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ISBN 978-3-902627-05-6

Spin-polarized transport in ferromagnetic multilayers:  
An unconditionally convergent FEM integrator

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Abstract

We propose and analyze a decoupled time-marching scheme for the coupling of the Landau-Lifshitz-Gilbert equation with a quasilinear diffusion equation for the spin accumulation. This model describes the interplay of magnetization and electron spin accumulation in magnetic and nonmagnetic multilayer structures. Despite the strong nonlinearity of the overall PDE system, the proposed integrator requires only the solution of two linear systems per time-step. Unconditional convergence of the integrator towards weak solutions is proved.

Keywords: micromagnetics, Landau-Lifshitz-Gilbert equation, spin accumulation, finite element method 
2000 MSC: 35K55, 65M60, 65Z05

1. Introduction

The interaction between electric current and magnetization in magnetic nanostructure devices and the control of this interaction have been realized through the prediction of the spin-transfer torque by Slonczewski and Berger \([14, 31]\). The transfer of spin angular momentum between the spin-polarized electrical current and the local magnetization has been observed in various magnetic devices, such as metallic spin-valves systems, magnetic tunnel junctions, and magnetic domain walls in permalloy nanowires \([29, 33]\). Based on these experiments, a number of technological applications have been proposed, e.g., STT-MRAMs, racetrack memories, and magnetic vortex oscillators \([26, 27]\).

The fundamental physics underlying these phenomena is understood as due to a spin torque that arises from the transfer of the spin angular momentum between conduction free electrons and magnetization. In the original works of Berger and Slonczewski \([14, 31]\), a homogeneous spin accumulation is assumed due to a current which flows through a first magnetic layer perpendicular to the interface into a second magnetic layer. The spin torque effect leads to an interaction...
between the spin-polarized current and the magnetization in the second layer. For magnetic multilayers it has been shown that a proper description of the magnetoresistance is essential to take into account the interplay between successive interfaces [22, 34, 32]. In order to calculate the spin torque transfer, the spin transport properties have to be calculated far beyond the interface.

The original model of Berger and Slonczewski has been extended by taking into account the diffusion process of the spin accumulation by Shpiro et al. for one-dimensional systems [30] and by García-Cervera and Wang [21, 20] for three-dimensional systems. There, the overall system of PDEs (SDLLG) is a quasilinear diffusion equation for the evolution of the spin accumulation coupled to the Landau-Lifshitz-Gilbert equation (LLG) for the magnetization dynamics. Existence of global weak solutions to LLG goes back to [5], while, in the same spirit, existence of global weak solutions to SDLLG is proved in [21].

The reliable numerical integration of LLG (and, in particular, SDLLG) faces several challenges due to the nonuniqueness of weak solutions, the explicit nonlinearity, and an inherent nonconvex modulus constraint. Numerical approximation schemes for weak solutions of LLG are first proposed in [3, 12]. First unconditional convergence results can be found in [13, 2], which consider the small-particle limit of LLG with exchange only. On the one hand, the integrator of [13] relies on the midpoint rule and reduced integration, and thus has to solve one nonlinear system of equations per time-step. On the other hand, the tangent plane integrator of [2], which extends the prior works [3, 12], relies on a reformulation of LLG which is solved for the discrete time derivative. Each time-step consists of the solution of one linear system of equations plus nodal projection. It has been generalized to linear-implicit time integration and full effective field in [4, 15].

Numerical integration of the coupling of LLG to other time-dependent PDEs has been analyzed in [6, 7] for the full Maxwell equations (MLLG), in [25, 24] for the eddy current formulation, and in [8] for LLG with magnetostriction. While [6] analyzes an extension of the midpoint scheme of [13], the works [7, 25, 24, 8] extend the tangent plane scheme from [2], and emphasis is on the decoupling of the time-marching scheme in [7, 24, 8].

In the models and works mentioned, e.g., MLLG, the coupling of LLG and Maxwell equations is weak in the sense that the magnetization of LLG only contributes to the right-hand side of the Maxwell system, while the magnetic field from the Maxwell equations gives a contribution to the effective field of LLG. In SDLLG the principal part of the differential operator of the spin diffusion equation depends nonlinearly on the magnetization. A first numerical integrator for SDLLG is proposed and empirically validated in [20]. While this scheme appears to be unconditionally stable, the work does not prove convergence of the discrete solution towards a weak solution of SDLLG.

In our work, we extend the tangent plane integrator to SDLLG and prove unconditional convergence. Altogether, the contributions of the current work can be summarized as follows:

- The proposed integrator is proven to converge (at least for a subsequence) towards a weak solution of SDLLG. This convergence is unconditional, i.e., there is no CFL-type coupling of the time and space discretizations. Despite the nonlinearity of SDLLG, each time-step requires only the solution of two successive linear systems, one for (the discrete time derivative of) the magnetization and one for the spin accumulation.

