relatively open subsets of Γ and each T is the image of T_{ref} under an* element map *$\gamma_T : \overline{T_{\text{ref}}} \to \overline{T}$. The element map γ_T is assumed to be bijective and C^1 on $\overline{T_{\text{ref}}}$.*

(ii) The elements cover Γ: $\bigcup_{T\in\mathcal{T}_h} \overline{T} = \overline{\Gamma}$.

(iii) "no hanging nodes": For each pair $(T, T') \in \mathcal{T}_h \times \mathcal{T}_h$, the intersection $\overline{T} \cap \overline{T'}$ is either empty or there are two j-faces f, $f' \subseteq \partial T_{\text{ref}}$ of T_{ref} with $j \in \{0, \ldots, d-2\}$ such that $\overline{T} \cap \overline{T'} = \gamma_T(f) = \gamma_{T'}(f')$.

(iv) Parametrizations of common boundary parts of neighboring elements are compatible: If $\emptyset \neq \overline{T} \cap \overline{T'} = \gamma_T(f) = \gamma_{T'}(f')$, then $\gamma_T^{-1} \circ \gamma_{T'} : f' \to f$ is an affine isomorphism.

We call the images of vertices of T_{ref} under the element maps nodes *of $\mathcal{T}_h$ and collect them in the set $\mathcal{N}_h$. The images of the $(d-2)$-faces of T_{ref} are called* facets *of $\mathcal{T}_h$ and collected in the set $\mathcal{F}_h$. For each $T \in \mathcal{T}_h$, we set $h(T) := \operatorname{diam}(T) := \sup_{x,y\in T} |x - y|$.*

*A regular triangulation is called κ-*shape regular*, if the element maps γ_T satisfy the following:*

(v) Let $G_T(x) := \gamma_T'(x)^\top \cdot \gamma_T'(x) \in \mathbb{R}^{(d-1)\times(d-1)}$ be the symmetric Gramian matrix of γ_T. The triangulation is κ-shape regular if for all $T \in \mathcal{T}_h$ the extremal eigenvalues $\lambda_{min}(G_T(x))$ and $\lambda_{max}(G_T(x))$ of $G_T(x)$ satisfy

$$\sup_{x\in T_{\text{ref}}} \left(\frac{h(T)^2}{\lambda_{min}(G_T(x))} + \frac{\lambda_{max}(G_T(x))}{h(T)^2} \right) \leq \kappa.$$

(vi) If $d = 2$, we require explicitly that the element sizes of neighboring elements are comparable:

$$h(T) \leq \kappa h(T') \qquad \text{for all } T, T' \text{ with } \overline{T} \cap \overline{T'} \neq \emptyset.$$

With each triangulation $\mathcal{T}_h$, we associate the local mesh size function $h \in L^\infty(\Gamma)$ which is defined elementwise by $h|_T := h(T)$ for all $T \in \mathcal{T}_h$. We note that for a κ-shape regular mesh we have

$$\max_{T\in\mathcal{T}_h} \frac{h(T)^{d-1}}{|T|} \lesssim 1, \tag{2.14}$$

where the implied constant depends solely on κ.

If Γ is the union of pieces of $(d-1)$-dimensional hyperplanes and the element maps are *affine*, then the Gramians are constants and the Definition 2.4 generalizes the classical concept of a shape-regular triangulation of Γ. In the non-affine case, the following example illustrates how triangulations as stipulated in Definition 2.4 can be created:

Example 2.5. *Let $\Gamma \subseteq \partial\Omega$ be an open surface piece and assume $\Gamma = \gamma(\widehat{\Gamma})$ for some reference configuration $\widehat{\Gamma} \subseteq \mathbb{R}^{d-1}$ and some sufficiently smooth map γ. Let $\widehat{\mathcal{T}}_h = \{\widehat{T}_1, \ldots, \widehat{T}_N\}$ be a standard, regular, shape-regular triangulation of $\widehat{\Gamma}$ with affine element maps $\widehat{\gamma}_{\widehat{T}_i}$, $i = 1, \ldots, N$. Then, the triangulation with elements $T = \gamma \circ \gamma_{\widehat{T}_i}(T_{\text{ref}})$ and element maps $\gamma \circ \gamma_{\widehat{T}_i}$ satisfies the hypotheses of Definition 2.4.*
This concept generalizes to surfaces consisting of several patches; it is worth emphasizing that in that case the patch parametrizations need to match at patch boundaries. ∎

Remark 2.6. *In Definition 2.4 the conditions on the mesh are formulated so as to ensure that the spaces $\mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h)$ below have good approximation properties. The conditions (iii) and (iv) in Definition 2.4 could be relaxed if only good approximation properties of the spaces $\mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$ are required.* ∎

For an element $T \in \mathcal{T}_h$, we define the element patch $\omega_h(T)$ by

$$\omega_h(T) := \left(\bigcup \{ \overline{T'} \,:\, T' \in \mathcal{T}_h \text{ with } \overline{T} \cap \overline{T'} \neq \emptyset \} \right)^{\circ}. \tag{2.15}$$

The assumptions on the element maps of a κ-shape regular triangulation imply that elements of a patch are comparable in size. Furthermore, the fact that Γ results from a Lipschitz dissection of $\partial\Omega$ imposes certain topological restrictions on the patches:

Lemma 2.7. *Let $\mathcal{T}_h$ be a regular, κ-shape regular mesh. Then there is a constant $C > 0$ that depends solely on κ and the Lipschitz character of $\partial\Omega$ such that the following holds:*

(i) $h(T) \leq Ch(T')$ for any two elements T, T' with $\overline{T} \cap \overline{T'} \neq \emptyset$.
(ii) The number of elements in an element patch is bounded by C.
(iii) For any two elements T, T' in the element patch $\omega_h(T'')$ there is a sequence $T = T_0, \ldots, T_n = T'$ of elements T_i, $i = 0, \ldots, n$, in $\omega_h(T'')$ such that two successive elements T_i, T_{i+1} share a common facet: $\overline{T_i} \cap \overline{T_{i+1}} \in \mathcal{F}_h$ for $i = 0, \ldots, n-1$.

Sketch of Proof. Statement (iii): We show (iii) first for the node patch

$$\omega_h(z) := \left(\bigcup \{ \overline{T} \,:\, T \in \mathcal{T}_h \text{ with } z \in \overline{T} \} \right)^{\circ}$$

and some node z of T''. This follows from the fact that Γ results from a Lipschitz dissection and considerations in $\mathbb{R}^{d-1}$ using local charts. After a Euclidean change of coordinates, we may assume that $\partial\Omega$ is (locally) a hypograph, i.e., there is a Lipschitz continuous function $\Lambda : B_r(0) \to \mathbb{R}$ such that the set $\{(x, \Lambda(x)) \colon x \in B_r(0)\} \subset \partial\Omega$. Without loss of generality, we assume the Euclidean coordinate change is such that $z = (0, \Lambda(0))$. One may also assume that Λ is defined on $\mathbb{R}^{d-1}$ (and Lipschitz continuous) so that the map $\widetilde{\Lambda} : \mathbb{R}^d \to \mathbb{R}^d$ given by $(x, t) \mapsto (x, \Lambda(x) + t)$ is bilipschitz.
We distinguish the cases $z \in \Gamma$ and $z \in \partial\Gamma$. Let z be an interior point of Γ. Then, the pull-backs $\widehat{T} := \widetilde{\Lambda}^{-1}(T)$, $T \subseteq \omega_h(z)$, are contained in the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ and (identifying this hyperplane with $\mathbb{R}^{d-1}$) completely cover a neighborhood of $0 \in \mathbb{R}^{d-1}$. This together with (iii) of Definition 2.4 shows the claim. If $z \in \partial\Gamma$, then the fact that the elements are contained in Γ and that Γ results from a Lipschitz dissection implies that near $0 \in \mathbb{R}^{d-1}$, the pull-backs $\widehat{T}$ are all on one side of a Lipschitz graph in $\mathbb{R}^{d-1}$. This together with (iii) of Definition 2.4 again implies the claim. Since $\omega_h(T'')$ is the union of the d node patches $\omega_h(z)$ associated with the d nodes of T'', this concludes the proof of (iii).

Statement (ii): Consider the case of an interior point $z \in \Gamma$. The assumption (iii) of Definition 2.4 and the fact that the map $\widetilde{\Lambda}$ is bilipschitz implies that the solid angles of the elements $\widehat{T}$ at 0 are bounded away from zero by a constant that depends solely on κ and $\widetilde{\Lambda}$. This implies the claim for a node patch $\omega_h(z)$ and thus for $\omega_h(T)$ with $T \in \mathcal{T}_h$.
Statement (i): For $d = 2$, this follows by definition. For $d \geq 3$ we first note that two elements sharing a facet $f \in \mathcal{F}_h$ have comparable size by (iii)—(v) of Definition 2.4. We conclude the proof with the aid of statements (iii) and (ii). □

2.4. Admissible weight functions and discrete spaces.

Definition 2.8 (σ-admissible weight functions and polynomial degree distributions)**.** *A function $w_h \in L^\infty(\Gamma)$ is* σ-admissible *with respect to $\mathcal{T}_h$ if*

$$\|w_h\|_{L^\infty(T)} \le \sigma\, w_h(x) \quad \textit{almost everywhere on } \omega_h(T).$$

A σ-admissible function $q_h \in L^\infty(\Gamma)$ is called a σ-admissible polynomial degree distribution *with respect to $\mathcal{T}_h$, if $q_h(T) := q_h|_T \in \mathbb{N}_0$ for all $T \in \mathcal{T}_h$.*

We write

$$\mathcal{P}^{\mathbf{q}}(\mathcal{T}_h) := \big\{\Psi_h \in L^2(\Gamma) \,:\, \forall T \in \mathcal{T}_h \quad \Psi_h \circ \gamma_T \text{ is a polynomial of degree } \le q_h(T)\big\}, \tag{2.16}$$

for the space of (discontinuous) piecewise polynomials of local degree $q_h(T)$. Moreover, we introduce spaces of continuous piecewise polynomials of local degree $q_h(T)+1$ by

$$\mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h) := \mathcal{P}^{\mathbf{q}+1}(\mathcal{T}_h) \cap H^1(\Gamma), \tag{2.17}$$

$$\widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h) := \mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h) \cap \widetilde{H}^1(\Gamma). \tag{2.18}$$

We note the inclusions $\mathcal{P}^{\mathbf{q}}(\mathcal{T}_h) \subset L^2(\Gamma) \subset \widetilde{H}^{-1/2}(\Gamma)$, $\widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h) \subset \widetilde{H}^1(\Gamma) \subset \widetilde{H}^{1/2}(\Gamma)$, and $\mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h) \subset H^1(\Gamma)$, as well as $\widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h) = \mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h)$ in case of $\Gamma = \partial\Omega$.

For $q \in \mathbb{N}_0$, the use of non-boldface superscripts in $\mathcal{P}^q(\mathcal{T}_h)$, $\mathcal{S}^{q+1}(\mathcal{T}_h)$, and $\widetilde{\mathcal{S}}^{q+1}(\mathcal{T}_h)$ indicates that a constant polynomial degree is employed.

3. Main result and applications

3.1. Inverse estimates. The following Theorem 3.1 is the main result of this work.

Theorem 3.1. *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ and let $w_h \in L^\infty(\Gamma)$ be a σ-admissible weight function with respect to $\mathcal{T}_h$. Then, it holds*

$$\|w_h \nabla_\Gamma \mathfrak{V}\psi\|_{L^2(\Gamma)} + \|w_h \mathfrak{K}'\psi\|_{L^2(\Gamma)} \le C_{\rm inv}\big(\|w_h/h^{1/2}\|_{L^\infty(\Gamma)}\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|w_h\psi\|_{L^2(\Gamma)}\big), \tag{3.1}$$

$$\|w_h \nabla_\Gamma \mathfrak{K}v\|_{L^2(\Gamma)} + \|w_h \mathfrak{W}v\|_{L^2(\Gamma)} \le C_{\rm inv}\big(\|w_h/h^{1/2}\|_{L^\infty(\Gamma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma)} + \|w_h\nabla_\Gamma v\|_{L^2(\Gamma)}\big), \tag{3.2}$$

for all functions $\psi \in L^2(\Gamma)$ and all $v \in \widetilde{H}^1(\Gamma)$. The constant $C_{\rm inv} > 0$ depends only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and σ.

In the following Corollary 3.2, we apply the estimates (3.1)–(3.2) of Theorem 3.1 to discrete functions $\Psi_h \in \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$ and $V_h \in \widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h)$. We mention that the restriction to $d \in \{2,3\}$ in Corollary 3.2 is due to the fact that the underlying reference [KMR14] restricts to this setting.

