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Mixed formulation for interface problems with distributed Lagrange multiplier

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Abstract

We study a mixed formulation for elliptic interface problems which has been recently introduced when dealing with a test problem arising from fluid-structure interaction applications. The formulation, which involves a Lagrange multiplier defined in the solid domain, can be approximated by standard finite elements on meshes which do not need to fit with the interface. In this paper we discuss a modification of the original formulation involving a different approach for the analysis and the numerical implementation of the Lagrange multiplier. New two-dimensional numerical results confirm the good performances of the proposed schemes.

Keywords: interface problems, finite elements, fluid-structure interactions, Lagrange multiplier
2000 MSC: 65N30, 35R05

1. Introduction

It is very frequent that real world applications can be mathematically described by equations which contain as basic ingredient interface or transmission problems. By interface problem we mean a partial differential equation posed in a domain subdivided into two or more subdomains by interfaces where the coefficients may jump: The coefficient discontinuities introduce transmission conditions among the different subdomains.

The applications we have in mind arise from the approximation of fluid-structure interactions through the finite element Immersed Boundary Method (IBM) [20, 9, 7]. The main motivation for the present research actually comes from recent developments in the modeling of the IBM. It is out of the aims of this paper to describe the links in more details, for which we refer the interested reader to [5, 2]. In particular, we observe that the terminology IBM has been used in several (sometimes very different) contexts. Here we refer to the framework introduced by Peskin [20].

It is well known that the approximation of interface problems generally requires a mesh which is compatible with the interface in order to achieve optimal convergence rates. In the case of moving boundaries, this requires to adapt the mesh at each time step and, in any case, limits the shape of the admissible interfaces. A quite large literature deals with this problem and several possible workarounds have been proposed. Some basic references can be found, e.g., in [15, 16, 14, 11, 10, 18, 3, 4, 21, 17, 1].

We remark that the main goal of this research is not simply to develop an additional, perhaps more sophisticated, method for the approximation of interface problems, but is to present a stepping stone on the way to more significant modifications of the finite element IBM. The results of this paper, in particular, will be useful for proving the stability properties of a distributed Lagrangian version of the IBM, see [8]. For this reason, it is out of the aims of this paper to compare the performance and the efficiency of our method with other existing schemes.

In [2] we have introduced a new scheme for the approximation of interface problems which makes use of a Lagrange multiplier and has some similarities with the fictitious domain approach with distributed Lagrange multiplier, see, e.g., [13, 12]). The aim of this paper is, on the one hand, to present and analyze a new variant of the method and, on the other hand, to extend the preliminary one-dimensional numerical results of [2] to a more interesting two-dimensional case. The modification of the method consists in a different treatment of a duality pairing which in the original approximation was interpreted numerically as a scalar product in L^2. Here, using the Riesz identification, we propose to replace the duality pairing with the scalar product in H^1. From our numerical experiments it turns out that the new scheme is more robust, in the sense that it provides a good approximation for quite general coefficient jumps. We consider the case when two materials are present; in this case we use two meshes: One for the whole domain and one for the region where only one material is present. We emphasize that the meshes are completely independent from each other and, in particular, do not need to match along the interface between the two subdomains.

The results of the paper are supported by a rigorous theoretical analysis, which gives sufficient conditions for the convergence of the scheme in terms of the problem coefficients and the ratio between the size of the two meshes.

The problem is presented in Section 2, together with the variational formulations corresponding to the use of the Lagrange multiplier. In the same Section the analysis of the mixed formulations is performed. In Section 3 the finite element approximation of both formulations is carried on and the error estimates are proved. The numerical tests, as well as the related results, are reported in Section 4.

2. Interface problem and its mixed formulations

Let $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded domain with Lipschitz continuous boundary. We assume that there exists a Lipschitz continuous interface $\Gamma \subset \Omega$ which splits Ω into two subdomains Ω_1 and Ω_2. Let $\beta_1 : \Omega_1 \to \mathbb{R}$ and $\beta_2 : \Omega_2 \to \mathbb{R}$ be two bounded continuous functions such that $0 < \underline{\beta} \le \beta_i$, $i = 1, 2$. Let us consider the following interface problem: given $f_1 : \Omega_1 \to \mathbb{R}$ and $f_2 : \Omega_2 \to \mathbb{R}$, find $u_1 : \Omega_1 \to \mathbb{R}$ and

$u_2 : \Omega_2 \to \mathbb{R}$ such that

$$\begin{aligned}
& -\operatorname{div}(\beta_1 \nabla u_1) = f_1 && \text{in } \Omega_1 \\
& -\operatorname{div}(\beta_2 \nabla u_2) = f_2 && \text{in } \Omega_2 \\
& u_1 = u_2 && \text{on } \Gamma \\
& \beta_1 \nabla u_1 \cdot n_1 + \beta_2 \nabla u_2 \cdot n_2 = 0 && \text{on } \Gamma \\
& u_1 = 0 && \text{on } \partial\Omega_1 \setminus \Gamma \\
& u_2 = 0 && \text{on } \partial\Omega_2 \setminus \Gamma,
\end{aligned} \tag{1}$$

where n_1 and n_2 are the unit vectors normal to Γ, pointing outwards with respect to Ω_1 and Ω_2, respectively.

Let $H^1(\Omega_i)$ be the Hilbert space of real function in $L^2(\Omega_i)$ with gradient in $L^2(\Omega_i)^d$, $i = 1, 2$. Then we shall consider the following spaces:

$$\begin{aligned}
& H^1_D(\Omega_i) = \{v \in H^1(\Omega_i) : v = 0 \text{ on } \partial\Omega_i \setminus \Gamma\} \quad i = 1, 2, \\
& \mathbf{W} = \{(v_1, v_2) \in H^1_D(\Omega_1) \times H^1_D(\Omega_2) : v_1|_\Gamma = v_2|_\Gamma\}.
\end{aligned} \tag{2}$$

The above problem can be written in variational form as follows: find $(u_1, u_2) \in \mathbf{W}$ such that

$$\begin{aligned}
\int_{\Omega_1} \beta_1 \nabla u_i \nabla v_1 \, dx + \int_{\Omega_2} \beta_2 \nabla u_2 \nabla v_2 \, dx = \int_{\Omega_1} f_1 v_1 \, dx + \int_{\Omega_2} f_2 v_2 \, dx \\
\forall (v_1, v_2) \in \mathbf{W}.
\end{aligned} \tag{3}$$

The finite element discretization of (3) requires the construction of meshes in Ω_1 and Ω_2 which fit with the interface Γ. Moreover, in order to enforce easily the continuity at the boundary of the two components one should choose matching grids so that the meshes share the nodes on Γ. If the interface problem above arises as a step in an advancing scheme in the resolution of fluid-structure interaction problems, the interface Γ might depend on time so that at each time step one has to change the mesh close to the interface. To avoid this inconvenience we have proposed in [2] the following *fictitious formulation with a distributed Lagrange multiplier*.

We set $\Omega = \Omega_1 \cup \Omega_2$. Let $v \in H^1_0(\Omega)$, then its restrictions $v_1 = v|_{\Omega_1}$ and $v_2 = v|_{\Omega_2}$ are such that the pair (v_1, v_2) belongs to $\mathbf{W}$. Let us consider extensions $\beta : \Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ of β_1 and f_1, such that $\beta|_{\Omega_1} = \beta_1$ and $f|_{\Omega_1} = f_1$ and β is continuous on Ω.

Let $u \in H^1_0(\Omega)$ be such that $u|_{\Omega_1} = u_1$ and $u|_{\Omega_2} = u_2$, then we can rewrite equation (3) as

$$\int_\Omega \beta \nabla u \nabla v \, dx + \int_{\Omega_2} (\beta_2 - \beta) \nabla u_2 \nabla v|_{\Omega_2} \, dx = \int_\Omega f v \, dx + \int_{\Omega_2} (f_2 - f) v|_{\Omega_2} \, dx \quad \forall v \in H^1_0(\Omega).$$

The above equation is equivalent to (3) if and only if $u|_{\Omega_2} = u_2$. Introducing a Lagrange multiplier associated to this constraint we arrive to the following problem. We use the following notation: $V = H^1_0(\Omega)$ and $V_2 = H^1(\Omega_2)$ and $\Lambda = [H^1(\Omega_2)]^*$, that is the dual space of $H^1(\Omega_2)$. These spaces are endowed with their natural norms.

Problem 1. *Given $f \in L^2(\Omega)$ and $f_2 \in L^2(\Omega_2)$, find $u \in V$, $u_2 \in V_2$ and $\lambda \in \Lambda$ such that*

$$\begin{aligned}
&\int_\Omega \beta \nabla u \nabla v \, dx + \langle \lambda, v|_{\Omega_2} \rangle = \int_\Omega f v \, dx && \forall v \in V \\
&\int_{\Omega_2} (\beta_2 - \beta) \nabla u_2 \nabla v_2 \, dx - \langle \lambda, v_2 \rangle = \int_{\Omega_2} (f_2 - f) v_2 \, dx && \forall v_2 \in V_2 \\
&\langle \mu, u|_{\Omega_2} - u_2 \rangle = 0 && \forall \mu \in \Lambda
\end{aligned} \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\Lambda = [H^1(\Omega_2)]^*$ and $V_2 = H^1(\Omega_2)$.

The equivalence of Problem 1 and Equation (3) has been proved in [2, Th. 2]. In particular the multiplier λ is given by the following expression.

$$\begin{aligned}
\langle \lambda, v_2 \rangle &= -\int_{\Omega_2} (\frac{\beta}{\beta_2} f_2 - f) v_2 \, dx + \int_\Gamma (\beta_2 - \beta) \nabla u_2 \cdot n_2 v_2 \, d\gamma \\
&= -\int_{\Omega_2} (\frac{\beta}{\beta_2} f_2 - f) v_2 \, dx - \int_\Gamma \beta (\nabla u \cdot n_1 + \nabla u_2 \cdot n_2) v_2 \, d\gamma.
\end{aligned} \tag{5}$$

The following proposition states existence and uniqueness of the solution of Problem 1.