- Our analysis thus provides, in particular, an alternate proof for the existence of (global) weak solutions of SDLLG, which has first been proved in [21]. In addition to [21], we prove that any weak limit of the proposed integrator satisfies an energy estimate similar to the theoretical behavior of (formal) strong solutions of SDLLG.
Unlike prior work on the tangent plane integrator, we adopt an idea from [11] and show that the nodal projection step of the tangent plane scheme is not necessary. In particular and unlike the cited works, our analysis can therefore avoid a technical angle condition on the triangulations used. This result also transfers to the models and analysis of [2, 4, 15, 7, 8, 24] and simplifies their (extended) tangent plane integrators.

1.1. Outline

The paper is organized as follows: In Section 2, we introduce and accurately describe the mathematical model, see (9) for the nondimensional formulation of SDLLG. In Section 3, we formulate a decoupled time-marching scheme (Algorithm 6) for the numerical integration of SDLLG and prove its well-posedness (Proposition 9). Section 4 contains the main result of our work (Theorem 12), which states unconditional convergence of the scheme towards weak solutions of SDLLG. Following [21], weak solutions of SDLLG have finite energy. In Section 5, we prove that any weak limit obtained by the proposed numerical integrator shows the same energy behavior as formal strong solutions of SDLLG (Theorem 24). Numerical examples as well as the empirical validation of the proposed algorithm are postponed to a forthcoming paper [1].

1.2. Notation

We use the standard notation [16] for Lebesgue and Sobolev spaces and norms. For any domain $D$, we denote the $L^2$ scalar product by $(r, s)_D = \int_D rs$ for all $r, s \in L^2(D)$. In the case of (spaces of) vector-valued functions, we use bold letters. For a sequence $\{x_n\}_{n \geq 1}$ in a Banach space $X$ and $x \in X$, we write $x_n \to x$ (resp. $x_n \rightharpoonup x$) in $X$ if the sequence converges strongly (resp. weakly) to $x$ in $X$. Similarly, we write $x_n \overset{\text{sub}}{\to} x$ (resp. $x_n \overset{\text{sub}}{\rightharpoonup} x$) in $X$ if there exists a subsequence of $\{x_n\}_{n \geq 1}$ which converges strongly (resp. weakly) to $x$ in $X$. Throughout the paper, $C$ denotes a generic positive constant, independent of the discretization parameters, not necessarily the same at each occurrence. Alternatively, we write $A \lesssim B$ to abbreviate $A \leq CB$. Given $a, b \in \mathbb{R}^3$, we denote by $a \otimes b \in \mathbb{R}^{3 \times 3}$ the tensor product defined by $(a \otimes b)_{jk} = a_j b_k$ for all $1 \leq j, k \leq 3$. By $|\cdot|$, we denote both the Frobenius norm of a matrix and the Euclidean norm of a vector. Since the meaning is clear from the argument, this does not lead to any ambiguity.

2. Model problem

In this section, we present the mathematical model, for which we introduce a nondimensional formulation, as well as the notion of a weak solution. We use physical units in the International System of Units (SI).

2.1. Physical background

We consider a magnetic multilayer. Let $\omega \subset \Omega$ be polyhedral Lipschitz domains in $\mathbb{R}^3$, where $\Omega$ corresponds to the volume occupied by the multilayer, and $\omega$ corresponds to the ferromagnetic part. A possible experimental setup is shown in Figure 2.1. Given some finite time $T > 0$, we consider the time-space domains $\omega_T := (0, T) \times \omega$ and $\Omega_T := (0, T) \times \Omega$.

In micromagnetics, the quantity of interest is the magnetization $M : \omega_T \to \mathbb{R}^3$, measured in ampere per meter (A/m). If the temperature is constant and far below from the Curie temperature of the ferromagnetic material, $M$ is a vector field of constant modulus $|M| = M_s$, with $M_s$ being the saturation magnetization (in A/m). In the absence of spin currents, the dynamics of $M$ is
described by the Landau-Lifshitz-Gilbert equation (LLG), which, in the so-called Gilbert form, reads
\[
\frac{\partial \mathbf{M}}{\partial t} = -\gamma \mu_0 \mathbf{M} \times \mathbf{H}_{\text{eff}} + \frac{\alpha}{M_s} \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} \quad \text{in } \omega_T.
\] (1)