Corollary 3.2. *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ. Suppose that $d \in \{2,3\}$ and that q_h is a σ-admissible polynomial degree distribution with respect to $\mathcal{T}_h$. Then, there exists a constant $\widetilde{C}_{\rm inv} > 0$ such that the following estimates hold:*

$$\|h^{1/2}(q_h+1)^{-1}\,\nabla_\Gamma \mathfrak{V}\Psi_h\|_{L^2(\Gamma)} + \|h^{1/2}(q_h+1)^{-1}\,\mathfrak{K}'\Psi_h\|_{L^2(\Gamma)} \le \widetilde{C}_{\rm inv}\|\Psi_h\|_{\widetilde{H}^{-1/2}(\Gamma)}, \tag{3.3}$$

$$\|h^{1/2}(q_h+1)^{-1}\,\nabla_\Gamma \mathfrak{K}V_h\|_{L^2(\Gamma)} + \|h^{1/2}(q_h+1)^{-1}\,\mathfrak{W}V_h\|_{L^2(\Gamma)} \le \widetilde{C}_{\rm inv}\|V_h\|_{\widetilde{H}^{1/2}(\Gamma)}, \tag{3.4}$$

for all discrete functions $\Psi_h \in \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$ and $V_h \in \widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h)$. The constant $\widetilde{C}_{\rm inv} > 0$ depends only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and the σ-admissibility of q_h, but is otherwise independent of the polynomial degrees and the mesh $\mathcal{T}_h$.

Proof. The starting point are the following two inverse estimates

$$\|h^{1/2}(q_h+1)^{-1}\,\Psi_h\|_{L^2(\Gamma)} \lesssim \|\Psi_h\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \Psi_h \in \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h), \tag{3.5}$$

$$\|h^{1/2}(q_h+1)^{-1}\nabla_\Gamma V_h\|_{L^2(\Gamma)} \lesssim \|V_h\|_{\widetilde{H}^{1/2}(\Gamma)} \quad \text{for all } V_h \in \widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h), \tag{3.6}$$

where the hidden constants depend solely on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and the σ-admissibility of q_h. The bound (3.5) is essentially taken from [Geo08, Thm. 3.9]. However, since the non-trivial interpolation argument is not worked out in [Geo08, Thm. 3.9] and since [Geo08, Thm. 3.9] is not concerned with open surfaces Γ, we present the details in Lemma A.1. We remark that its proof employs the characterization of fractional Sobolev norms in terms of the Aronstein-Slobodeckii norm. The bound (3.6) follows also from polynomial inverse estimates and an interpolation argument for spaces of piecewise polynomials, which is non-trivial—see [KMR14] for details. We also refer to [AFF$^+$14, Proposition 5] for the h-version of (3.6), in which the dependence on the polynomial degree q_h is left unspecified.

We define a weight function by $w_h := h^{1/2}(q_h+1)^{-1}$. Note that $\|w_h/h^{1/2}\|_{L^\infty(\Gamma)} \le 1$ and that w_h is τ-admissible, where τ depends only on κ and σ. The combination of (3.5) with (3.1) leads to (3.3). The bound (3.6) in conjunction with (3.2) yields (3.4). $\square$

3.2. Application to efficiency of residual error estimation.

3.2.1. *Weakly singular integral equations.* The next corollary proves that the estimate (3.1) provides stability of $\mathfrak{V}$ and $\mathfrak{K}'$ in weighted norms for subspaces $(1-P_h)L^2(\Gamma) \subseteq L^2(\Gamma)$, where P_h is some projection operator. Note that the following corollary is in particular applicable to the Galerkin projection onto $\mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$.

Corollary 3.3. *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ. Let X_h be a closed subspace of $\widetilde{H}^{-1/2}(\Gamma)$ with $\mathcal{P}^0(\mathcal{T}_h) \subseteq X_h \subset L^2(\Gamma)$. Let $\Pi_h : L^2(\Gamma) \to X_h$ be the L^2-orthogonal projection onto X_h and $\mathbb{P}_h : \widetilde{H}^{-1/2}(\Gamma) \to X_h \subseteq \widetilde{H}^{-1/2}(\Gamma)$ denote an arbitrary $\widetilde{H}^{-1/2}(\Gamma)$-stable projection onto X_h. Then, there is a constant $\widetilde{C}_{\rm inv} > 0$ depending only on the κ-shape regularity of $\mathcal{T}_h$, the stability constant of $\mathbb{P}_h$, on $\partial\Omega$ as well as Γ such that for all $\phi \in L^2(\Gamma)$ and $P_h \in \{\Pi_h, \mathbb{P}_h\}$*

$$\|h^{1/2}\nabla_\Gamma \mathfrak{V}(1-P_h)\phi\|_{L^2(\Gamma)} + \|h^{1/2}\mathfrak{K}'(1-P_h)\phi\|_{L^2(\Gamma)} \le \widetilde{C}_{\rm inv}\|h^{1/2}(1-P_h)\phi\|_{L^2(\Gamma)}. \tag{3.7}$$

Proof. For arbitrary $w \in H^{1/2}(\partial\Omega)$ we get by transformation to the reference element and standard approximation results that $\|(1-\Pi_h)w\|^2_{L^2(T)} \lesssim h(T)\|w\|^2_{H^{1/2}(T)}$, where we employ the Aronstein-Slobodeckii norm in the definition of $\|\cdot\|_{H^{1/2}(T)}$. Hence, by summation over all $T \in \mathcal{T}_h$, using the Aronstein-Slobodeckii characterization of $\|\cdot\|_{H^{1/2}(\partial\Omega)}$, and then the characterization (2.1) of the norm $\|\cdot\|_{H^{1/2}(\Gamma)}$, we arrive at

$$\|h^{-1/2}(1-\Pi_h)w\|_{L^2(\Gamma)} \lesssim \|w\|_{H^{1/2}(\Gamma)} \quad \text{for all } w \in H^{1/2}(\Gamma).$$

Orthogonality of Π_h and a duality argument then shows (see [CP06, Theorem 4.1] for the analogous proof on polygonal boundaries.)

$$\|(1-\Pi_h)\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|h^{1/2}(1-\Pi_h)\phi\|_{L^2(\Gamma)} \quad \text{for all } \phi \in L^2(\Gamma). \tag{3.8}$$

Combining this estimate with the inverse estimate (3.1) for $\psi = (1-\Pi_h)\phi$ and $w_h = h^{1/2}$, we get

$$\|h^{1/2}\nabla_\Gamma \mathfrak{V}(1-\Pi_h)\phi\|_{L^2(\Gamma)} + \|h^{1/2}\mathfrak{K}'(1-\Pi_h)\phi\|_{L^2(\Gamma)} \lesssim \|h^{1/2}(1-\Pi_h)\phi\|_{L^2(\Gamma)} \text{ for all } \phi \in L^2(\Gamma).$$

For an $\widetilde{H}^{-1/2}(\Gamma)$-stable projection $\mathbb{P}_h$, we note that the projection property of $\mathbb{P}_h$ implies $(1-\mathbb{P}_h)(1-\Pi_h) = (1-\mathbb{P}_h)$. This and elementwise stability of Π_h imply, for all $\phi \in L^2(\Gamma)$,

$$\|(1-\mathbb{P}_h)\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|(1-\Pi_h)\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \overset{(3.8)}{\lesssim} \|h^{1/2}(1-\Pi_h)\phi\|_{L^2(\Gamma)} \lesssim \|h^{1/2}\phi\|_{L^2(\Gamma)}.$$

Finally, we use the projection property $(1-\mathbb{P}_h)^2 = (1-\mathbb{P}_h)$ and argue as for Π_h to obtain

$$\|h^{1/2}\nabla_\Gamma \mathfrak{V}(1-\mathbb{P}_h)\phi\|_{L^2(\Gamma)} + \|h^{1/2}\mathfrak{K}'(1-\mathbb{P}_h)\phi\|_{L^2(\Gamma)} \lesssim \|h^{1/2}(1-\mathbb{P}_h)\phi\|_{L^2(\Gamma)} \text{ for all } \phi \in L^2(\Gamma).$$

This concludes the proof. □

One immediate consequence of Corollary 3.3 is the efficiency of the weighted residual error estimator η_h from [Car97, CMS01]: Suppose that $\mathfrak{V}$ is $\widetilde{H}^{-1/2}(\Gamma)$-elliptic (in the case $d = 2$, this can be enforced, for example, by the scaling requirement $\operatorname{diam}(\Omega) < 1$). For $f \in H^1(\Gamma)$, let $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ be the unique solution of the weakly singular integral equation $\mathfrak{V}\phi = f$. Let $X_h \subset L^2(\Gamma)$ be a discrete space which contains at least the piecewise constants, i.e., $\mathcal{P}^0(\mathcal{T}_h) \subseteq X_h$, and let $\Phi_h \in X_h$ be the unique Galerkin approximation of ϕ in X_h, i.e.,

$$\langle \mathfrak{V}(\phi - \Phi_h)\,,\, \Psi_h\rangle_\Gamma = 0 \quad \text{for all } \Psi_h \in X_h. \tag{3.9}$$

Under these assumptions (and, strictly speaking, for polyhedral Γ), [CMS01] proves the reliability estimate

$$C_{\rm rel}^{-1}\,\|\phi - \Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq \eta_{h,\mathfrak{V}} := \|h^{1/2}\nabla_\Gamma(f - \mathfrak{V}\Phi_h)\|_{L^2(\Gamma)}. \tag{3.10}$$

The constant $C_{\rm rel} > 0$ depends only on Γ and the κ-shape regularity of $\mathcal{T}_h$. The following corollary provides the converse efficiency estimate with respect to some slightly stronger weighted L^2-norm. We note that the additional assumption $\phi = \mathfrak{V}^{-1}f \in L^2(\Gamma)$ is in particular satisfied for $\Gamma = \partial\Omega$.

Corollary 3.4 (Efficiency of $\eta_{h,\mathfrak{V}}$ for weakly singular integral equations)**.** *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ. Assume $\phi = \mathfrak{V}^{-1}f \in L^2(\Gamma)$ and let $X_h \subseteq \widetilde{H}^{-1/2}(\Gamma)$ be a closed subspace with $\mathcal{P}^0(\mathcal{T}_h) \subseteq X_h \subset L^2(\Gamma)$. Let $\Phi_h \in X_h$ be given by* (3.9)*. Then the weighted residual error estimator from* (3.10) *satisfies*

$$\eta_{h,\mathfrak{V}} \leq C_{\rm eff}\,\|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)}, \tag{3.11}$$

where $C_{\rm eff} = \widetilde{C}_{\rm inv} > 0$ is the constant from Corollary 3.3.

Proof. With the Galerkin projection $\mathbb{P}_h : \widetilde{H}^{-1/2}(\Gamma) \to X_h$ and $\Phi_h = \mathbb{P}_h\phi$, Corollary 3.3 yields $\eta_{h,\mathfrak{V}} = \|h^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_h)\|_{L^2(\Gamma)} \lesssim \|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)}$. □

Remark 3.5 (Stronger efficiency of 2D BEM). *While the efficiency estimate* (3.11) *involves a slightly stronger norm on the right-hand side, particular situations (as, e.g., the 2D direct BEM formulation of the Dirichlet problem* [AFF+13b]*) permit to bound* $\|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)}$ *by* $\|\phi - \Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)}$ *up to higher-order terms. In* [AFF+13b]*, this is achieved by decomposing* ϕ *in a singular part associated with the vertices of* Ω *and a regular part; the higher-order terms depend only on the regular part of* ϕ. ∎

3.2.2. *Hypersingular integral equations.* Results similar to Corollary 3.3 also hold for the double-layer integral operator $\mathfrak{K}$ and the hypersingular integral operator $\mathfrak{W}$. Here, particularly interesting choices for the projection $\mathbb{P}_h$ are Scott-Zhang type projections onto $\widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h)$; see [SZ90] as well as the adaptation to BEM in [AFF+14, Section 3.2].

Corollary 3.6. *Let* $\mathcal{T}_h$ *be a regular,* κ*-shape regular triangulation of* Γ*. Let* $X_h \subseteq \widetilde{H}^{1/2}(\Gamma)$ *be a closed subspace with* $\widetilde{\mathcal{S}}^1(\mathcal{T}_h) \subseteq X_h \subseteq \widetilde{H}^1(\Gamma)$*. Let* $\mathbb{P}_h : \widetilde{H}^{1/2}(\Gamma) \to X_h$ *be an* $\widetilde{H}^{1/2}(\Gamma)$*-stable projection onto* X_h*. Then, for all* $v \in \widetilde{H}^1(\Gamma)$,

$$\|h^{1/2}\nabla_\Gamma \mathfrak{K}(1-\mathbb{P}_h)v\|_{L^2(\Gamma)} + \|h^{1/2}\mathfrak{W}(1-\mathbb{P}_h)\phi\|_{L^2(\Gamma)} \le \widetilde{C}_{\rm inv}\|h^{1/2}\nabla_\Gamma(1-\mathbb{P}_h)v\|_{L^2(\Gamma)}. \quad (3.12)$$

The constant $\widetilde{C}_{\rm inv} > 0$ *depends only on the* κ*-shape regularity of* $\mathcal{T}_h$*, the stability constant of* $\mathbb{P}_h$*, and* Γ.