Proposition 2.1. *Let β and β_2 be positive continuous functions on Ω and Ω_2, respectively, such that $\beta \geq \underline{\beta} > 0$ and $\beta_2 \geq \underline{\beta} > 0$. Given $f \in L^2(\Omega)$ and $f_2 \in L^2(\Omega_2)$, then there exists one and only one solution of Problem 1 such that*

$$\|u\|_V + \|u_2\|_{V_2} + \|\lambda\|_\Lambda \leq C \left(\|f\|_{0,\Omega} + \|f_2\|_{0,\Omega_2} \right). \tag{6}$$

The proof follows the same argument as the one given in [2] in the case of Ω_2 immersed in Ω, where $\Gamma = \partial\Omega_2$.

In particular, it is clear that Problem 1 has the structure of a saddle point problem. The variational formulation reads

$$\begin{aligned}
&a((u, u_2), (v, v_2)) + b(\lambda, (v, v_2)) = F(v, v_2) && \forall (v, v_2) \in V \times V_2 \\
&b(\mu, (u, u_2)) = 0 && \forall \mu \in \Lambda,
\end{aligned}$$

with

$$\begin{aligned}
a((u, u_2), (v, v_2)) &= \int_\Omega \beta \nabla u \cdot \nabla v \, dx + \int_{\Omega_2} (\beta_2 - \beta) \nabla u_2 \cdot \nabla v_2 \, dx \\
b(\lambda, (v, v_2)) &= \langle \lambda, v|_{\partial\Omega_2} - v_2 \rangle \\
F(v, v_2) &= \int_\Omega f v \, dx + \int_{\Omega_2} (f_2 - f) v_2 \, dx.
\end{aligned}$$

By introducing the kernel of the operator associated to the bilinear form in the third equation of (4)

$$\mathbb{K} = \{(v, v_2) \in V \times V_2 : v|_{\Omega_2} = v_2 \text{ in } \Omega_2\}, \tag{7}$$

the proof consists in checking the ellipticity on the kernel condition: there exists $\kappa_1 > 0$ such that

$$\int_\Omega \beta |\nabla v|^2 \, dx + \int_{\Omega_2} (\beta_2 - \beta) |\nabla v_2|^2 \, dx \geq \kappa_1 \left(\|v\|_V^2 + \|v_2\|_{V_2}^2 \right) \quad \forall (v, v_2) \in \mathbb{K} \tag{8}$$

and the inf-sup condition: there exists κ_2 such that

$$\sup_{(v,v_2)\in V\times V_2} \frac{\langle \mu, v|_{\Omega_2} - v_2\rangle}{(\|v\|_V^2 + \|v_2\|_{V_2}^2)^{1/2}} \geq \kappa_2 \|\mu\|_\Lambda. \tag{9}$$

Then, the well-known theory for saddle-point problems gives the result, see [6].

In view of the finite element discretization we observe that we can associate to any $\lambda \in \Lambda$ an element of $\varphi \in V_2$ which is the solution of the following variational equation:

$$(\varphi, v_2)_{V_2} = \langle \lambda, v_2\rangle \quad \forall v_2 \in V_2 \tag{10}$$

where $(\cdot,\cdot)_{V_2}$ denotes the scalar product in V_2, that is

$$(\varphi, v_2)_{V_2} = \int_{\Omega_2} (\nabla\varphi\nabla v_2 + \varphi v_2)\, dx.$$

It is easy to show that there exists a unique $\varphi \in V_2$ satisfying (10). Moreover, we have the following a priori estimate:

$$\|\varphi\|_{V_2}^2 = (\varphi,\varphi)_{V_2} = \langle\lambda,\varphi\rangle \leq \|\lambda\|_\Lambda \|\varphi\|_{V_2},$$

that is $\|\varphi\|_{V_2} \leq \|\lambda\|_\Lambda$. On the other hand, by the definition of the norm in Λ we have:

$$\|\lambda\|_\Lambda = \sup_{v_2\in V_2} \frac{\langle\lambda, v_2\rangle}{\|v_2\|_{V_2}} = \sup_{v_2\in V_2} \frac{(\varphi, v_2)_{V_2}}{\|v_2\|_{V_2}} \leq \|\varphi\|_{V_2}$$

so that

$$\|\varphi\|_{V_2} = \|\lambda\|_\Lambda.$$

Therefore we obtain the following equivalent formulation of Problem 1

Problem 2. *Given $f \in L^2(\Omega)$ and $f_2 \in L^2(\Omega_2)$, find $u \in V$, $u_2 \in V_2$ and $\varphi \in V_2$ such that*

$$\begin{aligned}
&\int_\Omega \beta\nabla u\nabla v\, dx + (\varphi, v_{\Omega_2})_{V_2} = \int_\Omega f v\, dx && \forall v \in V\\
&\int_{\Omega_2} (\beta_2 - \beta)\nabla u_2\nabla v_2\, dx - (\varphi, v_2)_{V_2} = \int_{\Omega_2} (f_2 - f) v_2\, dx && \forall v_2 \in V_2\\
&(\psi, u|_{\Omega_2} - u_2)_{V_2} = 0 && \forall \psi \in V_2.
\end{aligned} \tag{11}$$

In order to show existence and uniqueness it is sufficient to check the inf-sup condition, since the ellipticity on the kernel condition is given by (8). We have

$$\|\psi\|_{V_2} = \sup_{v_2\in V_2} \frac{(\psi, v_2)_{V_2}}{\|v_2\|_{V_2}} \leq \sup_{(v,v_2)\in V\times V_2} \frac{(\psi, v|_{\Omega_2} - v_2)_{V_2}}{(\|v\|_V^2 + \|v_2\|_{V_2}^2)^{1/2}}. \tag{12}$$

Remark 1. *Although Problem 1 and Problem 2 are equivalent at the continuous level, their finite element discretizations differ. We shall compare the performances of the two schemes in Section 4.*

3. Finite element approximation

Let $\mathcal{T}_h$ and $\mathcal{T}_{2,h}$ denote shape-regular families of decompositions of Ω and Ω_2, respectively. We denote by h and h_2 the maximum diameter of the elements in $\mathcal{T}_h$ and $\mathcal{T}_{2,h}$. To fix ideas, we are going to use first order elements, but this is not a restriction and many other choices can be made. So, for instance, in the case of triangular meshes, our finite element spaces will consist of linear polynomials, while in the case of quadrilateral meshes we shall use piecewise bilinear polynomials. Hence, we consider the following finite dimensional subspaces $V_h \subseteq V$, $V_{2,h} \subseteq V_2$ and $\Lambda_h \subseteq \Lambda$:

$$\begin{aligned} V_h &= \{v \in V : v|_K \in \mathcal{L}(K) \text{ for all } K \in \mathcal{T}_h\} \\ V_{2,h} &= \{v_2 \in V_2 : v_2|_K \in \mathcal{L}(K) \text{ for all } K \in \mathcal{T}_{2,h}\} \\ \Lambda_h &= V_{2,h}, \end{aligned} \tag{13}$$

where $\mathcal{L}$ denotes the space of first order Lagrangian finite elements.

We introduce first the finite element discretization of Problem 1. Since our finite elements $V_{2,h}$ and Λ_h are contained in $L^2(\Omega_2)$, we can interpret the duality pairing as scalar product in $L^2(\Omega_2)$, which we are going to denote, as usual, by $(\cdot, \cdot)$.

Problem 3. *Given $f \in L^2(\Omega)$ and $f_2 \in L^2(\Omega_2)$, find $u_h \in V_h$, $u_{2,h} \in V_{2,h}$ and $\lambda_h \in \Lambda_h$ such that*

$$\begin{aligned} &\int_\Omega \beta \nabla u_h \nabla v \, dx + (\lambda_h, v|_{\Omega_2}) = \int_\Omega f v \, dx && \forall v \in V_h \\ &\int_{\Omega_2} (\beta_2 - \beta) \nabla u_{2,h} \nabla v_2 \, dx - (\lambda_h, v_2) = \int_{\Omega_2} (f_2 - f) v_2 \, dx && \forall v_2 \in V_{2,h} \\ &(\mu, u_h|_{\Omega_2} - u_{2,h}) = 0 && \forall \mu \in \Lambda_h. \end{aligned} \tag{14}$$

In [2] we have checked the discrete ellipticity on the kernel and the inf-sup condition for the case of piecewise linear finite elements on triangular meshes. The proofs can be extended to the case of bilinear finite elements on quadrilaterals in a straightforward way. Therefore we can state the following results:

Proposition 3.1. *Let V_h, $V_{2,h}$ and Λ_h be given as in* (13) *and assume that $\beta_2 - \beta|_{\Omega_2} \geq \eta_0 > 0$ a.e. in Ω_2, then the* ellipticity on the discrete kernel *holds true, i.e., there exists a real number $\kappa_1^* > 0$, independent of h and h_2, such that for all $(v, v_2) \in \mathbb{K}_h^1$ it holds*

$$\int_\Omega \beta |\nabla v|^2 \, dx + \int_{\Omega_2} (\beta_2 - \beta) |\nabla v_2|^2 \, dx \geq \kappa_1^* \left(\|v\|_V^2 + \|v_2\|_{V_2}^2 \right), \tag{15}$$

where

$$\mathbb{K}_h^1 = \{(v, v_2) \in V_h \times V_{2,h} : \ (\mu, v|_{\Omega_2} - v_2) = 0 \ \forall \mu \in \Lambda_h\}. \tag{16}$$

Proposition 3.2. *Let V_h, $V_{2,h}$ and Λ_h be given as in* (13) *and assume that the mesh sequence is quasi-uniform, then the* discrete inf-sup condition *holds true, i.e., there exists a real number $\kappa_2^* > 0$, independent of h and h_2, such that for all $\mu \in \Lambda_h$ the following inequality holds true*

$$\sup_{(v, v_2) \in V_h \times V_{2,h}} \frac{(\mu, v|_{\Omega_2} - v_2)}{\left(\|v\|_V^2 + \|v_2\|_{V_2}^2 \right)^{1/2}} \geq \kappa_2^* \|\mu\|_\Lambda. \tag{17}$$

In order to derive the error estimates for the solution of Problem 1 we need to know the regularity of the solution. The solution of Problem 3 can be interpreted as a solution of an elliptic second order equation with discontinuous coefficient and we have that $u \in H^s(\Omega)$ with $1 < s < 3/2$. Moreover, since Γ is only Lipschitz continuous the restrictions $u|_{\Omega_1}$ and u_2 to the subdomains Ω_1 and Ω_2 belong to $H^r(\Omega_1)$ and $H^r(\Omega_2)$, respectively, with $3/2 < r \leq 2$, see [19, Ch. 2].

From Propositions 3.1 and 3.2, we obtain the following error estimates

Theorem 3.3. *Let $(u, u_2, \lambda) \in V \times V_2 \times \Lambda$ and $(u_h, u_{2,h}, \lambda_h) \in V_h \times V_{2,h} \times \Lambda_h$ be the solutions of Problems 1 and 3, respectively. Then, there exists a constant C, independent of h, such that*

$$\begin{aligned} &\|u - u_h\|_V + \|u_2 - u_{2,h}\|_{V_2} + \|\lambda - \lambda_h\|_\Lambda \\ &\quad \leq C h^{s-1} \left(\|(\beta/\beta_2) f_2 - f\|_0 + \|u|_{\Omega_1}\|_{H^r(\Omega_1)} + \|u_2\|_{H^r(\Omega_2)} \right). \end{aligned} \tag{18}$$

We now introduce the discretization of Problem 2 using the finite element spaces defined in (13).

Problem 4. *Given $f \in L^2(\Omega)$ and $f_2 \in L^2(\Omega_2)$, find $u_h \in V_h$, $u_{2,h} \in V_{2,h}$ and $\varphi_h \in V_{2,h}$ such that*

$$\begin{aligned} &\int_\Omega \beta \nabla u_h \nabla v \, dx + (\varphi_h, v_{\Omega_2})_{V_2} = \int_\Omega f v \, dx && \forall v \in V_h \\ &\int_{\Omega_2} (\beta_2 - \beta) \nabla u_{2,h} \nabla v_2 \, dx - (\varphi_h, v_2)_{V_2} = \int_{\Omega_2} (f_2 - f) v_2 \, dx && \forall v_2 \in V_{2,h} \\ &(\psi, u_h|_{\Omega_2} - u_{2,h})_{V_2} = 0 && \forall \psi \in V_{2,h}. \end{aligned} \tag{19}$$

Let us prove existence and uniqueness of the solution of Problem 4 together with error estimates. For this, once again, we need to check the ellipticity on the discrete kernel and the discrete inf-sup conditions.

The discrete kernel associated to the third equation in (19) is given by:

$$\mathbb{K}_h^2 = \{(v, v_2) \in V_h \times V_{2,h} : (\psi, v|_{\Omega_2} - v_2)_{V_2} = 0 \ \forall \psi \in V_{2,h}\}. \tag{20}$$

Let $\Pi_h : V_2 \to V_{2,h}$ the projection operator in V_2 that is: for all $v_2 \in V_2$, $\Pi_h v_2 \in V_{2,h}$ is such that

$$(\psi, v_2 - \Pi_h v_2)_{V_2} = 0 \quad \forall \psi \in V_{2,h}. \tag{21}$$

Then we can characterize the elements in $\mathbb{K}_h^2$ as the pairs $(v, v_2) \in V_h \times V_{2,h}$ such that $v_2 = \Pi_h(v|_{\Omega_2})$. For brevity we shall write $\Pi_h v$ instead of $\Pi_h(v|_{\Omega_2})$.

Proposition 3.4. *Let V_h and $V_{2,h}$ be given as in* (13)*, and assume that $\beta_2 - \beta|_{\Omega_2} \geq \eta_0 > 0$ a.e. in Ω_2, then the* ellipticity on the discrete kernel *holds true, i.e., there exists a real number $\kappa_3^* > 0$, independent of h and h_2, such that for all $(v, v_2) \in \mathbb{K}_h^2$ it holds*

$$\int_\Omega \beta |\nabla v|^2 \, dx + \int_{\Omega_2} (\beta_2 - \beta) |\nabla v_2|^2 \, dx \geq \kappa_3^* \left(\|v\|_V^2 + \|v_2\|_{V_2}^2 \right). \tag{22}$$

The proof is straightforward and follows the same argument as the one of Proposition 3.1, see [2].

Let us now prove the discrete inf-sup condition for Problem 4.

Proposition 3.5. *There exists $\kappa_4^* > 0$, independent of h and h_2, such that*

$$\sup_{(v,v_2)\in V_h\times V_{2,h}} \frac{(\mu, v|_{\Omega_2} - v_2)_{V_2}}{(\|v\|_V^2 + \|v_2\|_{V_2}^2)^{1/2}} \geq \kappa_4^* \|\mu\|_{V_2} \quad \forall \mu \in V_{2,h}. \tag{23}$$

Proof. By definition of the norm in V_2 and of the projection operator Π_h, we obtain

$$\begin{aligned}\|\mu\|_{V_2} &= \sup_{v_2\in V_2} \frac{(\mu, v_2)_{V_2}}{\|v_2\|_{V_2}} \leq \sup_{v_2\in V_2} \frac{(\mu, \Pi_h v_2)_{V_2}}{\|\Pi_h v_2\|_{V_2}} \\ &\leq \sup_{v_2\in V_{2,h}} \frac{(\mu, v_2)_{V_2}}{\|v_2\|_{V_2}} \leq \sup_{(v,v_2)\in V_h\times V_{2,h}} \frac{(\mu, v|_{\Omega_2} - v_2)_{V_2}}{(\|v\|_V^2 + \|v_2\|_{V_2}^2)^{1/2}}.\end{aligned}$$

□

Due to the theory on approximation of saddle point problems, Propositions 3.4 and 3.5 imply the following optimal error estimates:

Theorem 3.6. *Let $(u, u_2, \varphi) \in V \times V_2 \times V_2$ and $(u_h, u_{2,h}, \varphi_h) \in V_h \times V_{2,h} \times V_{2,h}$ be the solutions of Problems 2 and 4, respectively. Assume that the assumptions of Propositions 3.4 and 3.5 are satisfied, then there exists a constant C independent of h such that:*

$$\begin{aligned}&\|u - u_h\|_V + \|u_2 - u_{2,h}\|_{V_2} + \|\lambda - \lambda_h\|_{V_2} \\ &\quad \leq Ch^{s-1} \left(\|(\beta/\beta_2) f_2 - f\|_0 + \|u|_{\Omega_1}\|_{H^r(\Omega_1)} + \|u_2\|_{H^r(\Omega_2)} \right).\end{aligned} \tag{24}$$

Our numerical tests will show that the requirement on $\beta_2 - \beta|_{\Omega_2}$ (see Proposition 3.4) is actually not necessary, provided that h_2 is small enough with respect to h. We now show that this observation can be proved rigorously and the mesh conditions can be made explicit.

Indeed, taking into account the characterization of the elements in $\mathbb{K}_h^2$, we can rewrite the left hand side of (22) as follows

$$\begin{aligned}&\int_\Omega \beta |\nabla v|^2 \, dx + \int_{\Omega_2} (\beta_2 - \beta) |\nabla v_2|^2 \, dx \\ &\quad = \int_{\Omega_1} \beta |\nabla v|^2 \, dx + \int_{\Omega_2} \beta |\nabla v|^2 \, dx + \int_{\Omega_2} (\beta_2 - \beta) |\nabla v_2|^2 \, dx \\ &\quad = \int_{\Omega_1} \beta |\nabla v|^2 \, dx + \int_{\Omega_2} \beta_2 |\nabla v_2|^2 \, dx + \int_{\Omega_2} \beta \left(|\nabla v|^2 - |\nabla \Pi_h v|^2 \right) dx.\end{aligned} \tag{25}$$

Therefore, in order to prove ellipticity, we need to find an estimate from below, possibly depending on h and h_2, of the quantity

$$\int_{\Omega_2} \beta \left(|\nabla v|^2 - |\nabla \Pi_h v|^2 \right) dx$$

in terms of the first two integrals in the last line of (25).

We are going to prove the ellipticity in the discrete kernel for all the values of β and β_2 in the case of piecewise linear elements under additional assumptions on the meshes.

Proposition 3.7. *Let V_h and $V_{2,h}$ consist of piecewise linear polynomials. Assume that the mesh $\mathcal{T}_h$ is quasi-uniform. Then for $h_2/h^{d/2}$ sufficiently small, the ellipticity in the discrete kernel holds true for Problem 4, i.e., there exists $\alpha > 0$, independent of h and h_2, such that*

$$\int_\Omega \beta|\nabla v|^2\,dx + \int_{\Omega_2}(\beta_2-\beta)|\nabla v_2|^2\,dx \geq \alpha(\|v\|_V^2 + \|v_2\|_{V_2}^2) \quad \forall (v,v_2)\in\mathbb{K}_h^2. \tag{26}$$

Proof. Due to the discussion above we need to estimate the quantity

$$\int_{\Omega_2}\beta\left(|\nabla v|^2 - |\nabla\Pi_h v|^2\right)\,dx.$$

Since β is bounded we have

$$\begin{aligned}\int_{\Omega_2}\beta\left||\nabla v|^2 - |\nabla\Pi_h v|^2\right|\,dx &\leq \overline{\beta}\int_{\Omega_2}|(\nabla v-\nabla\Pi_h v)(\nabla v+\nabla\Pi_h v)|\,dx\\ &\leq \overline{\beta}\|\nabla v-\nabla\Pi_h v\|_{L^2(\Omega_2)}\|\nabla v+\nabla\Pi_h v\|_{L^2(\Omega_2)}\\ &\leq \overline{\beta}\|\nabla v-\nabla\Pi_h v\|_{L^2(\Omega_2)}\left(\|\nabla v\|_{L^2(\Omega_2)}+\|\nabla v_2\|_{L^2(\Omega_2)}\right),\end{aligned} \tag{27}$$

where we have denoted by $\overline{\beta}$ an upper bound for β.