Proof. Arguing along the lines of the proof of Corollary 3.3, we first consider the Scott-Zhang projection $J_h : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{\mathcal{S}}^1(\mathcal{T}_h)$ onto $\widetilde{\mathcal{S}}^1(\mathcal{T}_h)$. According to [AFF+14, Lemma 7] (strictly speaking, this result is formulated for polygonal boundaries only, but the proof transfers with minor changes to the present case), it holds

$$\|(1-J_h)w\|_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \min_{W_h \in \widetilde{\mathcal{S}}^1(\mathcal{T}_h)} \|h^{1/2}\nabla(w-W_h)\|_{L^2(\Gamma)} \le \|h^{1/2}\nabla(1-J_h)w\|_{L^2(\Gamma)} \quad (3.13)$$

for all $w \in \widetilde{H}^1(\Gamma)$. The hidden constant depends only on Γ and the κ-shape regularity of $\mathcal{T}_h$. Combining this with the inverse estimate (3.2) for $v = (1-J_h)w$, we arrive at

$$\|h^{1/2}\nabla_\Gamma\mathfrak{K}(1-J_h)w\|_{L^2(\Gamma)} + \|h^{1/2}\mathfrak{W}(1-J_h)w\|_{L^2(\Gamma)} \lesssim \|h^{1/2}\nabla_\Gamma(1-J_h)w\|_{L^2(\Gamma)} \text{ for all } w \in \widetilde{H}^1(\Gamma).$$

The same arguments apply for any $\widetilde{H}^{1/2}$-stable projection $\mathbb{P}_h : \widetilde{H}^{1/2}(\Gamma) \to X_h$, but additionally employ its stability. □

As in Section 3.2.1, an immediate consequence of Corollary 3.6 is the efficiency of the weighted residual error estimator η_h from [Car97, CMPS04] for the hypersingular integral equation: Suppose that $\widetilde{H}^{1/2}(\Gamma)$ does not contain any characteristic function χ_ω for $\omega \subseteq \Gamma$ (this is in particular satisfied if $\partial\Omega$ is connected and $\Gamma \subsetneqq \partial\Omega$). Then, $\mathfrak{W} : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is an elliptic isomorphism. For $f \in L^2(\Gamma)$, let $u \in \widetilde{H}^{1/2}(\Gamma)$ be the unique solution of the hypersingular integral equation $\mathfrak{W}u = f$. Let $X_h \subseteq \widetilde{H}^1(\Gamma)$ be a discrete space which contains at least the piecewise affines, i.e., $\widetilde{\mathcal{S}}^1(\mathcal{T}_h) \subseteq X_h$. In addition, let $U_h \in X_h$ be the unique Galerkin approximation of u in X_h, i.e.,

$$\langle \mathfrak{W}(u-U_h)\,,\, V_h\rangle_\Gamma = 0 \quad \text{for all } V_h \in X_h. \quad (3.14)$$

Under these assumptions (and, strictly speaking, for polyhedral Γ), [CMPS04] proves the reliability estimate

$$C_{\rm rel}^{-1}\,\|u-U_h\|_{\widetilde{H}^{1/2}(\Gamma)} \le \eta_{h,\mathfrak{W}} := \|h^{1/2}(f-\mathfrak{W}U_h)\|_{L^2(\Gamma)}. \quad (3.15)$$

The constant $C_{\mathrm{rel}} > 0$ depends only on Γ and the κ-shape regularity of $\mathcal{T}_h$. The following corollary provides the converse efficiency estimate with respect to some slightly stronger weighted H^1-seminorm.

Corollary 3.7 (Efficiency of η_h for hypersingular integral equations)**.** *Let $u = \mathfrak{W}^{-1}f \in \widetilde{H}^1(\Gamma)$. Let $X_h \subseteq \widetilde{H}^{1/2}(\Gamma)$ be closed with $\widetilde{\mathcal{S}}^1(\mathcal{T}_h) \subseteq X_h \subseteq \widetilde{H}^1(\Gamma)$. Let $U_h \in X_h$ be given by* (3.14)*. Then the weighted residual error estimator from* (3.15) *satisfies*

$$\eta_{h,\mathfrak{W}} \leq C_{\mathrm{eff}} \, \|h^{1/2}\nabla_\Gamma(u - U_h)\|_{L^2(\Gamma)}, \tag{3.16}$$

where $C_{\mathrm{eff}} = \widetilde{C}_{\mathrm{inv}} > 0$ is the constant from Corollary 3.6.

Proof. With the Galerkin projection $\mathbb{P}_h : \widetilde{H}^{1/2}(\Gamma) \to X_h$ and $U_h = \mathbb{P}_h u$, Corollary 3.6 yields $\eta_{h,\mathfrak{W}} = \|h^{1/2}\mathfrak{W}(u - U_h)\|_{L^2(\Gamma)} \lesssim \|h^{1/2}\nabla_\Gamma(u - U_h)\|_{L^2(\Gamma)}$. ☐

Remark 3.8. *If $\Gamma = \partial\Omega$ is connected, the kernel of $\mathfrak{W}$ is the space f constant functions on Γ. Therefore, $\mathfrak{W} : H^{1/2}_\star(\partial\Omega) \to H^{-1/2}_\star(\partial\Omega)$ is an elliptic isomorphism, where $H^s_\star(\partial\Omega) := \big\{v \in H^s(\partial\Omega) \,:\, \langle v\,,\,1\rangle_{\partial\Omega} = 0\big\}$ for $|s| \leq 1$. Recall that $\mathfrak{W} : H^s_\star(\partial\Omega) \to H^{s-1}_\star(\partial\Omega)$ is an isomorphism for all $0 \leq s \leq 1$. For $f \in H^0_\star(\partial\Omega)$, the solution $u := \mathfrak{W}^{-1}f$ thus has additional regularity $u \in H^1_\star(\partial\Omega)$, and Corollary 3.7 holds accordingly.*

3.2.3. *Remarks on the extension to hp-BEM.* The above efficiency statements are formulated for the h-version BEM. They do generalize to the hp-version.
Since the corresponding reliability estimates have only been formulated for closed surfaces $\Gamma = \partial\Omega$ and affine element maps in [KM14], we restrict the following result to this setting:

Corollary 3.9. *Let $d \in \{2, 3\}$, $\Gamma = \partial\Omega$, and $\mathcal{T}_h$ a regular, κ-shape regular triangulation of Γ. Assume that the element maps are affine. Let q_h be a σ-admissible polynomial degree distribution. Then there exists $C > 0$ depending only on $\partial\Omega$, the κ-shape regularity of $\mathcal{T}_h$, and σ such that the following holds:*

(i) Let $\mathfrak{V} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ be an isomorphism. Let $\phi = \mathfrak{V}^{-1}f$ for some $f \in H^1(\partial\Omega)$. Set $X_h := \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$ and let $\Phi_{hp} \in X_h$ be the Galerkin solution given by (3.9)*. Then:*

$$C^{-1}\|\phi - \Phi_{hp}\|_{H^{-1/2}(\partial\Omega)} \leq \eta_{hp,\mathfrak{V}} := \|(h/(1+q_h))^{1/2}\nabla_\Gamma(f - \mathfrak{V}\Phi_{hp})\|_{L^2(\partial\Omega)}, \tag{3.17}$$

$$\eta_{hp,\mathfrak{V}} \leq C\|(h/(1+q_h))^{1/2}(\phi - \Phi_{hp})\|_{L^2(\partial\Omega)}. \tag{3.18}$$

(ii) Let Γ be connected and $u = \mathfrak{W}^{-1}f$ for some $f \in H^0_\star(\partial\Omega)$. Set $X_h := \mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h) \cap H^1_\star(\partial\Omega)$ and let $U_{hp} \in X_h$ be the Galerkin solution given by (3.14)*. Then:*

$$C^{-1}\|u - U_{hp}\|_{H^{1/2}(\partial\Omega)} \leq \eta_{hp,\mathfrak{W}} := \|(h/(1+q_h))^{1/2}(f - \mathfrak{W}U_{hp})\|_{L^2(\partial\Omega)}, \tag{3.19}$$

$$\begin{aligned}\eta_{hp,\mathfrak{W}} \leq C\Big[&\|(h/(1+q_h))^{1/2}\nabla_\Gamma(u - U_{hp})\|_{L^2(\partial\Omega)} \\ &+ \|(h/(1+q_h))^{1/2}(u - U_{hp})\|_{L^2(\partial\Omega)}\Big].\end{aligned} \tag{3.20}$$

Proof. The reliability bounds (3.17), (3.19) are taken from [KM14, Cor. 3.9, Cor. 3.12].
For the proof of (3.18), we let Π_{hp} be the $L^2(\partial\Omega)$-projection and $\mathbb{P}_{hp}$ be the Galerkin projection. We note that the analogue of (3.8) is

$$\|(1 - \mathbb{P}_{hp})\psi\|_{H^{-1/2}(\partial\Omega)} \lesssim \|(1 - \Pi_{hp})\psi\|_{H^{-1/2}(\partial\Omega)} \lesssim \|(h/(1+q_h))^{1/2}\psi\|_{L^2(\partial\Omega)} \qquad \forall \psi \in L^2(\partial\Omega) \tag{3.21}$$

(cf. [KM14, Thm. 3.8]). Hence, proceeding as in the proof of Corollary 3.3 with $w_h = (h/(1+q_h))^{1/2}$ we get

$$\|w_h \mathfrak{V}(1-\mathbb{P}_{hp})\phi\|_{L^2(\partial\Omega)} \lesssim \|(1+q_h)^{-1/2}\|_{L^\infty(\partial\Omega)} \|(1-\mathbb{P}_{hp})^2\phi\|_{H^{-1/2}(\partial\Omega)} + \|w_h(1-\mathbb{P}_h)\phi\|_{L^2(\partial\Omega)}$$
$$\stackrel{(3.21)}{\lesssim} \|(1+q_h)^{-1/2}\|_{L^\infty(\partial\Omega)} \|w_h(1-\mathbb{P}_{hp})\phi\|_{L^2(\partial\Omega)} + \|w_h(1-\mathbb{P}_{hp})\phi\|_{L^2(\partial\Omega)}.$$

The proof of (3.20) proceeds along similar lines. The key is the analog of (3.13). Combining [KM14, Lem. 3.7] and the proof of [KM14, Lem. 3.10] produces an approximation operator $J'_{hp} : H^1(\partial\Omega) \to \mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h)$ with

$$\|(1-J'_{hp})v\|_{H^{1/2}(\partial\Omega)} \lesssim \|(h/(1+q_h))^{1/2}\nabla_\Gamma v\|_{L^2(\partial\Omega)} + \|(h/(1+q_h))^{1/2} v\|_{L^2(\partial\Omega)}.$$

Finally, an operator $J_{hp} : H^1_*(\partial\Omega) \to X_{hp}$ is then obtained by setting $J_{hp}v := J'_{hp}v - \overline{J'_{hp}v}$, where the overbar denotes the average over $\partial\Omega$. It is easy to see that J_{hp} has the same approximation properties as J'_{hp} on the space $H^1_*(\partial\Omega)$. Proceeding as in the proof of Corollary 3.3 or 3.6 finishes the proof. □

4. Far-field and near-field estimates for the simple-layer potential

The proof of Theorem 3.1 is based on decomposing the pertinent potentials into "far-field" and "near-field" contributions. In the present section, we analyze the decomposition for the simple-layer potential and provide inverse estimates for both components. Section 4.2 is concerned with inverse estimates for the near-field parts, which essentially follow from scaling arguments, whereas Section 4.3 deals with the far-field part. Throughout the section, we let

$$\psi \in L^2(\Gamma) \text{ and assume that } \psi \text{ is extended by zero to } \partial\Omega \setminus \overline{\Gamma}, \tag{4.1}$$

i.e., we identify ψ with $E_{0,\Gamma}\psi$.