We denote by $v_I\in V_{2,h}$ the interpolation of $v_{|\Omega_2}$ with respect to the mesh $\mathcal{T}_{2,h}$. Since Π_h is the projection operator onto $V_{2,h}$, we obtain

$$\begin{aligned}\|\nabla v-\nabla\Pi_h v\|_{L^2(\Omega_2)} &\leq \|v-\Pi_h v\|_{V_2} \leq \inf_{w\in V_{2,h}}\|v-w\|_{V_2}\\ &\leq \|v-v_I\|_{V_2} = \left(\|v-v_I\|_{L^2(\Omega_2)}^2 + (\|\nabla v-\nabla v_I\|_{L^2(\Omega_2)}^2\right)^{1/2}.\end{aligned} \tag{28}$$

By standard interpolation estimate we have that $\|v-v_I\|_{L^2(\Omega_2)}\leq Ch_2\|\nabla v\|_{L^2(\Omega_2)}$, hence it remains to estimate the last term in the above relation. We set

$$\begin{aligned}\mathcal{T}_{2,h}^1 &= \{K\in\mathcal{T}_{2,h} : K \text{ is included in an element of } \mathcal{T}_h\},\\ \mathcal{T}_{2,h}^2 &= \mathcal{T}_{2,h}\setminus\mathcal{T}_{2,h}^1.\end{aligned}$$

Since $v\in V_h$ is piecewise linear, we have for its interpolation in $V_{2,h}$ that $v_I|_K = v|_K$ if $K\in\mathcal{T}_{2,h}^1$. Therefore we can compute

$$|\nabla v-\nabla v_I|_{L^2(\Omega_2)}^2 = \sum_{K\in\mathcal{T}_{2,h}}\|\nabla v-\nabla v_I\|_{L^2(K)}^2 = \sum_{K\in\mathcal{T}_{2,h}^2}\|\nabla v-\nabla v_I\|_{L^2(K)}^2.$$

Due to our construction $|\nabla v_I|\leq\max_K|\nabla v|$, so that for all $K\in\mathcal{T}_{2,h}^2$ it holds by inverse inequality

$$\int_K|\nabla v-\nabla v_I|^2\,dx \leq h_2^d\max_K|\nabla v-\nabla v_I|^2 \leq 2h_2^d\|\nabla v\|_{L^\infty(K)}^2 \leq C\frac{h_2^d}{h^{d/2}}\|\nabla v\|_{L^2(K)}^2.$$

In conclusion we have

$$|\nabla v - \nabla v_I|^2_{L^2(\Omega_2)} \leq C \frac{h_2^d}{h^{d/2}} \sum_{K \in \mathcal{T}^2_{2,h}} \|\nabla v\|^2_{L^2(K)} = C \frac{h_2^d}{h^{d/2}} \|\nabla v\|^2_{L^2(\Omega_2)}. \tag{29}$$

Combining the inequalities (27), (28), and (29), we get

$$\|\nabla v - \nabla \Pi_h v\|_{L^2(\Omega_2)} \leq C \left(h_2 + \frac{h_2^d}{h^{d/2}} \right) \|\nabla v\|_{L^2(\Omega_2)} = C \frac{h_2}{h^{d/2}} \left(h^{d/2} + h_2^{d-1} \right) \|\nabla v\|_{L^2(\Omega_2)}. \tag{30}$$

Then we can compute

$$\begin{aligned} \|\nabla v\|_{L^2(\Omega_2)} &\leq \|\nabla \Pi_h v\|_{L^2(\Omega_2)} + \|\nabla v - \nabla \Pi_h v\|_{L^2(\Omega_2)} \\ &\leq \|\nabla \Pi_h v\|_{L^2(\Omega_2)} + C \frac{h_2}{h^{d/2}} \left(h^{d/2} + h_2^{d-1} \right) \|\nabla v\|_{L^2(\Omega_2)}. \end{aligned}$$

Taking $h_2/h^{d/2}$ small enough such that $C\left(h_2/h^{d/2}\right)\left(h^{d/2} + h_2^{d-1}\right) \leq 1/2$ we get

$$\|\nabla v\|_{L^2(\Omega_2)} \leq 2\|\nabla \Pi_h v\|_{L^2(\Omega_2)} = 2\|\nabla v_2\|_{L^2(\Omega_2)}. \tag{31}$$

This last equation implies that

$$\begin{aligned} \int_\Omega \beta |\nabla v|^2 \, dx + \int_{\Omega_2} (\beta_2 - \beta) |\nabla v_2|^2 \, dx \geq \underline{\beta} \left(\|\nabla v\|^2_{L^2(\Omega_1)} + \|\nabla v_2\|^2_{L^2(\Omega_2)} \right) \\ - 2C \frac{h_2}{h^{d/2}} \left(h^{d/2} + h_2^{d-1} \right) \|\nabla v_2\|^2_{L^2(\Omega_2)}, \end{aligned}$$

and the estimate

$$\int_\Omega \beta |\nabla v|^2 \, dx + \int_{\Omega_2} (\beta_2 - \beta) |\nabla v_2|^2 \, dx \geq (\underline{\beta}/2) \left(\|\nabla v\|^2_{L^2(\Omega_1)} + \|\nabla v_2\|^2_{L^2(\Omega_2)} \right)$$

if $2C\left(h_2/h^{d/2}\right)\left(h^{d/2} + h_2^{d-1}\right) \leq \underline{\beta}/2$, where $\underline{\beta}$ denotes a (positive) lower bound of β.

Thanks to (31) we conclude that there exists a constant $\alpha_1 > 0$ such that the following estimate holds true

$$\int_\Omega \beta |\nabla v|^2 \, dx + \int_{\Omega_2} (\beta_2 - \beta) |\nabla v_2|^2 \, dx \geq \alpha_1 (\|\nabla v\|^2_{L^2(\Omega)} + \|\nabla v_2\|^2_{L^2(\Omega_2)}).$$

In order to obtain the ellipticity on the discrete kernel it remains to show that the right hand side of the above estimate controls also the L^2-norms of v and v_2. We observe that $v \in V_h \subset H^1_0(\Omega)$ so that by Poincaré inequality we have that $\|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_{L^2(\Omega)}$. If also $\partial\Omega_2 \setminus \Gamma$ is not empty, by Poincaré inequality we have that $\|\nabla v_2\|_{L^2(\Omega_2)}$ is equivalent to $\|v_2\|_{V_2}$ so that we obtain (26). Let us consider the case $\partial\Omega_2 \setminus \Gamma$ is empty. Taking into accont that $(v, v_2) \in \mathbb{K}^2_h$, (30), and (31), we have that

$$\|v_2\|_{L^2(\Omega_2)} = \|\Pi_h v\|_{L^2(\Omega_2)} \leq \|v\|_{L^2(\Omega_2)} + 2C \frac{h_2}{h^{d/2}} \left(h^{d/2} + h_2^{d-1} \right) \|\nabla v_2\|_{L^2(\Omega_2)}$$

and the estimate (26) is completely proved. □

4. Numerical experiments

In this section, we investigate the performance of the proposed methods for the discretization of (1) empirically. The following numerical results extend the one proposed in [2]. Specifically, we consider the new approach for the evaluation of the term involving the Lagrange multiplier introduced in this paper. Moreover, we show that the numerical results of the schemes are confirmed in the two-dimensional case.

For the sake of simplicity, we consider the case (of practical interest when homogeneous materials are involved) in which the functions β_1 and β_2 in (1) are constant in each subdomains. Regarding their values, we consider two different combinations of values for β_1 and β_2, namely $\beta_1 = 1, \beta_2 = 10$, and $\beta_1 = 10, \beta_2 = 1$. For the one-dimensional case, to illustrate the independence on the jump values as already noticed in [2], we also evaluate the cases $\beta_1 = 1, \beta_2 = 10000$, and $\beta_1 = 10000, \beta_2 = 1$.

We compare the performances of the standard Galerkin method, for which we consider both the cases of a uniform grid and a grid compatible with the discontinuity of the coefficients, and the two versions of fictitious domain method with distributed Lagrange multiplier introduced in Problem 3 and in Problem 4.

For all the methods, the convergence plots describe the evolution of the relative error with respect to the meshsize in the H^1 seminorm and in the L^2 norm. In particular, the plots relating to the fictitious domain approach depict the behavior of the errors for $|u - u_h|_{H^1(\Omega)}/|u|_{H^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}/\|u\|_{L^2(\Omega)}$ with respect to the meshsize h. We compare the empirical convergence behavior with the one expected from the theory, i.e., order $1/2$ for the H^1-error and order 1 for the L^2-error (resp. order 1 for the H^1-error and order 2 for the L^2-error when the standard Galerkin method with a compatible mesh is used). The plots of the corresponding quantities for the error $u_2 - u_{2,h}$ versus h_2 exhibit the same behavior, so we omit them.

Moreover, in both the one-dimensional and the two-dimensional cases, we have tried different values for the ratio between the meshsizes of the triangulations of Ω and Ω_2. Specifically, we have considered the cases $h/h_2 = 1/4, 1/2, 1, 2, 4$.

4.1. One-dimensional case

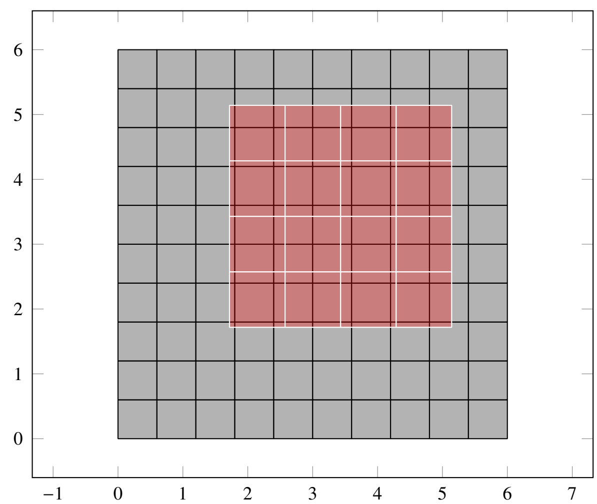
Given $a < b < c < d$, we consider $\Omega_1 = (a, b) \cup (c, d)$ and $\Omega_2 = (b, c)$ in (1). It follows that $\Omega = (a, d)$ and $\Gamma = \{b, c\}$. The numerical results refer to the domain defined by the parameters $a = 0$, $b = \exp(1)$, $c = 1 + \pi$, $d = 6$ (Ω_2 immersed in Ω). The choice of the interface points b and c as irrational numbers ensures that they never happen to be nodes of the mesh unless we explicitly impose it. For the discretization, we use the standard $\mathcal{P}_1$ finite element space of globally continuous and piecewise linear functions. In this simple one-dimensional case, it is possible to compute the analytical expression of the exact solution and the errors can thus be evaluated exactly.