4.1. Decomposition into near-field and far-field. For a parameter $\delta > 0$, we define for each element $T \in \mathcal{T}_h$ the neighborhood U_T of T by

$$T \subset U_T := \bigcup_{x\in T} B_{2\delta h(T)}(x). \tag{4.2}$$

Since $\partial\Omega$ is Lipschitz and Γ stems from a Lipschitz dissection and by κ-shape regularity of $\mathcal{T}_h$, we can fix the parameter $\delta > 0$ and find $M \in \mathbb{N}$ (both δ and M are independent of $\mathcal{T}_h$) such that the following two conditions are satisfied:

(a) $\Gamma \cap U_T$ is contained in the patch $\omega_h(T)$ of T (see (2.15) for the definition), i.e.,

$$\Gamma \cap U_T \subseteq \omega_h(T). \tag{4.3}$$

(b) The covering $\Gamma \subseteq \bigcup_{T\in\mathcal{T}_h} U_T$ is locally finite with a uniform bound, i.e.,

$$\#\{U_T : T \in \mathcal{T}_h \text{ and } x \in U_T\} \le M \quad \text{for all } x \in \mathbb{R}^d. \tag{4.4}$$

Finally, we fix a bounded domain $U \subset \mathbb{R}^d$ such that

$$U_T \subset U \quad \text{for all } T \in \mathcal{T}_h. \tag{4.5}$$

It will be important that U is chosen independently of $\mathcal{T}_h$. To deal with the non-locality of the integral operators, we define for $T \in \mathcal{T}_h$ the near-field $u_{\mathfrak{V},T}^{\text{near}}$ and the far-field $u_{\mathfrak{V},T}^{\text{far}}$ of the simple-layer potential $u_{\mathfrak{V}} := \widetilde{\mathfrak{V}}\psi$ by

$$u_{\mathfrak{V},T}^{\text{near}} := \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\cap U_T}) \quad \text{and} \quad u_{\mathfrak{V},T}^{\text{far}} := \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\setminus U_T}), \tag{4.6}$$

where χ_ω denotes the characteristic function of the set $\omega \subseteq \mathbb{R}^d$. We have the obvious identity

$$u_{\mathfrak{V}} = \widetilde{\mathfrak{V}}\psi = u_{\mathfrak{V},T}^{\text{near}} + u_{\mathfrak{V},T}^{\text{far}} \quad \text{for all } T \in \mathcal{T}_h. \tag{4.7}$$

In our analysis, we will treat $u_{\mathfrak{V},T}^{\text{near}}$ and $u_{\mathfrak{V},T}^{\text{far}}$ separately, starting with the simpler case of $u_{\mathfrak{V},T}^{\text{near}}$.

4.2. Inverse estimates for the near-field part $u_{\mathfrak{V},T}^{\text{near}}$. The near-field parts of a potential can be treated with local arguments and the stability properties of the associated boundary integral operators.

Lemma 4.1. *There exists a constant $\widetilde{C}_{\text{near}} > 0$ depending only on $\partial\Omega$, Γ, and the κ-shape regularity of $\mathcal{T}_h$ such that for arbitrary $T \in \mathcal{T}_h$ and $\Psi_h^T \in \mathcal{P}^0(\mathcal{T}_h)$ with $\operatorname{supp}(\Psi_h^T) \subseteq \overline{\omega_h(T)}$ it holds*

$$\|\nabla\widetilde{\mathfrak{V}}\Psi_h^T\|_{L^2(U_T)} \le \widetilde{C}_{\text{near}}\|h^{1/2}\Psi_h^T\|_{L^2(\omega_h(T))}.$$

Proof. We fix an element $T \in \mathcal{T}_h$. We recall that Ψ_h^T is piecewise constant and compute for $x \in \Omega$

$$(\nabla\widetilde{\mathfrak{V}}\Psi_h^T)(x) = \sum_{T' \in \omega_h(T)} \Psi_h^T|_{T'} \int_{T'} \nabla_x G(x,y)\,dy \quad \text{for all } x \in \mathbb{R}^d \setminus \Gamma.$$

The number of elements T' in the patch $\omega_h(T)$ is bounded in terms of the shape regularity constant κ (cf. Lemma 2.7). With some constant that depends only on κ, we bound

$$|(\nabla\widetilde{\mathfrak{V}}\Psi_h^T)(x)|^2 \lesssim \sum_{T' \in \omega_h(T)} |\Psi_h^T|_{T'}|^2 \Big(\int_{T'} \big|\nabla_x G(x,y)\big|\,dy\Big)^2. \tag{4.8}$$

Next, we show for elements $T' \subseteq \omega_h(T)$

$$\int_{U_T} \Big(\int_{T'} \big|\nabla_x G(x,y)\big|\,dy\Big)^2 dx \lesssim h(T)^d. \tag{4.9}$$

This follows from a local Lipschitz parametrization of $\partial\Omega$. We assume that (after possibly a Euclidean change of coordinates) that $\{(x', \Lambda(x'))\colon x' \in B_{2r}(0)\}$ is a part of $\partial\Omega$ that contains $\omega_h(T)$. The function Λ is Lipschitz continuous, and we remark in passing that by [Ste70, Thm. 3, Sect. VI] we may assume that $\Lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz continuous. (If such a local consideration is not possible, then, since the number of local charts is finite by definition of bounded Lipschitz domains, we must have $\operatorname{diam}(\omega_h(T)) = O(1)$ so that (4.9) is trivially true.) We may also assume that $U_T \subset \{(x', \Lambda(x') + t) \,|\, x' \in B_{2r}(0), t \in \mathbb{R}\}$. The key observation is that the mapping $\widetilde{\Lambda} : \mathbb{R}^d \to \mathbb{R}^d$ given by $(x', t) \mapsto (x', \Lambda(x') + t)$ is bilipschitz. We conclude that $\widetilde{\Lambda}^{-1}U_T =: \widetilde{U}_T$ and $\widetilde{\Lambda}^{-1}T' =: \widetilde{T}' \subseteq B_r(0) \times \{0\}$ satisfy for some x_0 and some $c > 0$, which depends solely on the bilipschitz mapping $\widetilde{\Lambda}$,

$$\widetilde{U}_T \subseteq B_{ch(T)}(x_0) \times [-ch(T), ch(T)], \qquad \widetilde{T}' \subseteq B_{ch(T)}(x_0) \times \{0\}.$$

Finally, using $\nabla_x G(x,y) \simeq |x-y|^{-(d-1)}$, the definition of the surface integral, and the change of variables formula for bilipschitz mappings from [EG92, Sec. 3.3.3], we get

$$\begin{aligned}
\int_{x\in U_T} \left(\int_{y\in T'} |\nabla_x G(x,y)|\, dy \right)^2 dx &\simeq \int_{\widetilde{x}\in \widetilde{U}_T} \left(\int_{\widetilde{y}\in \widetilde{T}'} |\widetilde{x}-\widetilde{y}|^{-(d-1)}\, d\widetilde{y} \right)^2 d\widetilde{x} \\
&\lesssim \int_{\xi\in B_{ch(T)}(x_0)} \int_{t=-ch(T)}^{ch(T)} \left(\int_{\eta\in B_{ch(T)}(x_0)} \left(|\xi-\eta|^2+t^2\right)^{-(d-1)/2} d\eta \right)^2 dt\, d\xi \\
&\simeq h(T)^d \int_{\xi\in B_c(x_0)} \int_{t=-c}^{c} \left(\int_{\eta\in B_c(x_0)} \left(|\xi-\eta|^2+t^2\right)^{-(d-1)/2} d\eta \right)^2 dt\, d\xi \\
&\simeq h(T)^d,
\end{aligned}$$

where the last estimate follows by a direct estimation of the integrals, which is independent of $h(T)$. We have thus shown (4.9). Inserting (4.9) in (4.8) gives

$$\int_{U_T} |(\nabla \widetilde{\mathfrak{V}} \Psi_h^T)(x)|^2\, dx \lesssim \sum_{T'\in\omega_h(T)} |\Psi_h^T|_{T'}|^2 h(T)^d \simeq \|h^{1/2}\Psi_h^T\|^2_{L^2(\omega_h(T))}. \qquad \square$$

Proposition 4.2 (Near-field bound for $\widetilde{\mathfrak{V}}$)**.** *Let w_h be a σ-admissible weight function. There exists a constant $C_{\mathrm{near}} > 0$ depending only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and σ, such that the near-field part $u^{\mathrm{near}}_{\mathfrak{V},T}$ satisfies $u^{\mathrm{near}}_{\mathfrak{V},T} \in H^1(U)$ and $\gamma_0^{\mathrm{int}} u^{\mathrm{near}}_{\mathfrak{V},T} \in H^1(\Gamma)$ together with*

$$\sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \gamma_0^{\mathrm{int}} u^{\mathrm{near}}_{\mathfrak{V},T}\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|\nabla u^{\mathrm{near}}_{\mathfrak{V},T}\|^2_{L^2(U_T)} \le C_{\mathrm{near}}\, \|w_h\psi\|^2_{L^2(\Gamma)}. \quad (4.10)$$

Proof. The stability (2.9) of $\mathfrak{V}: L^2(\partial\Omega) \to H^1(\partial\Omega)$ proved in [Ver84] gives, for each $T\in\mathcal{T}_h$,

$$\|\nabla_\Gamma \gamma_0^{\mathrm{int}} u^{\mathrm{near}}_{\mathfrak{V},T}\|_{L^2(T)} \le \|\mathfrak{V}(\psi\chi_{U_T\cap\Gamma})\|_{H^1(\partial\Omega)} \lesssim \|\psi\chi_{U_T\cap\Gamma}\|_{L^2(\partial\Omega)} = \|\psi\|_{L^2(U_T\cap\Gamma)}.$$

Summing the last estimate over all $T\in\mathcal{T}_h$ and using (4.3)–(4.4), and σ-admissibility of w_h, we arrive at

$$\sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \gamma_0^{\mathrm{int}} u^{\mathrm{near}}_{\mathfrak{V},T}\|^2_{L^2(T)} \lesssim \sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\psi\|^2_{L^2(U_T\cap\Gamma)} \simeq \|w_h\psi\|^2_{L^2(\Gamma)},$$

where all estimates depend only on the κ-shape regularity of $\mathcal{T}_h$ and the admissibility constant σ. This bounds the first term on the left-hand side of (4.10). To bound the second term, let Π_h denote the $L^2(\Gamma)$-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$. We decompose the near-field as $u^{\mathrm{near}}_{\mathfrak{V},T} = \widetilde{\mathfrak{V}}(\Pi_h(\psi\chi_{\Gamma\cap U_T})) + \widetilde{\mathfrak{V}}\big((1-\Pi_h)\psi\chi_{\Gamma\cap U_T}\big)$. The condition $\operatorname{supp}(\psi\chi_{\Gamma\cap U_T}) \subseteq \overline{\omega_h(T)}$ implies $\operatorname{supp}(\Pi_h(\psi\chi_{\Gamma\cap U_T})) \subseteq \overline{\omega_h(T)}$ and therefore, taking $\Psi_h^T = \Pi_h(\psi\chi_{\Gamma\cap U_T})$ in Lemma 4.1, we conclude

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} &\|w_h/h^{1/2}\|^2_{L^\infty(T)} \|\nabla\widetilde{\mathfrak{V}}(\Pi_h(\psi\chi_{\Gamma\cap U_T}))\|^2_{L^2(U_T)} \\
&\lesssim \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|h^{1/2}\Pi_h(\psi\chi_{\Gamma\cap U_T})\|^2_{L^2(\omega_h(T))} \qquad (4.11) \\
&\lesssim \|w_h\psi\|^2_{L^2(\Gamma)},
\end{aligned}$$

where we used the local L^2-stability of Π_h in the last estimate. Recalling the stability $\widetilde{\mathfrak{V}} : \widetilde{H}^{-1/2}(\partial\Omega) \to H^1(U)$ of (2.7), the equality (2.4), and the approximation property (3.8) of Π_h, we get

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|\nabla\widetilde{\mathfrak{V}}\big((1-\Pi_h)\psi\chi_{\Gamma\cap U_T}\big)\|^2_{L^2(U_T)} \\
&\qquad\qquad \overset{(2.7)}{\lesssim} \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|(1-\Pi_h)\psi\chi_{\Gamma\cap U_T}\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \qquad (4.12)\\
&\qquad\qquad \overset{(3.8)}{\lesssim} \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|h^{1/2}(\psi\chi_{\Gamma\cap U_T})\|^2_{L^2(\Gamma)} \simeq \|w_h\psi\|^2_{L^2(\Gamma)}.
\end{aligned}
$$

Combining (4.11)–(4.12), we bound $\sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla u^{\rm near}_{\mathfrak{V},T}\|^2_{L^2(U_T)}$ to conclude the estimate in (4.10). $\square$

4.3. Estimates for the far-field part $u^{\rm far}_{\mathfrak{V},T}$. The following lemma is taken from [FKMP13]. For the convenience of the reader and since the same argument underlies the proof of the analogous lemma for the double-layer potential (Lemma 5.3 below), we recall its proof here.