In Figure 2 we show the convergence rate when the standard Galerkin $\mathcal{P}_1$ finite element method is used with $\beta_1 = 1$ and $\beta_2 = 10$. As expected, the scheme is of first order in H^1 (resp. second order in L^2) when a compatible mesh is used, while it loses half power of h in H^1 and one full power of h in L^2 when the mesh does not match with the discontinuity. Figures 6 and 7 show the convergence history when the material properties are $\beta_1 = 1$ and $\beta_2 = 10$. In this case, both our schemes behave correctly. For this material choice, all the data are provided in Tables 1, 2, 3, and 4.

Figure 3 shows the standard Galerkin solution when $\beta_1 = 10$ and $\beta_2 = 1$. The convergence rates of our two proposed schemes are reported in Figures 8 and 9. In this case, the numerical results are different: For the method from Problem 3 the cases $h \approx h_2/4$, $h \approx h_2/2$ and $h \approx h_2$ are critical. On the other hand, for the method from Problem 4, the convergence is guaranteed, independently of the ratio between the meshsizes used in the discretization.

Analogously, Figures 4, 5, 10, 11, 12, and 13 refer to the cases $\beta_1 = 1, \beta_2 = 10000$, and $\beta_1 = 10000, \beta_2 = 1$. The qualitative behavior of the methods is unchanged.

Figure 1: Domain and meshes for the fictitious domain approach in 2D.



4.2. Two-dimensional case

Given $a < b < c < d$ and $e < f < g < h$, we consider the boundary value problem (1), where the domains are defined by $\Omega_2 = (b, c) \times (f, g)$ and $\Omega_1 = \Omega \setminus \Omega_2$, with $\Omega = (a, d) \times (e, h)$. The numerical results refer to the choice $a = 0$, $b = \exp(1)$, $c = 1 + \pi$, $d = 6$, $e = 0$, $f = \exp(1)$, $g = 1 + \pi$, $h = 6$, i.e., Ω and Ω_2 are both square domains. For the discretization, we use regular partitions of the domains into squares and the standard Q_1 finite element space of globally continuous and piecewise bilinear functions. A possible configuration of the triangulations of Ω and Ω_2 in the fictitious domain approach is depicted in Figure 1. In this case, since an analytic expression of the exact solution is not available, we use as reference solution its standard Galerkin Q_1-FEM approximation, obtained by using a regular mesh of 400×400 quadrilaterals (160801 grid points) compatible with the discontinuity of β (meshsize $h \approx 0.0212$).

The two-dimensional numerical results totally agree with those obtained in the one-dimensional case. As before, the standard Galerkin method shows the expected convergence rate for both the cases $\beta_1 = 1, \beta_2 = 10$, and $\beta_1 = 10, \beta_2 = 1$, see Figures 14 and 15.

For the plots related to the fictitious domain approach when $\beta_1 = 1, \beta_2 = 10$, we refer to Figures 16 and 17. The computed convergence rates are the same as the one of the standard Galerkin method and they are independent of the value of the ratio h/h_2 used in the discretization.

When $\beta_1 = 10$ and $\beta_2 = 1$, the convergence of the scheme from Problem 4 is guaranteed, see Figure 19. In the case of the method from Problem 3, the numerical results which correspond to the choices $h/h_2 = 1/4, 1/2, 1$ seem to reveal that the scheme is unstable, while the method converges for the choices $h/h_2 = 2, 4$, see Figure 18.

4.3. Conclusions

Our new two-dimensional numerical results confirm the performance of the methods obtained in the one-dimensional case. The computed convergence rates show that the behavior of the fictitious domain approach is comparable to the one of the standard Galerkin method. The oscillations in the convergence plot reflect the fact that the mesh is non-matching. Clearly, if the mesh matches with the interface, we see no oscillations in the convergence plot and obtain a full order of convergence.

The numerical experiments also show that, if the duality pair is evaluated as the H^1 scalar product, then the method behaves robustly with respect to the values of the parameters β_1 and β_2, i.e., no condition on the ratio between the meshsizes h and h_2 needs to be assumed. On the other hand, when the duality pair is evaluated as the L^2 scalar product, if $\beta_1 > \beta_2$, then h_2/h must be small enough. Unfortunately, this robustness of the method, clear from numerical evidence, is not fully covered by our theory at the moment.

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Table 1: 1D - L^2-error - Galerkin method vs FD/DLM method (L2) - $\beta_1 = 1, \beta_2 = 10$

	Galerkin method		FD/DLM P1 method (L2)				
h	unif.	comp.	$h \approx h_2/4$	$h \approx h_2/2$	$h \approx h_2$	$h \approx 2h_2$	$h \approx 4h_2$
$1.88 \cdot 10^{-2}$	$2.69 \cdot 10^{-4}$	$8.08 \cdot 10^{-6}$	$1.29 \cdot 10^{-3}$	$4.34 \cdot 10^{-4}$	$2.63 \cdot 10^{-4}$	$2.68 \cdot 10^{-4}$	$2.67 \cdot 10^{-4}$
$9.38 \cdot 10^{-3}$	$2.61 \cdot 10^{-4}$	$2.02 \cdot 10^{-6}$	$6.44 \cdot 10^{-4}$	$2.18 \cdot 10^{-4}$	$2.47 \cdot 10^{-4}$	$2.62 \cdot 10^{-4}$	$2.61 \cdot 10^{-4}$
$4.69 \cdot 10^{-3}$	$2.47 \cdot 10^{-4}$	$5.05 \cdot 10^{-7}$	$3.25 \cdot 10^{-4}$	$1.24 \cdot 10^{-4}$	$1.22 \cdot 10^{-4}$	$2.48 \cdot 10^{-4}$	$2.46 \cdot 10^{-4}$
$2.34 \cdot 10^{-3}$	$1.24 \cdot 10^{-4}$	$1.26 \cdot 10^{-7}$	$1.62 \cdot 10^{-4}$	$6.24 \cdot 10^{-5}$	$1.78 \cdot 10^{-5}$	$1.15 \cdot 10^{-5}$	$2.27 \cdot 10^{-5}$
$1.17 \cdot 10^{-3}$	$7.98 \cdot 10^{-5}$	$3.16 \cdot 10^{-8}$	$8.12 \cdot 10^{-5}$	$3.28 \cdot 10^{-5}$	$1.34 \cdot 10^{-5}$	$1.43 \cdot 10^{-5}$	$5.94 \cdot 10^{-5}$
$5.86 \cdot 10^{-4}$	$4.07 \cdot 10^{-5}$	$7.82 \cdot 10^{-9}$	$4.08 \cdot 10^{-5}$	$1.69 \cdot 10^{-5}$	$5.96 \cdot 10^{-6}$	$3.33 \cdot 10^{-5}$	$4.12 \cdot 10^{-5}$
$2.93 \cdot 10^{-4}$	$1.25 \cdot 10^{-5}$	$2 \cdot 10^{-9}$	$2.02 \cdot 10^{-5}$	$7.28 \cdot 10^{-6}$	$9.84 \cdot 10^{-6}$	$1.26 \cdot 10^{-5}$	$1.24 \cdot 10^{-5}$
$1.46 \cdot 10^{-4}$	$1.04 \cdot 10^{-5}$	$8.55 \cdot 10^{-10}$	$1.02 \cdot 10^{-5}$	$4.22 \cdot 10^{-6}$	$1.69 \cdot 10^{-6}$	$7.06 \cdot 10^{-6}$	$1.05 \cdot 10^{-5}$

Table 2: 1D - L^2-error - Galerkin method vs FD/DLM method (H1) - $\beta_1 = 1, \beta_2 = 10$

	Galerkin method		FD/DLM P1 method (H1)				
h	uniform	compatible	$h \approx h_2/4$	$h \approx h_2/2$	$h \approx h_2$	$h \approx 2h_2$	$h \approx 4h_2$
$1.88 \cdot 10^{-2}$	$2.69 \cdot 10^{-4}$	$8.08 \cdot 10^{-6}$	$1.98 \cdot 10^{-4}$	$2.39 \cdot 10^{-4}$	$2.66 \cdot 10^{-4}$	$2.66 \cdot 10^{-4}$	$2.66 \cdot 10^{-4}$
$9.38 \cdot 10^{-3}$	$2.61 \cdot 10^{-4}$	$2.02 \cdot 10^{-6}$	$2.01 \cdot 10^{-4}$	$2.23 \cdot 10^{-4}$	$2.57 \cdot 10^{-4}$	$2.58 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$
$4.69 \cdot 10^{-3}$	$2.47 \cdot 10^{-4}$	$5.05 \cdot 10^{-7}$	$1.5 \cdot 10^{-4}$	$1.64 \cdot 10^{-4}$	$2.18 \cdot 10^{-4}$	$2.45 \cdot 10^{-4}$	$2.45 \cdot 10^{-4}$
$2.34 \cdot 10^{-3}$	$1.24 \cdot 10^{-4}$	$1.26 \cdot 10^{-7}$	$2.12 \cdot 10^{-5}$	$2.22 \cdot 10^{-5}$	$2.33 \cdot 10^{-5}$	$3.91 \cdot 10^{-5}$	$6.38 \cdot 10^{-5}$
$1.17 \cdot 10^{-3}$	$7.98 \cdot 10^{-5}$	$3.16 \cdot 10^{-8}$	$2 \cdot 10^{-5}$	$2.09 \cdot 10^{-5}$	$2.29 \cdot 10^{-5}$	$4.54 \cdot 10^{-5}$	$6.75 \cdot 10^{-5}$
$5.86 \cdot 10^{-4}$	$4.07 \cdot 10^{-5}$	$7.82 \cdot 10^{-9}$	$1.65 \cdot 10^{-5}$	$1.76 \cdot 10^{-5}$	$2.14 \cdot 10^{-5}$	$3.69 \cdot 10^{-5}$	$3.95 \cdot 10^{-5}$
$2.93 \cdot 10^{-4}$	$1.25 \cdot 10^{-5}$	$2 \cdot 10^{-9}$	$8.86 \cdot 10^{-6}$	$9.51 \cdot 10^{-6}$	$1.19 \cdot 10^{-5}$	$1.24 \cdot 10^{-5}$	$1.24 \cdot 10^{-5}$
$1.46 \cdot 10^{-4}$	$1.04 \cdot 10^{-5}$	$8.55 \cdot 10^{-10}$	$3.77 \cdot 10^{-6}$	$3.99 \cdot 10^{-6}$	$4.68 \cdot 10^{-6}$	$8.77 \cdot 10^{-6}$	$1.03 \cdot 10^{-5}$