Lemma 4.3 (Caccioppoli inequality for $u^{\rm far}_{\mathfrak{V},T}$)**.** *With $\Omega^{\rm ext} = \mathbb{R}^d \setminus \overline{\Omega}$, the function $u^{\rm far}_{\mathfrak{V},T}$ from (4.6) satisfies $u^{\rm far}_{\mathfrak{V},T}|_\Omega \in C^\infty(\Omega)$, $u^{\rm far}_{\mathfrak{V},T}|_{\Omega^{\rm ext}} \in C^\infty(\Omega^{\rm ext})$, and $u^{\rm far}_{\mathfrak{V},T}|_{U_T} \in C^\infty(U_T)$. Moreover, there exists a constant $C_{\rm cacc} > 0$ depending only on $\partial\Omega$, Γ, and the κ-shape regularity of $\mathcal{T}_h$ such that Hessian matrix $D^2u^{\rm far}_{\mathfrak{V},T}$ satisfies*

$$
\|D^2u^{\rm far}_{\mathfrak{V},T}\|_{L^2(B_{\delta h(T)}(x))} \le C_{\rm cacc}\,\frac{1}{h(T)}\,\|\nabla u^{\rm far}_{\mathfrak{V},T}\|_{L^2(B_{2\delta h(T)}(x))} \quad \text{for all } x\in T\in\mathcal{T}_h. \tag{4.13}
$$

Proof. The statements $u^{\rm far}_{\mathfrak{V},T}|_\Omega \in C^\infty(\Omega)$ and $u^{\rm far}_{\mathfrak{V},T}|_{\Omega^{\rm ext}} \in C^\infty(\Omega^{\rm ext})$ are taken from [SS11, Theorem 3.1.1], and we therefore focus on $u^{\rm far}_{\mathfrak{V},T}|_{U_T} \in C^\infty(U_T)$ and the estimate (4.13). According to [SS11, Proposition 3.1.7], [SS11, Theorem 3.1.16], and [SS11, Theorem 3.3.1], the function $u^{\rm far}_{\mathfrak{V},T} \in H^1_{\ell oc}(\mathbb{R}^d) := \big\{v : \mathbb{R}^d \to \mathbb{R} \,:\, v|_K \in H^1(K) \text{ for all } K\subset\mathbb{R}^d \text{ compact}\big\}$ solves the transmission problem

$$
\begin{array}{rcll}
-\Delta u^{\rm far}_{\mathfrak{V},T} &=& 0 & \text{in } \Omega\cup\Omega^{\rm ext},\\
{[u^{\rm far}_{\mathfrak{V},T}]} &=& 0 & \text{in } H^{1/2}(\partial\Omega),\\
{[\gamma_1 u^{\rm far}_{\mathfrak{V},T}]} &=& -\psi\chi_{\Gamma\setminus U_T} & \text{in } H^{-1/2}(\partial\Omega).
\end{array} \tag{4.14}
$$

In particular, (4.14) states that the jump of $u^{\rm far}_{\mathfrak{V},T}$ as well as the jump of the normal derivative vanish on $\partial\Omega\cap U_T$. This implies that $u_{\mathfrak{V},T}$ is harmonic in U_T by the following classical argument: First, we observe that $u^{\rm far}_{\mathfrak{V},T}$ is distributionally harmonic in U_T, since a two-fold integration by parts that uses these jump conditions shows for $v\in C_0^\infty(U_T)$ that $\langle u^{\rm far}_{\mathfrak{V},T}, -\Delta v\rangle_\Omega = 0$. Weyl's lemma (see, e.g., [Mor08, Theorem 2.3.1]) then implies that $u^{\rm far}_{\mathfrak{V},T}$ is therefore strongly harmonic and $u^{\rm far}_{\mathfrak{V},T}\in C^\infty(U_T)$.

The Caccioppoli inequality (4.13) now expresses interior regularity for elliptic problems. Indeed, for each $u\in H^1(B_{r+\varepsilon})$ such that $u\in H^2(B_r)$ and $\Delta u = f$ on $B_{r+\varepsilon}$ with balls $B_r \subseteq B_{r+\varepsilon}$ with radii $0<r<r+\varepsilon$ and some $f\in L^2(B_{r+\varepsilon})$, [Mor08, Lemma 5.7.1] shows

$$
\|D^2u\|_{L^2(B_r)} \lesssim \Big(\|f\|_{L^2(B_{r+\varepsilon})} + \frac{1}{\varepsilon}\,\|\nabla u\|_{L^2(B_{r+\varepsilon})} + \frac{1}{\varepsilon^2}\|u\|_{L^2(B_{r+\varepsilon})}\Big). \tag{4.15}
$$

The hidden constant depends solely on the spatial dimension and is independent of $r, \varepsilon > 0$, and u, f. We apply (4.15) with $r = \delta h(T) = \varepsilon$, $f = 0$, and $u = u^{\mathrm{far}}_{\mathfrak{V},T} - c_T$, where $c_T = \frac{1}{|B_{2\delta h(T)}(x)|} \int_{B_{2\delta h(T)}(x)} u^{\mathrm{far}}_{\mathfrak{V},T}(y)\, dy$. An additional Poincaré inequality finally leads to (4.13). Note that δ and hence C_{cacc} depend only on $\partial\Omega$, Γ, and the κ-shape regularity of $\mathcal{T}_h$. $\square$

The non-local character of the operator $\widetilde{\mathfrak{V}}$ is represented by the far-field part. Lemma 4.3 allows us to show a local inverse estimate for the far-field part of the simple-layer operator:

Lemma 4.4 (Local far-field bound for $\widetilde{\mathfrak{V}}$)**.** *For all $T \in \mathcal{T}_h$, it holds*

$$\|h^{1/2} \nabla_\Gamma \gamma_0^{\mathrm{int}} u^{\mathrm{far}}_{\mathfrak{V},T}\|_{L^2(T)} \leq \|h^{1/2} \nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|_{L^2(T)} \leq C_{\mathrm{far}} \|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|_{L^2(U_T)}. \tag{4.16}$$

The constant $C_{\mathrm{far}} > 0$ depends only on Γ, $\partial\Omega$ and the κ-shape regularity constant of $\mathcal{T}_h$.

Proof. By Lemma 4.3 we have $u^{\mathrm{far}}_{\mathfrak{V},T} \in C^\infty(U_T)$. The first estimate in (4.16) follows from the fact that, for smooth functions, the surface gradient $\nabla_\Gamma(\cdot)$ is the orthogonal projection of the gradient $\nabla(\cdot)$ onto the tangent plane, i.e., $\nabla_\Gamma \gamma_0^{\mathrm{int}} u(x) = \nabla u(x) - \big(\nabla u(x) \cdot \boldsymbol{\nu}(x)\big)\, \boldsymbol{\nu}(x)$, see [Ver84].

The second estimate in (4.16) is proved with a trace inequality and the Caccioppoli inequality (4.13) in the following way. We fix an element $T \in \mathcal{T}_h$.
1. step: We provide a trace inequality. Let $B = B_r(x)$ be a ball with center $x \in T \subseteq \partial\Omega$ and radius $r > 0$. Let $B' = B_{3r/2}(x)$ and $B'' := B_{5r/4}(x)$. We define a smooth cut-off function $\widetilde{\chi}_B \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \widetilde{\chi}_B \subseteq B'$ and $\widetilde{\chi}_B \equiv 1$ on B by

$$\widetilde{\chi}_B := \chi_{B''} \star \rho_{r/4},$$

where $\rho_\varepsilon(x)$ is a standard mollifier of the form $\rho_\varepsilon(x) = \varepsilon^{-d} \rho_1(x/\varepsilon)$ for a fixed $\rho_1 \in C_0^\infty(\mathbb{R}^d)$ with $\rho_1 \geq 0$, $\operatorname{supp} \rho_1 \subseteq B_1(0)$ and $\int_{\mathbb{R}^d} \rho_1(x)\, dx = 1$. We note that for a $C > 0$ depending solely on the choice of ρ_1, we have

$$\|\nabla \widetilde{\chi}_B\|_{L^\infty(\mathbb{R}^d)} \leq C r^{-1}.$$

With this cut-off function in hand, we estimate for sufficiently regular functions v and the standard multiplicative trace inequality for $\partial\Omega$

$$\begin{aligned} \|v\|^2_{L^2(B \cap \partial\Omega)} &\leq \|\widetilde{\chi}_B v\|^2_{L^2(\partial\Omega)} \lesssim \|\widetilde{\chi}_B v\|^2_{L^2(\Omega)} + \|\widetilde{\chi}_B v\|_{L^2(\Omega)} \|\nabla(\widetilde{\chi}_B v)\|_{L^2(\Omega)} \\ &\lesssim r^{-1} \|v\|^2_{L^2(B')} + \|v\|_{L^2(B')} \|\nabla v\|_{L^2(B')}. \end{aligned} \tag{4.17}$$

2. step: The set $\mathcal{F} := \left\{ \overline{B_{\delta h(T)/2}(x)} \mid x \in T \right\}$ is a closed cover of T with $\sup_{B \in \mathcal{F}} \operatorname{diam}(B) < \infty$, and T is the set of their midpoints. According to Besicovitch's covering theorem, cf. [EG92, Sect. 1.5.2], there is a constant N_d, which depends only on the spatial dimension d, as well as countable subsets $\mathcal{G}_j \subseteq \mathcal{F}$, $j = 1, \dots, N_d$, the elements of every $\mathcal{G}_j$ being pairwise disjoint, such that $T \subseteq \bigcup_{j=1}^{N_d} \bigcup_{B \in \mathcal{G}_j} B$. Let $\widehat{\mathcal{G}}_j$ be the set of balls obtained by doubling the radius of the balls of $\mathcal{G}_j$, i.e., $\widehat{\mathcal{G}}_j := \left\{ B_{\delta h(T)}(x) \mid B_{\delta h(T)/2}(x) \in \mathcal{G}_j \right\}$. As the elements of $\mathcal{G}_j$ are pairwise disjoint and all balls have the same radius $\delta h(T)/2$, there is a constant $\widehat{N}_d$, also depending only on the spatial dimension d, such that at most $\widehat{N}_d$ elements of $\widehat{\mathcal{G}}_j$ overlap. If we write $B := B_{\delta h(T)/2}(x)$, $B' := B_{3/4\delta h(T)}(x)$, and $\widehat{B} := B_{\delta h(T)}(x)$, the multiplicative trace inequality (4.17) and the Caccioppoli inequality (4.13) show

$$\|\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(B\cap T)}^2 \overset{(4.17)}{\lesssim} \frac{1}{h(T)}\|\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(B')}^2 + \|\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(B')}\|D^2 u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(B')} \overset{(4.13)}{\lesssim} \frac{1}{h(T)}\|\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(\widehat{B})}^2.$$

3. step: We use the last estimate to get

$$\|\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(T)}^2 \le \sum_{j=1}^{N_d}\sum_{B\in\mathcal{G}_j}\|\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(B\cap T)}^2 \lesssim \frac{1}{h(T)}\sum_{j=1}^{N_d}\sum_{\widehat{B}\in\widehat{\mathcal{G}}_j}\|\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(\widehat{B})}^2 \lesssim \frac{N_d\widehat{N}_d}{h(T)}\|\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(U_T)}^2.$$

This concludes the proof of (4.16). □

Summation of the elementwise estimates of Lemma 4.4 yields the following result:

Proposition 4.5 (Far-field bound for $\widetilde{\mathfrak{V}}$)**.** *There is a constant $C_{\mathrm{far}} > 0$ depending only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and the σ-admissibility of the weight function w_h such that*

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\|w_h\nabla_\Gamma\gamma_0^{\mathrm{int}}u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(T)}^2 &\le \sum_{T\in\mathcal{T}_h}\|w_h\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(T)}^2\\ &\le C_{\mathrm{far}}\Big(\|w_h/h^{1/2}\|_{L^\infty(\Gamma)}^2\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 + \|w_h\psi\|_{L^2(\Gamma)}^2\Big).\end{aligned}$$

Proof. We use the local far-field bound (4.16) of Lemma 4.4 and $u_{\mathfrak{V},T}^{\mathrm{far}} = \widetilde{\mathfrak{V}}\psi - u_{\mathfrak{V},T}^{\mathrm{near}}$ to see

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\|w_h\nabla_\Gamma\gamma_0^{\mathrm{int}}u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(T)}^2 &\le \sum_{T\in\mathcal{T}_h}\|w_h\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(T)}^2 \overset{(4.16)}{\lesssim} \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|_{L^\infty(T)}^2\|\nabla u_{\mathfrak{V},T}^{\mathrm{far}}\|_{L^2(U_T)}^2\\ &\overset{(4.7)}{\lesssim} \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|_{L^\infty(T)}^2\|\nabla\widetilde{\mathfrak{V}}\psi\|_{L^2(U_T)}^2 + \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|_{L^\infty(T)}^2\|\nabla u_{\mathfrak{V},T}^{\mathrm{near}}\|_{L^2(U_T)}^2.\end{aligned}\tag{4.18}$$

The first term on the right-hand side in (4.18) is estimated by stability of $\widetilde{\mathfrak{V}}$, the finite overlap property (4.4), and (2.4)

$$\sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|_{L^\infty(T)}^2\|\nabla\widetilde{\mathfrak{V}}\psi\|_{L^2(U_T)}^2 \overset{(4.4)}{\lesssim} \|w_h/h^{1/2}\|_{L^\infty(\Gamma)}^2\|\nabla\widetilde{\mathfrak{V}}\psi\|_{L^2(U)}^2 \lesssim \|w_h/h^{1/2}\|_{L^\infty(\Gamma)}^2\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}^2.$$

The second term in (4.18) is bounded with the near-field bound (4.10). □

5. Far-field and near-field estimates for the double-layer potential

Section 4 studied far-field and near-field estimates for the simple-layer potential. Corresponding results for the double-layer potential are derived in the present section. Throughout this section, let

$$v \in \widetilde{H}^1(\Gamma) \subset H^1(\partial\Omega).$$

In particular, $v \in \widetilde{H}^{1/2}(\Gamma)$ with $\|v\|_{\widetilde{H}^{1/2}(\Gamma)} = \|v\|_{H^{1/2}(\partial\Omega)}$. Since $H^{1/2}$ does not allow jumps, the splitting into near-field and far-field contribution of the double-layer potential $u_{\mathfrak{K}} := \widetilde{\mathfrak{K}}v$ cannot be done by characteristic functions, but requires smoother cut-off functions and greater technical care.