Table 3: 1D - H^1-error - Galerkin method vs FD/DLM method (L2) - $\beta_1 = 1, \beta_2 = 10$

	Galerkin method		FD/DLM P1 method (L2)				
h	unif.	comp.	$h \approx h_2/4$	$h \approx h_2/2$	$h \approx h_2$	$h \approx 2h_2$	$h \approx 4h_2$
$1.88 \cdot 10^{-2}$	$3.02 \cdot 10^{-3}$	$2.78 \cdot 10^{-3}$	$3.05 \cdot 10^{-2}$	$2.06 \cdot 10^{-2}$	$2.89 \cdot 10^{-3}$	$2.82 \cdot 10^{-3}$	$2.83 \cdot 10^{-3}$
$9.38 \cdot 10^{-3}$	$1.81 \cdot 10^{-2}$	$1.39 \cdot 10^{-3}$	$1.85 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$	$1.77 \cdot 10^{-2}$	$1.82 \cdot 10^{-2}$	$1.81 \cdot 10^{-2}$
$4.69 \cdot 10^{-3}$	$1.23 \cdot 10^{-2}$	$6.95 \cdot 10^{-4}$	$1.47 \cdot 10^{-2}$	$1.15 \cdot 10^{-2}$	$1.04 \cdot 10^{-2}$	$1.22 \cdot 10^{-2}$	$1.22 \cdot 10^{-2}$
$2.34 \cdot 10^{-3}$	$6.23 \cdot 10^{-3}$	$3.47 \cdot 10^{-4}$	$9.14 \cdot 10^{-3}$	$5.02 \cdot 10^{-3}$	$4.88 \cdot 10^{-4}$	$8.12 \cdot 10^{-4}$	$1.84 \cdot 10^{-3}$
$1.17 \cdot 10^{-3}$	$6.13 \cdot 10^{-3}$	$1.74 \cdot 10^{-4}$	$6.07 \cdot 10^{-3}$	$3.1 \cdot 10^{-3}$	$9.97 \cdot 10^{-4}$	$1.93 \cdot 10^{-3}$	$4.76 \cdot 10^{-3}$
$5.86 \cdot 10^{-4}$	$3.91 \cdot 10^{-3}$	$8.68 \cdot 10^{-5}$	$5.83 \cdot 10^{-3}$	$4.61 \cdot 10^{-3}$	$4.19 \cdot 10^{-3}$	$3.55 \cdot 10^{-3}$	$3.94 \cdot 10^{-3}$
$2.93 \cdot 10^{-4}$	$3.14 \cdot 10^{-3}$	$4.34 \cdot 10^{-5}$	$3.5 \cdot 10^{-3}$	$2.76 \cdot 10^{-3}$	$2.85 \cdot 10^{-3}$	$3.15 \cdot 10^{-3}$	$3.13 \cdot 10^{-3}$
$1.46 \cdot 10^{-4}$	$1.92 \cdot 10^{-3}$	$2.17 \cdot 10^{-5}$	$2.97 \cdot 10^{-3}$	$2.36 \cdot 10^{-3}$	$2.14 \cdot 10^{-3}$	$1.67 \cdot 10^{-3}$	$1.93 \cdot 10^{-3}$

Table 4: 1D - H^1-error - Galerkin method vs FD/DLM method (H1) - $\beta_1 = 1, \beta_2 = 10$

	Galerkin method		FD/DLM P1 method (H1)				
h	uniform	compatible	$h \approx h_2/4$	$h \approx h_2/2$	$h \approx h_2$	$h \approx 2h_2$	$h \approx 4h_2$
$1.88 \cdot 10^{-2}$	$3.02 \cdot 10^{-3}$	$2.78 \cdot 10^{-3}$	$7.53 \cdot 10^{-3}$	$5.06 \cdot 10^{-3}$	$2.85 \cdot 10^{-3}$	$2.85 \cdot 10^{-3}$	$2.84 \cdot 10^{-3}$
$9.38 \cdot 10^{-3}$	$1.81 \cdot 10^{-2}$	$1.39 \cdot 10^{-3}$	$1.56 \cdot 10^{-2}$	$1.62 \cdot 10^{-2}$	$1.79 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$	$1.81 \cdot 10^{-2}$
$4.69 \cdot 10^{-3}$	$1.23 \cdot 10^{-2}$	$6.95 \cdot 10^{-4}$	$9.7 \cdot 10^{-3}$	$1 \cdot 10^{-2}$	$1.12 \cdot 10^{-2}$	$1.21 \cdot 10^{-2}$	$1.22 \cdot 10^{-2}$
$2.34 \cdot 10^{-3}$	$6.23 \cdot 10^{-3}$	$3.47 \cdot 10^{-4}$	$1.35 \cdot 10^{-3}$	$1.37 \cdot 10^{-3}$	$1.58 \cdot 10^{-3}$	$2.22 \cdot 10^{-3}$	$3.35 \cdot 10^{-3}$
$1.17 \cdot 10^{-3}$	$6.13 \cdot 10^{-3}$	$1.74 \cdot 10^{-4}$	$1.77 \cdot 10^{-3}$	$1.91 \cdot 10^{-3}$	$2.23 \cdot 10^{-3}$	$3.6 \cdot 10^{-3}$	$5.2 \cdot 10^{-3}$
$5.86 \cdot 10^{-4}$	$3.91 \cdot 10^{-3}$	$8.68 \cdot 10^{-5}$	$3.53 \cdot 10^{-3}$	$3.55 \cdot 10^{-3}$	$3.62 \cdot 10^{-3}$	$3.67 \cdot 10^{-3}$	$3.83 \cdot 10^{-3}$
$2.93 \cdot 10^{-4}$	$3.14 \cdot 10^{-3}$	$4.34 \cdot 10^{-5}$	$2.53 \cdot 10^{-3}$	$2.64 \cdot 10^{-3}$	$3.02 \cdot 10^{-3}$	$3.11 \cdot 10^{-3}$	$3.12 \cdot 10^{-3}$
$1.46 \cdot 10^{-4}$	$1.92 \cdot 10^{-3}$	$2.17 \cdot 10^{-5}$	$1.81 \cdot 10^{-3}$	$1.82 \cdot 10^{-3}$	$1.84 \cdot 10^{-3}$	$1.75 \cdot 10^{-3}$	$1.9 \cdot 10^{-3}$

Figure 2: 1D - Standard Galerkin P1 method - $\beta_1 = 1, \beta_2 = 10$

(a) uniform mesh (b) compatible mesh

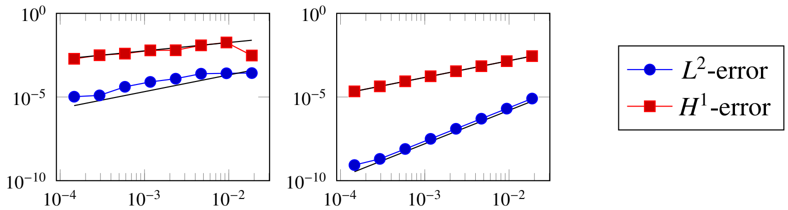

Figure 3: 1D - Standard Galerkin P1 method - $\beta_1 = 10, \beta_2 = 1$

(a) uniform mesh (b) compatible mesh

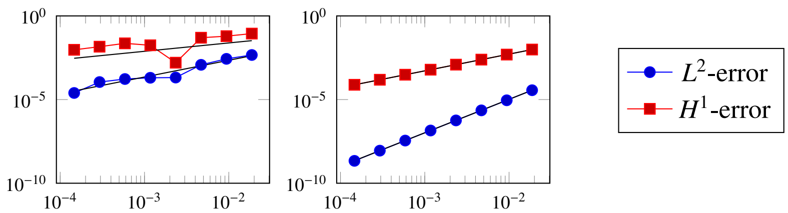

Figure 4: 1D - Standard Galerkin P1 method - $\beta_1 = 1, \beta_2 = 10000$

(a) uniform mesh (b) compatible mesh

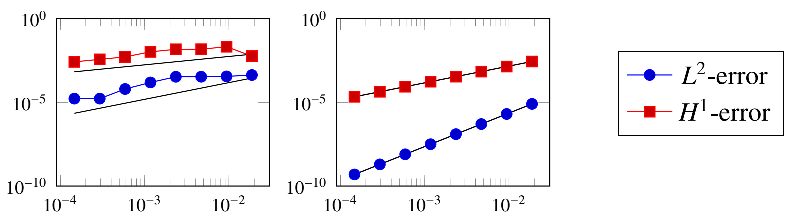

Figure 5: 1D - Standard Galerkin P1 method - $\beta_1 = 10000, \beta_2 = 1$

(a) uniform mesh (b) compatible mesh

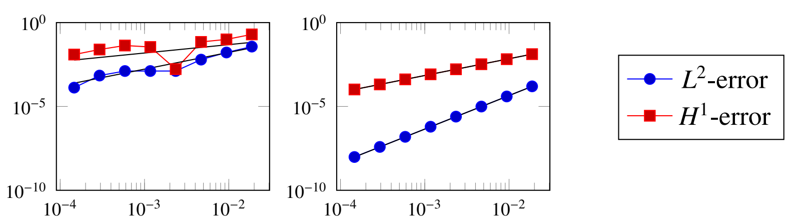

Figure 6: 1D - FD/DLM P1 method (L2) - $\beta_1 = 1, \beta_2 = 10$

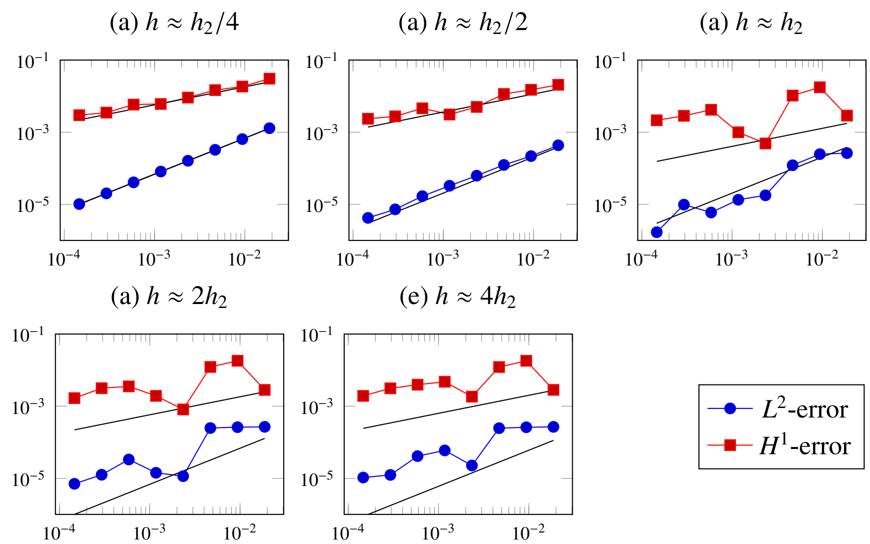

Figure 7: 1D - FD/DLM P1 method (H1) - $\beta_1 = 1, \beta_2 = 10$

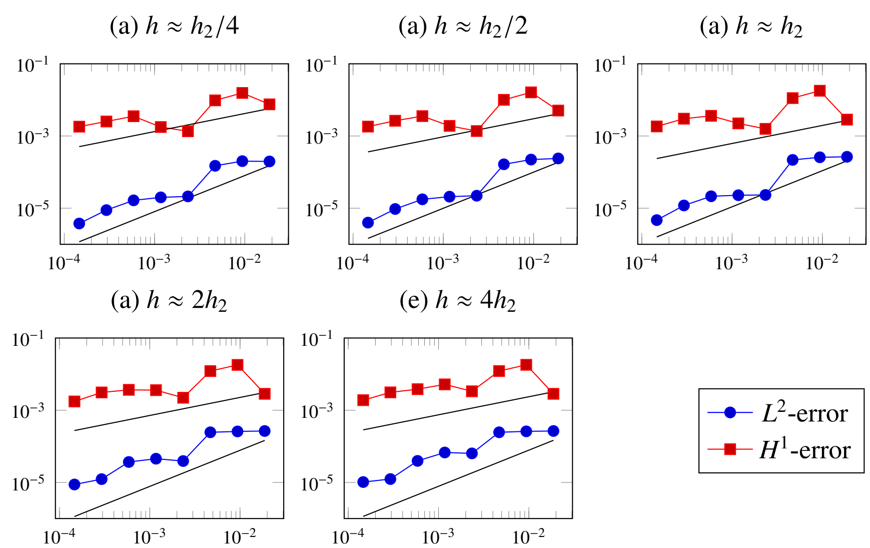

Figure 8: 1D - FD/DLM P1 method (L2) - $\beta_1 = 10, \beta_2 = 1$

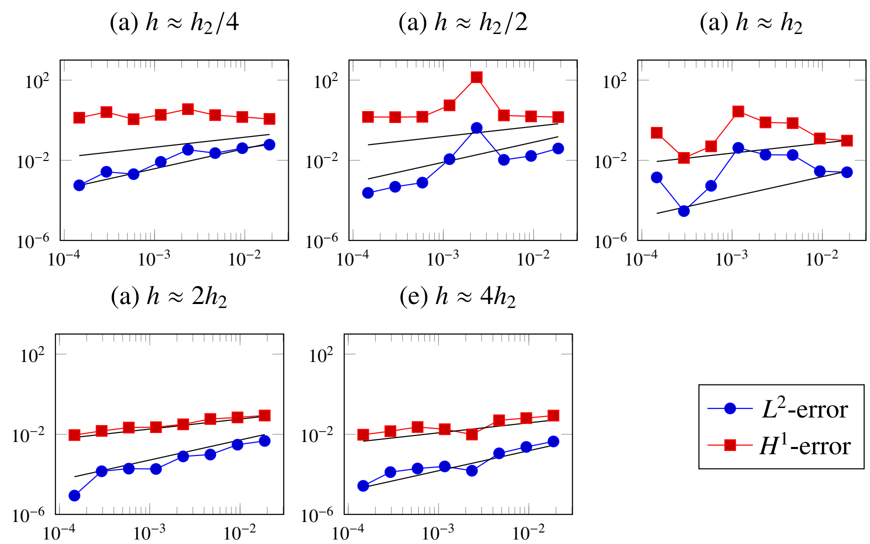

Figure 9: 1D - FD/DLM P1 method (H1) - $\beta_1 = 10, \beta_2 = 1$

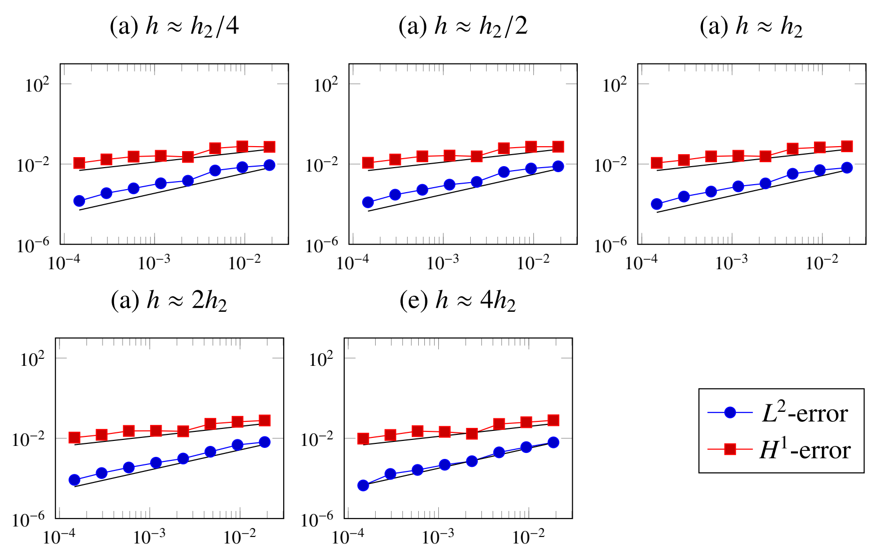

Figure 10: 1D - FD/DLM P1 method (L2) - $\beta_1 = 1, \beta_2 = 10000$