5.1. Decomposition into near-field and far-field. We use the notation introduced in Section 4.1 concerning the neighborhoods U_T. In order to define the near-field and far-field parts for the double-layer potential, we need appropriate cut-off functions: For each $T \in \mathcal{T}_h$, we define $\eta_T \in C_0^\infty(\mathbb{R}^d)$ with the aid of the standard mollifier ρ_ε that was already used in the proof of Lemma 4.4:

$$\eta_T := \chi_{\widetilde{U}_T} \star \rho_{\delta/4h(T)}, \qquad \widetilde{U}_T := \cup_{x\in T} B_{\delta/2h(T)}, \qquad U'_T := \cup_{x\in T} B_{\delta/4h(T)}. \tag{5.1}$$

This function satisfies:

$$\operatorname{supp} \eta_T \subseteq U_T, \quad \eta_T|_{U'_T} \equiv 1, \quad \|\eta_T\|_{L^\infty(\mathbb{R}^d)} \leq 1, \quad \|\nabla \eta_T\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{h(T)}, \tag{5.2}$$

where the implied constant depends on the κ-shape regularity of the triangulation through the parameter δ. We note that the assumptions on U_T imply $(\operatorname{supp} \eta_T) \cap \Gamma \subseteq \overline{\omega_h(T)}$.

The following lemma may be viewed as an extension of [DS80, Thm. 7.1] to the case of curved elements.

Lemma 5.1 (Poincaré-Friedrichs inequality on patches)**.** *Let $v \in \widetilde{H}^1(\Gamma)$. For each $T \in \mathcal{T}_h$, there is a constant $v_T \in \mathbb{R}$ such that $(v - v_T)\eta_T \in \widetilde{H}^1(\Gamma)$, $(v - v_T)(1 - \eta_T) \in H^1(\partial\Omega)$, and*

$$\|v - v_T\|_{L^2(\omega_h(T))} \leq C_1 \|h \nabla_\Gamma v\|_{L^2(\omega_h(T))}, \tag{5.3}$$

$$\|(v - v_T)\eta_T\|_{H^{1/2}(\partial\Omega)} \leq C_1 \|h^{1/2} \nabla_\Gamma v\|_{L^2(\omega_h(T))}, \tag{5.4}$$

$$\|(v - v_T)\eta_T\|_{H^1(\partial\Omega)} \leq C_1 \|\nabla_\Gamma v\|_{L^2(\omega_h(T))}. \tag{5.5}$$

The constant v_T satisfies $v_T = 0$ if $\partial\omega_h(T) \cap \partial\Gamma$ contains a facet of the triangulation. The constant $C_1 > 0$ depends only on $\partial\Omega$ and the κ-shape regularity constant of $\mathcal{T}_h$.

Proof. It is clear that $(v - v_T)(1 - \eta_T) \in H^1(\partial\Omega)$, since $v \in \widetilde{H}^1(\Gamma)$ and η_T is smooth. The remaining statements require more care.

1. step: For $v \in \widetilde{H}^1(\Gamma)$ and a facet $f \in \mathcal{F}_h$ of the triangulation $\mathcal{T}_h$ (recall that facets are images of $(d-2)$-faces of T_{ref} under the element map) denote by $\ell_f(v)$ the average of v on f. As $\ell_f(1) = 1$, we can use the Deny-Lions lemma on the reference element, and the assumptions on the element maps then imply

$$\|v - \ell_f(v)\|_{L^2(T)} \lesssim h(T) \|\nabla_\Gamma v\|_{L^2(T)} \qquad \text{if } f \text{ is a facet of } T, \tag{5.6}$$

$$|\ell_{f_1}(v) - \ell_{f_2}(v)| \lesssim h(T)^{1-(d-1)/2} \|\nabla_\Gamma v\|_{L^2(T)} \qquad \text{if } f_1,\, f_2 \text{ are two facets of } T. \tag{5.7}$$

2. step: Fix an element $T \in \mathcal{T}_h$.

- If $\eta_T|_{\partial\Gamma} \equiv 0$, then select an arbitrary facet f_T of the element patch $\omega_h(T)$.
- If $\eta_T|_{\partial\Gamma} \not\equiv 0$, then we claim that there exists a facet f of $\omega_h(T)$ with $f \subseteq \partial\Gamma$. To see this, let $x_0 \in \partial\Gamma$ with $\eta_T(x_0) \neq 0$. By continuity of η_T and since $\partial\Gamma$ is covered by facets of the triangulation, we may assume that x_0 is in the interior of a boundary facet f_T. This facet belongs to a unique element T_f of the triangulation; by continuity of η_T, we may assume $\operatorname{supp} \eta_T \cap T_f \neq \emptyset$. Since $(\operatorname{supp} \eta_T) \cap \overline{\Gamma} \subseteq \overline{\omega_h(T)}$, we conclude $T_f \subseteq \omega_h(T)$ and thus the boundary facet f_T is a facet of $\omega_h(T)$.

Set $v_T := \ell_{f_T}(v)$. An immediate consequence of $v \in \widetilde{H}^1(\Gamma)$ is that $v_T = 0$ if η_T does not vanish on $\partial\Gamma$. Since η_T is smooth, we conclude $(v - v_T)\eta_T \in \widetilde{H}^1(\Gamma)$. In fact, viewed as a function on $\partial\Omega$, we have

$$\operatorname{supp}((v - v_T)\eta_T) \subseteq \overline{\omega_h(T)}. \tag{5.8}$$

3. step: The bounds (5.6), (5.7) in conjunction with Lemma 2.7 imply

$$\|v - v_T\|_{L^2(\omega_h(T))} \lesssim \|h\nabla_\Gamma v\|_{L^2(\omega_h(T))}, \tag{5.9}$$
$$\|\nabla_\Gamma(v - v_T)\|_{L^2(\omega_h(T))} = \|\nabla_\Gamma v\|_{L^2(\omega_h(T))}, \tag{5.10}$$

where (5.9) is already the claim (5.3). The product rule, (5.2), (5.8), the estimate (5.9), the trivial bound $h(T) \lesssim |T|^{1/(d-1)} \le |\Gamma|^{1/(d-1)} \lesssim 1$ yield

$$\|\nabla_\Gamma\big((v - v_T)\eta_T\big)\|_{L^2(\partial\Omega)} \le \|(v - v_T)\nabla_\Gamma\eta_T\|_{L^2(\omega_h(T))} + \|\eta_T\nabla_\Gamma(v - v_T)\|_{L^2(\omega_h(T))} \lesssim \|\nabla_\Gamma v\|_{L^2(\omega_h(T))},$$

which proves (5.5). It remains to verify (5.4). To that end, we recall the interpolation inequality $\|u\|^2_{H^{1/2}(\partial\Omega)} \lesssim \|u\|_{L^2(\partial\Omega)}\|u\|_{H^1(\partial\Omega)}$ for all $u \in H^1(\partial\Omega)$. Since $\|(v - v_T)\eta_T\|_{L^2(\partial\Omega)} \le \|v - v_T\|_{L^2(\omega_h(T))}$, we get

$$\begin{aligned}\|(v - v_T)\eta_T\|_{H^{1/2}(\partial\Omega)} &\lesssim \|(v - v_T)\eta_T\|^{1/2}_{L^2(\partial\Omega)}\|(v - v_T)\eta_T\|^{1/2}_{H^1(\partial\Omega)}\\ &\lesssim \|h\nabla_\Gamma v\|^{1/2}_{L^2(\omega_h(T))}\|\nabla_\Gamma v\|^{1/2}_{L^2(\omega_h(T))}\\ &\simeq \|h^{1/2}\nabla_\Gamma v\|_{L^2(\omega_h(T))},\end{aligned}$$

where the last estimate hinges on κ-shape regularity of $\mathcal{T}_h$ (cf. Lemma 2.7, (i)). $\square$

Let $v \in \widetilde{H}^1(\Gamma)$. For each $T \in \mathcal{T}_h$, let v_T be the constant from Lemma 5.1. For each $T \in \mathcal{T}_h$ we define the near-field and the far-field part of the double-layer potential $u_{\mathfrak{K}} = \widetilde{\mathfrak{K}}v$ by

$$u^{\text{near}}_{\mathfrak{K},T} := \widetilde{\mathfrak{K}}\big((v - v_T)\eta_T\big) \quad \text{and} \quad u^{\text{far}}_{\mathfrak{K},T} := \widetilde{\mathfrak{K}}\big((v - v_T)(1 - \eta_T)\big). \tag{5.11}$$

Note that $(v - v_T)\eta_T \in \widetilde{H}^1(\Gamma) \subseteq H^1(\partial\Omega)$ and $(v - v_T)(1 - \eta_T) \in H^1(\partial\Omega)$ so that $u^{\text{near}}_{\mathfrak{K},T}, u^{\text{far}}_{\mathfrak{K},T} \in H^1(U\backslash\partial\Omega)$ are well-defined. Since $\widetilde{\mathfrak{K}}1 \equiv -1$ in Ω and $\widetilde{\mathfrak{K}}1 \equiv 0$ in Ω^{ext}, we have, for every $T \in \mathcal{T}_h$, the identities

$$u_{\mathfrak{K}} + v_T = u^{\text{near}}_{\mathfrak{K},T} + u^{\text{far}}_{\mathfrak{K},T} \quad \text{in } \Omega \quad \text{and} \quad u_{\mathfrak{K}} = u^{\text{near}}_{\mathfrak{K},T} + u^{\text{far}}_{\mathfrak{K},T} \quad \text{in } \Omega^{\text{ext}}. \tag{5.12}$$

5.2. Inverse estimates for the near-field part $u^{\text{near}}_{\mathfrak{K},T}$. The following proposition provides an estimate for the near-field part of the double-layer potential.