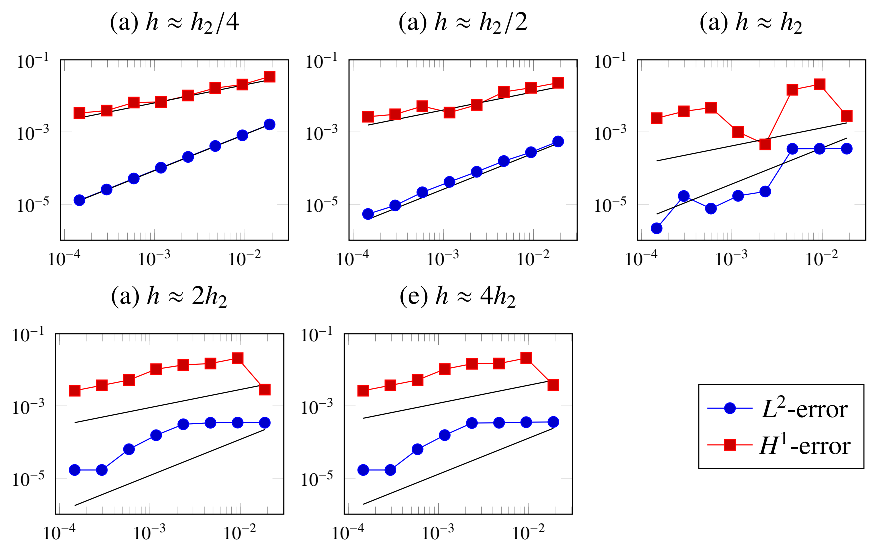

Figure 11: 1D - FD/DLM P1 method (H1) - $\beta_1 = 1, \beta_2 = 10000$

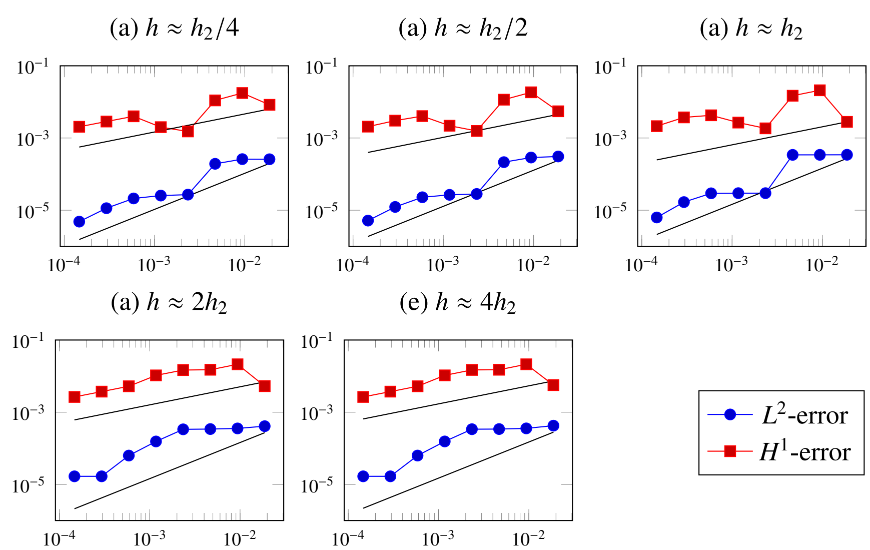

Figure 12: 1D - FD/DLM P1 method (L2) - $\beta_1 = 10000, \beta_2 = 1$

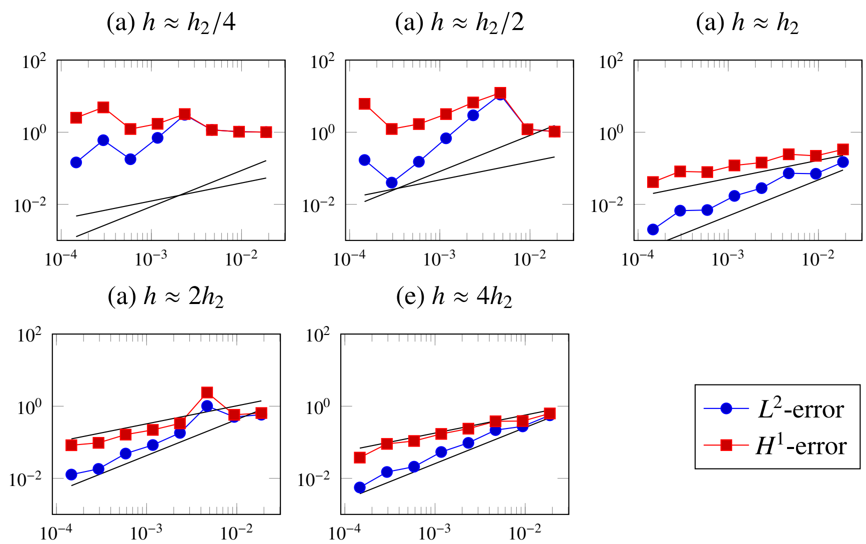

Figure 13: 1D - FD/DLM P1 method (H1) - $\beta_1 = 10000, \beta_2 = 1$

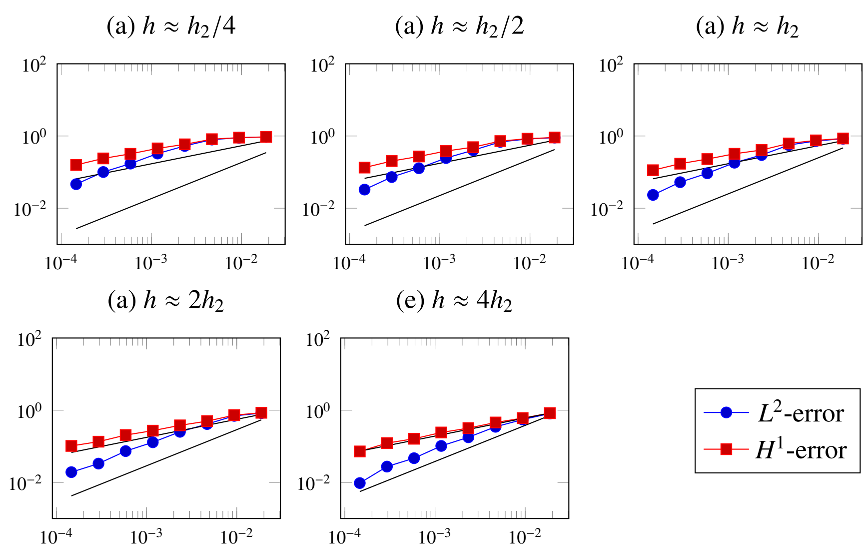

Figure 14: 2D - Standard Galerkin Q1 method - $\beta_1 = 1, \beta_2 = 10$

(a) uniform mesh (b) compatible mesh

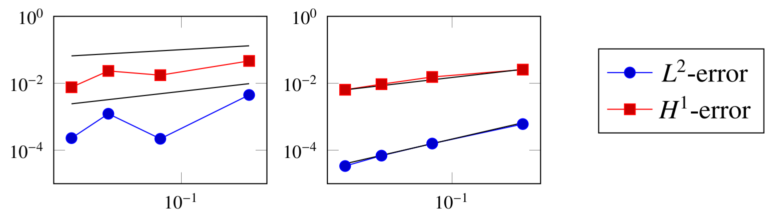

Figure 15: 2D - Standard Galerkin Q1 method - $\beta_1 = 10, \beta_2 = 1$

(a) uniform mesh (b) compatible mesh

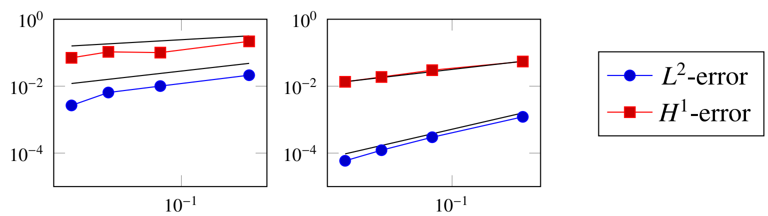

Figure 16: 2D - FD/DLM Q1 method (L2) - $\beta_1 = 1, \beta_2 = 10$

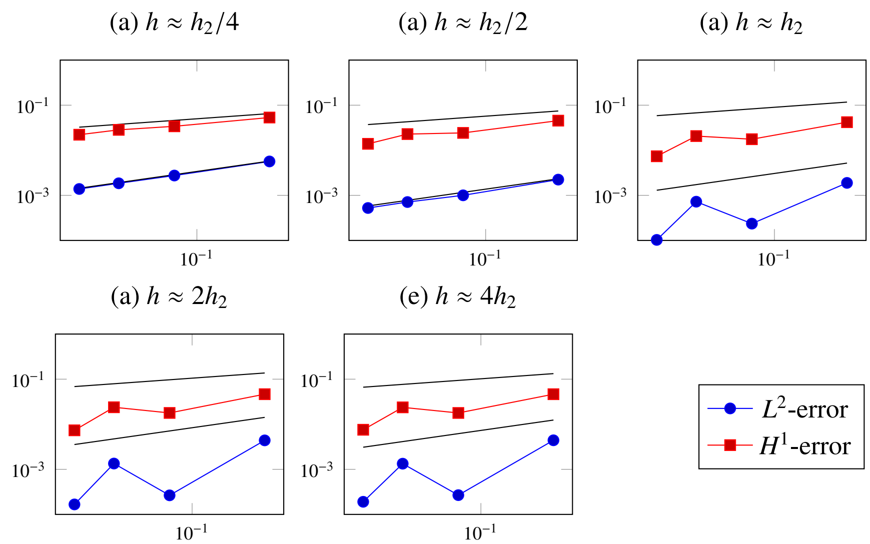

Figure 17: 2D - FD/DLM Q1 method (H1) - $\beta_1 = 1, \beta_2 = 10$

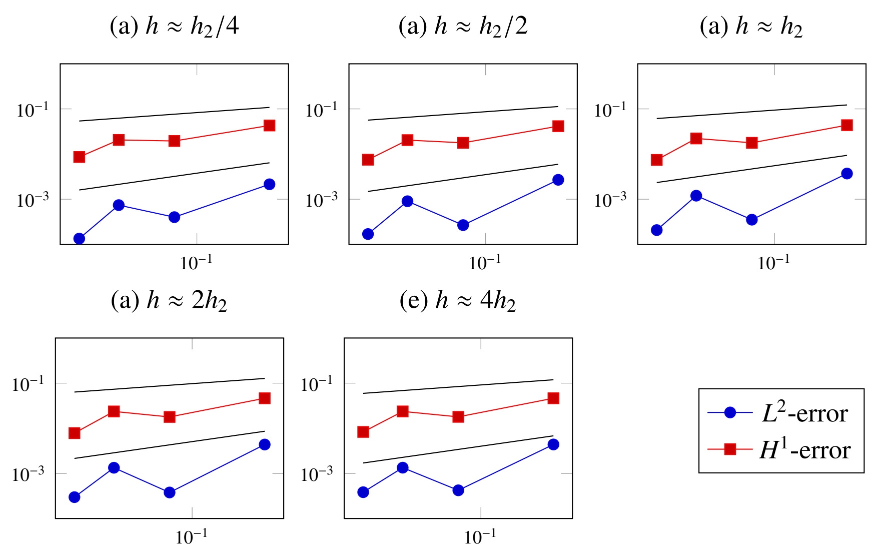

Figure 18: 2D - FD/DLM Q1 method (L2) - $\beta_1 = 10, \beta_2 = 1$

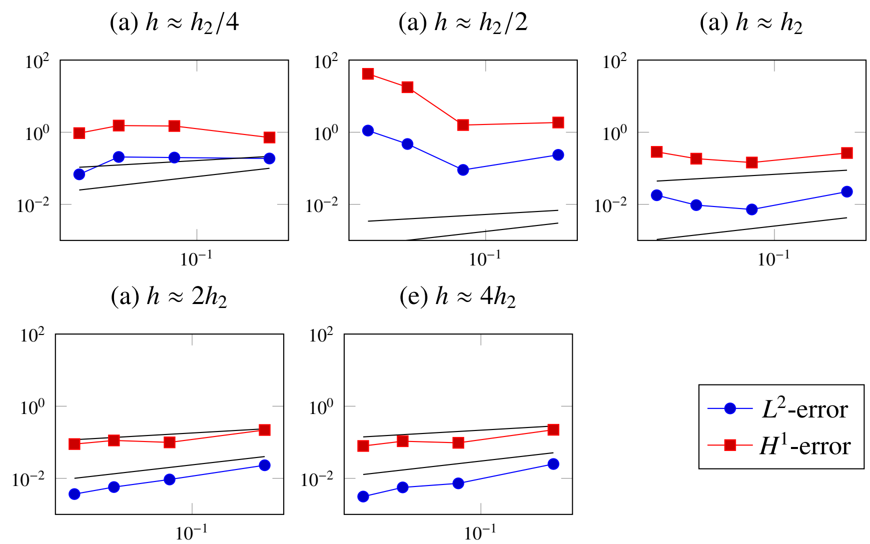

Figure 19: 2D - FD/DLM Q1 method (H1) - $\beta_1 = 10, \beta_2 = 1$