Proposition 5.2 (Near-field bound for $\widetilde{\mathfrak{K}}$)**.** *Let w_h be a σ-admissible weight function. There exists a constant $C_{\text{near}} > 0$ depending only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and σ such that the near-field part $u^{\text{near}}_{\mathfrak{K},T}$ satisfies $\gamma_0^{\text{int}}u^{\text{near}}_{\mathfrak{K},T} \in H^1(\Gamma)$, $u^{\text{near}}_{\mathfrak{K},T}|_\Omega \in H^1(\Omega)$, and $u^{\text{near}}_{\mathfrak{K},T}|_{U\backslash\overline{\Omega}} \in H^1(U \setminus \overline{\Omega})$ with*

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)}\Big(\|h^{1/2}\nabla_\Gamma\gamma_0^{\text{int}}u^{\text{near}}_{\mathfrak{K},T}\|^2_{L^2(T)} + \|\nabla u^{\text{near}}_{\mathfrak{K},T}\|^2_{L^2(U_T\cap\Omega)} + \|\nabla u^{\text{near}}_{\mathfrak{K},T}\|^2_{L^2(U_T\cap\Omega^{\text{ext}})}\Big)&\\ \le C_{\text{near}}\,\|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)}.&\end{aligned} \tag{5.13}$$

Proof. Recall stability (2.10) of $\gamma_0^{\text{int}}\widetilde{\mathfrak{K}} = \mathfrak{K} - \frac{1}{2} : H^1(\partial\Omega) \to H^1(\partial\Omega)$. Taking into account (5.2) and the Poincaré-type estimate (5.5), we observe

$$\|\nabla_\Gamma\gamma_0^{\text{int}}u^{\text{near}}_{\mathfrak{K},T}\|_{L^2(T)} \le \|\nabla_\Gamma\gamma_0^{\text{int}}u^{\text{near}}_{\mathfrak{K},T}\|_{L^2(\Gamma)} \overset{(2.10)}{\lesssim} \|(v - v_T)\eta_T\|_{H^1(\partial\Omega)} \overset{(5.5)}{\lesssim} \|\nabla_\Gamma v\|_{L^2(\omega_h(T))}.$$

Summation over all $T \in \mathcal{T}_h$ shows

$$\sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|h^{1/2}\nabla_\Gamma \gamma_0^{\mathrm{int}} u^{\mathrm{near}}_{\widetilde{\mathfrak{K}},T}\|^2_{L^2(T)} \lesssim \|w_h \nabla_\Gamma v\|^2_{L^2(\Gamma)}. \tag{5.14}$$

Next, we use the continuity of $\widetilde{\mathfrak{K}} : H^{1/2}(\partial\Omega) \to H^1(U \setminus \partial\Omega)$ from (2.7) and get

$$\|\nabla u^{\mathrm{near}}_{\widetilde{\mathfrak{K}},T}\|^2_{L^2(U_T\cap\Omega)} + \|\nabla u^{\mathrm{near}}_{\widetilde{\mathfrak{K}},T}\|^2_{L^2(U_T\cap\Omega^{\mathrm{ext}})} \overset{(2.7)}{\lesssim} \|(v - v_T)\eta_T\|^2_{H^{1/2}(\partial\Omega)} \overset{(5.4)}{\lesssim} \|h^{1/2}\nabla_\Gamma v\|^2_{L^2(\omega_h(T))}.$$

Summation over all $T \in \mathcal{T}_h$ gives

$$\sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \Big(\|\nabla u^{\mathrm{near}}_{\widetilde{\mathfrak{K}},T}\|^2_{L^2(U_T\cap\Omega)} + \|\nabla u^{\mathrm{near}}_{\widetilde{\mathfrak{K}},T}\|^2_{L^2(U_T\cap\Omega^{\mathrm{ext}})} \Big) \lesssim \|w_h \nabla_\Gamma v\|^2_{L^2(\Gamma)}. \tag{5.15}$$

Combining (5.14)–(5.15), we conclude the proof. $\square$

5.3. Estimates for the far-field part $u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}$. As for the simple-layer potential, we have a Caccioppoli inequality for the double-layer potential, which underlies the analysis of the far-field contribution.

For the next result, recall U'_T from (5.1).

Lemma 5.3 (Caccioppoli inequality for $u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}$)**.** *For the constant C_{cacc} of Lemma 4.3, the functions $u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}$ of (5.11) satisfy $u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}|_\Omega \in C^\infty(\Omega)$, $u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}|_{\Omega^{\mathrm{ext}}} \in C^\infty(\Omega^{\mathrm{ext}})$, and $u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}|_{U'_T} \in C^\infty(U'_T)$ together with*

$$\|D^2 u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}\|_{L^2(B_{\delta h(T)/8}(x))} \le C_{\mathrm{cacc}} \frac{1}{h(T)} \|\nabla u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}\|_{L^2(B_{\delta h(T)/4}(x))} \quad \text{for all } x \in T \in \mathcal{T}_h. \tag{5.16}$$

Proof. The proof is very similar to that of Lemma 4.3. One observes that the far-field $u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}$ solves the transmission problem

$$\begin{aligned} -\Delta u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T} &= 0 && \text{in } \Omega \cup \Omega^{\mathrm{ext}}, \\ [u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}] &= (v - v_T)(1-\eta_T) && \text{in } H^{1/2}(\partial\Omega), \\ [\gamma_1 u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}] &= 0 && \text{in } H^{-1/2}(\partial\Omega). \end{aligned}$$

We note that $(1-\eta_T)|_{\Gamma\cap U'_T} = 0$ by construction of η_T in (5.2). Hence, the same reasoning as in the proof of Lemma 4.3 can be applied to reach the conclusion (5.16). $\square$

Lemma 5.4 (Local far-field bound for $\widetilde{\mathfrak{K}}$)**.** *For all $T \in \mathcal{T}_h$*

$$\|h^{1/2}\nabla_\Gamma \gamma_0^{\mathrm{int}} u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}\|_{L^2(T)} \le \|h^{1/2}\nabla u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}\|_{L^2(T)} \le C_{\mathrm{far}} \|\nabla u^{\mathrm{far}}_{\widetilde{\mathfrak{K}},T}\|_{L^2(U'_T)}. \tag{5.17}$$

The constant $C_{\mathrm{far}} > 0$ depends only on $\partial\Omega$ and the κ-shape regularity constant of $\mathcal{T}_h$.

Proof. The lemma is shown in exactly the same way as the corresponding bound for the simple-layer potential $\mathfrak{V}$ in Lemma 4.4, appealing to the Caccioppoli inequality (5.16) instead of (4.13). $\square$

Proposition 5.5 (Far-field bound for $\widetilde{\mathfrak{K}}$)**.** *Let w_h be a σ-admissible weight function. There is a constant $C_{\rm far} > 0$ depending only on $\partial\Omega$, the κ-shape regularity constant of $\mathcal{T}_h$, and σ such that*

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} &\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|h^{1/2}\nabla_\Gamma\gamma_0^{\rm int}u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)}\\ &\qquad\qquad\le \sum_{T\in\mathcal{T}_h}\|w_h\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)}\\ &\qquad\qquad\le C_{\rm far}\left(\|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)} + \|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)}\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)}\right).\end{aligned}$$

Proof. Lemma 5.4 implies

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\|w_h\nabla_\Gamma\gamma_0^{\rm int}u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)} &\le \sum_{T\in\mathcal{T}_h}\|w_h\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)} \lesssim \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(U'_T)}\\ = \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(U'_T\cap\Omega)} &+ \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(U'_T\cap\Omega^{\rm ext})}.\end{aligned}\tag{5.18}$$

With the identities (5.12) and a triangle inequality, we therefore obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}&\|w_h\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)}\\ &\overset{(5.18)}{\lesssim}\sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\left(\|\nabla\widetilde{\mathfrak{K}}(v-v_T)\|^2_{L^2(U'_T\cap\Omega)} + \|\nabla\widetilde{\mathfrak{K}}v\|^2_{L^2(U'_T\cap\Omega^{\rm ext})}\right)\\ &\qquad+\sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\left(\|\nabla u^{\rm near}_{\mathfrak{K},T}\|^2_{L^2(U'_T\cap\Omega)} + \|\nabla u^{\rm near}_{\mathfrak{K},T}\|^2_{L^2(U'_T\cap\Omega^{\rm ext})}\right).\end{aligned}\tag{5.19}$$

The near-field contribution is bounded by Proposition 5.2. Furthermore, noting $\nabla\widetilde{\mathfrak{K}}v_T = \nabla(-v_T) = 0$ in Ω, we get

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}&\|w_h\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)}\\ &\overset{(5.13)}{\lesssim}\|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)}\sum_{T\in\mathcal{T}_h}\left(\|\nabla\widetilde{\mathfrak{K}}v\|^2_{L^2(U'_T\cap\Omega)} + \|\nabla\widetilde{\mathfrak{K}}v\|^2_{L^2(U'_T\cap\Omega^{\rm ext})}\right) + \|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)}\\ &\overset{(2.7)}{\lesssim}\|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)}\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)} + \|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)},\end{aligned}$$

where we have used continuity (2.7) of $\widetilde{\mathfrak{K}}$, the overlap property (4.4), and $\|v\|_{\widetilde{H}^{1/2}(\Gamma)} = \|v\|_{H^{1/2}(\partial\Omega)}$. ∎

6. Proof of Theorem 3.1

Finally, we are in a position to prove the inverse estimates (3.1), (3.2) of Theorem 3.1.

Proof of the inverse estimate (3.1). Let $\psi\in L^2(\Gamma)$, extend ψ by zero to the entire boundary $\partial\Omega$, and recall the notation from Section 4.1. First, we treat the simple-layer potential $\mathfrak{V}$.

With the bounds of Propositions 4.2 and 4.5 we get

$$\begin{aligned}
\|w_h \nabla_\Gamma \mathfrak{V}\psi\|^2_{L^2(\Gamma)} &= \sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \mathfrak{V}\psi\|^2_{L^2(T)} \\
&\lesssim \sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \gamma_0^{\mathrm{int}} u^{\mathrm{far}}_{\mathfrak{V},T}\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \gamma_0^{\mathrm{int}} u^{\mathrm{near}}_{\mathfrak{V},T}\|^2_{L^2(T)} \\
&\lesssim \|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)} \|\psi\|^2_{\widetilde{H}^{-1/2}(\Gamma)} + \|w_h\psi\|^2_{L^2(\Gamma)}.
\end{aligned} \tag{6.1}$$

The estimate for the adjoint double-layer potential $\mathfrak{K}'$ follows by similar arguments. We split the left-hand side into near-field and far-field contributions to obtain

$$\|w_h \mathfrak{K}'\psi\|^2_{L^2(\Gamma)} \lesssim \sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{U_T\cap\Gamma})\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{\Gamma\setminus U_T})\|^2_{L^2(T)}. \tag{6.2}$$

The continuity $\mathfrak{K}' : L^2(\partial\Omega) \to L^2(\partial\Omega)$ stated in (2.11) yields for the near-field contribution

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{U_T\cap\Gamma})\|^2_{L^2(T)} &\leq \sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{U_T\cap\Gamma})\|^2_{L^2(\partial\Omega)} \\
&\overset{(2.11)}{\lesssim} \sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\psi\|^2_{L^2(U_T\cap\Gamma)} \lesssim \|w_h\psi\|^2_{L^2(\Gamma)}.
\end{aligned}$$

For the far-field contribution, we write $u^{\mathrm{far}}_{\mathfrak{V},T} = \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\setminus U_T})$ and note that $\mathfrak{K}' = -1/2 + \gamma_1^{\mathrm{int}}\widetilde{\mathfrak{V}}$ and clearly $(\psi\chi_{\Gamma\setminus U_T})|_T = 0$. Therefore, on T we have $\mathfrak{K}'(\psi\chi_{\Gamma\setminus U_T}) = \gamma_1^{\mathrm{int}} u^{\mathrm{far}}_{\mathfrak{V},T}$. Furthermore, by the smoothness of $u^{\mathrm{far}}_{\mathfrak{V},T}$ near T (see Lemma 4.3), we have $\gamma_1^{\mathrm{int}} u^{\mathrm{far}}_{\mathfrak{V},T} = \partial_{\boldsymbol{\nu}} u^{\mathrm{far}}_{\mathfrak{V},T}$ on T (cf. Remark 2.2) and get

$$\|\mathfrak{K}'(\psi\chi_{\Gamma\setminus U_T})\|_{L^2(T)} = \|\gamma_1^{\mathrm{int}} u^{\mathrm{far}}_{\mathfrak{V},T}\|_{L^2(T)} = \|\partial_{\boldsymbol{\nu}} u^{\mathrm{far}}_{\mathfrak{V},T}\|_{L^2(T)} \lesssim \|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|_{L^2(T)}.$$

The far-field contribution in (6.2) can therefore be bounded by Proposition 4.5

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{\Gamma\setminus U_T})\|^2_{L^2(T)} &\lesssim \sum_{T\in\mathcal{T}_h} \|w_h \nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|^2_{L^2(T)} \\
&\lesssim \|w_h\psi\|^2_{L^2(\Gamma)} + \|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)} \|\psi\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.
\end{aligned}$$

Altogether, this gives

$$\|w_h \mathfrak{K}'\psi\|_{L^2(\Gamma)} \lesssim \|w_h\psi\|_{L^2(\Gamma)} + \|w_h/h^{1/2}\|_{L^\infty(\Gamma)} \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}. \qquad \square$$

Proof of inverse estimate (3.2). First, we treat the double-layer potential $\mathfrak{K}$. Let $v \in \widetilde{H}^1(\Gamma)$, extend v by zero to $v \in H^1(\partial\Omega)$, and recall the notation from Section 5.1. We recall the stability of $\mathfrak{K} = \frac{1}{2} + \gamma_0^{\mathrm{int}}\widetilde{\mathfrak{K}} : H^1(\partial\Omega) \to H^1(\partial\Omega)$, from which we conclude $\gamma_0^{\mathrm{int}}\widetilde{\mathfrak{K}}v \in H^1(\Gamma)$. Therefore,

$$\|w_h \nabla_\Gamma \mathfrak{K}v\|_{L^2(\Gamma)} = \|w_h \nabla_\Gamma \big(\tfrac{1}{2} + \gamma_0^{\mathrm{int}}\widetilde{\mathfrak{K}}\big)v\|_{L^2(\Gamma)} \leq \frac{1}{2}\|w_h \nabla_\Gamma v\|_{L^2(\Gamma)} + \|w_h \nabla_\Gamma \gamma_0^{\mathrm{int}} u_{\mathfrak{K}}\|_{L^2(\Gamma)} \tag{6.3}$$

with $u_{\mathfrak{K}} = \widetilde{\mathfrak{K}}v$. There holds $u_{\mathfrak{K}} + v_T = u^{\rm near}_{\mathfrak{K},T} + u^{\rm far}_{\mathfrak{K},T}$ in Ω, cf. (5.12). For the second term on the right-hand side in (6.3), we obtain

$$
\begin{aligned}
\|w_h \nabla_\Gamma \gamma_0^{\rm int} u_{\mathfrak{K}}\|^2_{L^2(\Gamma)} &\le \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|h^{1/2}\nabla_\Gamma\gamma_0^{\rm int}(u_{\mathfrak{K}}+v_T)\|^2_{L^2(T)} \\
&\overset{(5.12)}{\lesssim} \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|h^{1/2}\nabla_\Gamma\gamma_0^{\rm int} u^{\rm near}_{\mathfrak{K},T}\|^2_{L^2(T)} \\
&\quad + \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|h^{1/2}\nabla_\Gamma\gamma_0^{\rm int} u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)}.
\end{aligned} \tag{6.4}
$$

The first sum can be bounded by Proposition 5.2, whereas the second sum can be bounded by Proposition 5.5. Altogether, this yields

$$
\|w_h\nabla_\Gamma \mathfrak{K} v\|_{L^2(\Gamma)} \lesssim \|w_h/h^{1/2}\|_{L^\infty(\Gamma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma)} + \|w_h\nabla_\Gamma v\|_{L^2(\Gamma)}
$$

and concludes the first part of the proof.

The result for the hypersingular integral operator $\mathfrak{W}$ is shown with similar arguments. Let again $v \in \widetilde{H}^1(\Gamma)$ and v_T as in Lemma 5.1. Note that $\mathfrak{W}v_T = 0$. Splitting now into near-field and far-field yields

$$
\begin{aligned}
\|w_h\mathfrak{W}v\|^2_{L^2(\Gamma)} &= \sum_{T\in\mathcal{T}_h} \|w_h\mathfrak{W}(v-v_T)\|^2_{L^2(T)} \\
&\lesssim \sum_{T\in\mathcal{T}_h} \|w_h\mathfrak{W}((v-v_T)\eta_T)\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_h} \|w_h\mathfrak{W}((v-v_T)(1-\eta_T))\|^2_{L^2(T)}.
\end{aligned} \tag{6.5}
$$

The near-field contribution is bounded by the stability of $\mathfrak{W}: H^1(\partial\Omega) \to L^2(\partial\Omega)$ stated in (2.12) and the Poincaré-type estimate (5.5)

$$
\|\mathfrak{W}((v-v_T)\eta_T)\|^2_{L^2(T)} \overset{(2.12)}{\lesssim} \|(v-v_T)\eta_T\|^2_{H^1(\omega_h(T))} \overset{(5.5)}{\lesssim} \|\nabla_\Gamma v\|^2_{L^2(\omega_h(T))}.
$$

The sum over all elements gives

$$
\sum_{T\in\mathcal{T}_h} \|w_h\mathfrak{W}((v-v_T)\eta_T)\|^2_{L^2(T)} \lesssim \sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)}\|\nabla_\Gamma v\|^2_{L^2(\omega_h(T))} \lesssim \|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)}.
$$

It remains to bound the second term on the right-hand side in (6.5). In view of the support properties of η_T, the potential $u^{\rm far}_{\mathfrak{K},T} = \widetilde{\mathfrak{K}}((v-v_T)(1-\eta_T))$ is smooth near T (cf. Lemma 5.3) so that $\gamma_1^{\rm int}u^{\rm far}_{\mathfrak{K},T} = \partial_{\boldsymbol{\nu}} u^{\rm far}_{\mathfrak{K},T}$ on T. Furthermore, since $\mathfrak{W} = -\gamma_1^{\rm int}\widetilde{\mathfrak{K}}$ we see

$$
\|\mathfrak{W}((v-v_T)(1-\eta_T))\|^2_{L^2(T)} = \|\gamma_1^{\rm int}u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)} = \|\partial_{\boldsymbol{\nu}} u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)} \le \|\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)}.
$$

We use Proposition 5.5 to conclude

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h} \|w_h\mathfrak{W}((v-v_T)(1-\eta_T))\|^2_{L^2(T)} &\le \sum_{T\in\mathcal{T}_h} \|w_h\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)} \\
&\lesssim \|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)} + \|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)}\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)}.
\end{aligned}
$$

Altogether, we obtain

$$
\|w_h\mathfrak{W}v\|_{L^2(\Gamma)} \lesssim \|w_h\nabla_\Gamma v\|_{L^2(\Gamma)} + \|w_h/h^{1/2}\|_{L^\infty(\Gamma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma)}. \qquad \square
$$

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Appendix A

Lemma A.1. *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ. Suppose that $d \geq 2$ and that q_h is a σ-admissible polynomial degree distribution with respect to $\mathcal{T}_h$. Then, there exists a constant $\widetilde{C}_{\rm inv} > 0$, which depends solely on $\partial\Omega$, the κ-shape regularity of $\mathcal{T}_h$, and σ, such that*

$$\|h^{1/2}(q_h+1)^{-1}\,\Psi_h\|_{L^2(\Gamma)} \leq \widetilde{C}_{\rm inv}\|\Psi_h\|_{H^{-1/2}(\Gamma)} \quad \textit{for all } \Psi_h \in \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h). \tag{A.1}$$

Proof. For each $T \in \mathcal{T}_h$, let $\widehat{\chi}_{T,q_h(T)}$ be a smooth function on $T_{\rm ref}$ with the following properties for some fixed $\delta > 0$ (see, e.g., the proofs of [Geo08, Lem. 3.7, Prop. 3.8] or the arguments below):

$$\operatorname{supp}\widehat{\chi}_{T,q_h(T)} \subseteq \{x \in T_{\rm ref} \colon\ \operatorname{dist}(x, \partial T_{\rm ref}) > \delta/(q_h(T)+1)^2\}, \tag{A.2}$$

$$0 \leq \widehat{\chi}_{T,q_h(T)} \leq 1 \quad \text{in } T_{\rm ref}, \qquad \|\nabla\widehat{\chi}_{T,q_h(T)}\|_{L^\infty(T_{\rm ref})} \lesssim (q_h(T)+1)^{-2}, \tag{A.3}$$

$$\widehat{\chi}_{T,q_h(T)} \equiv 1 \quad \text{in } \{x \in T_{\rm ref} \colon\ \operatorname{dist}(x, \partial T_{\rm ref}) > 3\delta/(q_h(T)+1)^2\}, \tag{A.4}$$

$$\|\pi\|_{L^2(T_{\rm ref})} \leq C\|\pi\widehat{\chi}_{T,p(T)}\|_{L^2(T_{\rm ref})} \quad \forall \text{ polynomials } \pi \text{ of degree } q_h(T), \tag{A.5}$$

$$\|\pi\widehat{\chi}_{T,q_h(T)}\|_{H^1(T_{\rm ref})} \leq C(1+q_h(T))^2\|\pi\|_{L^2(T_{\rm ref})} \quad \forall \text{ polynomials } \pi \text{ of degree } q_h(T). \tag{A.6}$$

$\widehat{\chi}_{T,q_h(T)}$ is obtained from a mollification of the characteristic function of $T_{\rm ref} \setminus S_{2\delta/(q_h(T)+1)^2}$, where $S_\varepsilon := \{x \in T_{\rm ref} \colon\ \operatorname{dist}(x, \partial T_{\rm ref}) < \varepsilon\}$. The parameter $\delta > 0$ is dictated by the requirement (A.5). For this, we use the shorthand $\varepsilon(\delta) = 3\delta/(q_h(T)+1)^2$ and observe that we assume $\widehat{\chi}_{T,q_h(T)} \equiv 1$ on $T_{\rm ref} \setminus S_{\varepsilon(\delta)}$ so that we are done once we have established $\|\pi\|_{L^2(S_{\varepsilon(\delta)})} \lesssim \|\pi\|_{L^2(T_{\rm ref}\setminus S_{\varepsilon(\delta)})}$. [LMWZ10, Lemma 2.1] and the polynomial inverse estimate $\|\pi\|_{H^1(T_{\rm ref})} \lesssim (q_h(T)+1)^2\|\pi\|_{L^2(T_{\rm ref})}$, yield

$$\begin{aligned}\|\pi\|^2_{L^2(S_{\varepsilon(\delta)})} &\lesssim \varepsilon(\delta)\|\pi\|_{L^2(T_{\rm ref})}\|\pi\|_{H^1(T_{\rm ref})} \lesssim \varepsilon(\delta)(q_h(T)+1)^2\|\pi\|^2_{L^2(T_{\rm ref})} \\ &= \varepsilon(\delta)(q_h(T)+1)^2\left[\|\pi\|^2_{L^2(T_{\rm ref}\setminus S_{\varepsilon(\delta)})} + \|\pi\|^2_{L^2(S_{\varepsilon(\delta)})}\right].\end{aligned}$$

Taking δ sufficiently small produces $\|\pi\|_{L^2(S_{\varepsilon(\delta)})} \lesssim \|\pi\|_{L^2(T_{\rm ref}\setminus S_{\varepsilon(\delta)})}$ as desired.
Define $\chi_{T,q_h(T)}$ with $\operatorname{supp}\chi_{T,q_h(T)} \subseteq T$ by $\chi_{T,q_h(T)} \circ \gamma_T = \widehat{\chi}_{T,q_h(T)}$. Given $\Psi_h \in \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$, define $\widetilde{H}^1(\Gamma) \ni v := \sum_{T\in\mathcal{T}_h} v_T$ in an elementwise fashion by requiring $\operatorname{supp} v_T \subseteq T$ and

$$v_T|_T := \frac{h(T)}{(1+q_h(T))^2}(\Psi_h|_T)\chi_{T,q_h(T)}, \tag{A.7}$$

Note that $v_T \in \widetilde{H}^1(\Gamma)$ by the support properties of $\chi_{T,q_h(T)}$. An interpolation inequality and the estimate (A.6) on the reference element give

$$\begin{aligned}\|v_T\|^2_{\widetilde{H}^{1/2}(\Gamma)} &\leq \|v_T\|^2_{\widetilde{H}^{1/2}(\partial\Omega)} \lesssim \|v_T\|_{L^2(\partial\Omega)}\|v_T\|_{H^1(\partial\Omega)} = \|v_T\|_{L^2(T)}\|v_T\|_{H^1(T)} \\ &\overset{(A.6),(A.5)}{\lesssim} \frac{(1+q_h(T))^2}{h(T)}\|v_T\|^2_{L^2(T)}.\end{aligned}$$

This implies

$$\sum_{T\in\mathcal{T}_h} \|v_T\|^2_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \left\| \frac{q_h+1}{h^{1/2}} v \right\|^2_{L^2(\Gamma)}. \tag{A.8}$$

From [SS11, Lemma 4.1.49] we get from (A.8)

$$\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)} = \left\| \sum_{T\in\mathcal{T}_h} v_T \right\|^2_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \sum_{T\in\mathcal{T}_h} \|v_T\|^2_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \left\| \frac{1+q_h}{h^{1/2}} v \right\|^2_{L^2(\Gamma)}. \tag{A.9}$$

Finally, we estimate

$$\begin{aligned}
\left\| \frac{h^{1/2}}{1+q_h} \Psi_h \right\|^2_{L^2(\Gamma)} &= \sum_{T\in\mathcal{T}_h} \left\| \frac{h(T)^{1/2}}{1+q_h(T)} \Psi_h \right\|^2_{L^2(T)} \overset{(A.5)}{\lesssim} \sum_{T\in\mathcal{T}_h} \left\| \frac{h(T)^{1/2}}{1+q_h(T)} \chi_{T,q_h(T)} \Psi_h \right\|^2_{L^2(T)} \\
&\overset{(A.2)}{\lesssim} \sum_{T\in\mathcal{T}_h} \| \frac{h(T)^{1/2}}{1+q_h(T)} \sqrt{\chi_{T,q_h(T)}} \Psi_h \|^2_{L^2(T)} = \sum_{T\in\mathcal{T}_h} (v_T, \Psi_h)_{L^2(T)} = (v, \Psi_h)_{L^2(\Gamma)} \\
&\leq \|\Psi_h\|_{H^{-1/2}(\Gamma)} \|v\|_{\widetilde{H}^{1/2}(\Gamma)} \overset{(A.9)}{\lesssim} \|\Psi_h\|_{H^{-1/2}(\Gamma)} \left\| \frac{1+q_h}{h^{1/2}} v \right\|_{L^2(\Gamma)} \\
&\overset{(A.7)}{\lesssim} \|\Psi_h\|_{H^{-1/2}(\Gamma)} \left\| \frac{h^{1/2}}{1+q_h} \Psi_h \right\|_{L^2(\Gamma)}.
\end{aligned}$$

□

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