Here, \(\gamma = 1.76 \times 10^{11} \text{ rad/(s T)}\) (radian per second per tesla) and \(\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2\) (newton per square ampere) are the gyromagnetic ratio and the permeability of vacuum, respectively, while \(\alpha > 0\) is the nondimensional empiric Gilbert damping parameter. The effective field \(\mathbf{H}_{\text{eff}} : \Omega_T \to \mathbb{R}^3\), measured in A/m, depends on \(\mathbf{M}\) and is proportional to the negative functional derivative of the total magnetic Gibbs free energy with respect to \(\mathbf{M}\), i.e.,
\[
\mu_0 \mathbf{H}_{\text{eff}}(\mathbf{M}) = -\frac{\delta E(\mathbf{M})}{\delta \mathbf{M}}.
\] (2)

In (2) the energy functional reads
\[
E(\mathbf{M}) = \frac{A}{M_s^2} \int_\omega |\nabla \mathbf{M}|^2 + K \int_\omega \phi(\mathbf{M}/M_s) - \mu_0 \int_\omega \mathbf{H}_e \cdot \mathbf{M} + \frac{\mu_0}{2} \int_{\mathbb{R}^3} |\nabla u|^2
\] (3)

and consists of four terms, which correspond to the exchange energy, the anisotropy energy, Zeeman’s energy, and the magnetostatic energy, respectively. In (3), \(A > 0\) is the so-called exchange stiffness constant, measured in joule per meter (J/m), and \(K > 0\) is the anisotropic constant (in J/m\(^3\)), while \(\phi : \mathbb{S}^2 \to \mathbb{R}\) is a (nondimensional) smooth function, which takes into account the anisotropy of the ferromagnetic material. Moreover, \(\mathbf{H}_e\) is a given external field (in A/m), while \(u : \mathbb{R}^3 \to \mathbb{R}\) refers to the magnetostatic potential, which is the unique solution of the full-space transmission problem
\[
\Delta u = \nabla \cdot \mathbf{M} \quad \text{in } \omega,
\]
\[
\Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus \omega,
\]
\[
[u] = 0 \quad \text{on } \partial \omega,
\]
\[
[\partial_n u] = -\mathbf{M} \cdot \mathbf{n} \quad \text{on } \partial \omega,
\]
\[
u(x) = O(1/|x|) \quad \text{as } |x| \to \infty.
\]

Combining (2) and (3), we obtain the following expression for the effective field
\[
\mathbf{H}_{\text{eff}}(\mathbf{M}) = \frac{2A}{\mu_0 M_s^2} \Delta \mathbf{M} - \frac{K}{\mu_0 M_s} \nabla \phi(\mathbf{M}/M_s) + \mathbf{H}_e + \mathbf{H}_s,
\] (4)

where \(\mathbf{H}_s = -\nabla u\) denotes the stray field (in A/m).
The dynamics of the spin accumulation $S : \Omega_T \rightarrow \mathbb{R}^3$, measured in A/m, is described by the diffusion equation

$$\frac{\partial S}{\partial t} = -\nabla \cdot J_S - \frac{2D_0}{\lambda_{sf}^2} S - \frac{2D_0}{M_s \lambda_{sf}^2} S \times M \quad \text{in} \ \Omega_T,$$  

(5)

where $D_0 : \Omega \rightarrow \mathbb{R}$ is the diffusion coefficient (in m$^2$/s), $\lambda_{sf}$ is the characteristic length of the spin-flip relaxation, and $\lambda_s$ is related to the mean free path of an electron (both measured in m). The spin current $J_S : \Omega_T \rightarrow \mathbb{R}^{3\times 3}$, measured in A/s, is defined by

$$J_S = \frac{\beta \mu_B}{eM_s} M \otimes J = 2D_0 \left( \nabla S - \frac{\beta \gamma}{M_s^2} M \otimes (\nabla \cdot M) \right) \quad \text{in} \ \Omega_T,$$  

(6)

where $\mu_B = 9.2741 \times 10^{-24}$ A m$^2$ is the Bohr magneton, $e = -1.602 \times 10^{-19}$ A s is the charge of the electron, and $J : \Omega_T \rightarrow \mathbb{R}^3$ is the applied current density field (in A/m$^2$).

To describe the dynamics of the magnetization, we take into account the interaction between the spin accumulation and the magnetization. Thus, we consider an augmented version of (1), namely

$$\frac{\partial M}{\partial t} = -\gamma M \times (\mu_B H_{\text{eff}}(M) + JS) + \frac{\alpha}{M_s} M \times \frac{\partial M}{\partial t} \quad \text{in} \ \omega_T,$$  

(7)

where the constant $J$ in N/A$^2$ is the strength of the interaction between the spin accumulation and the magnetization. Finally, to complete the setting, (5)–(7) are supplemented by initial conditions

$$M(0) = M^0 \quad \text{in} \ \omega \quad \text{and} \quad S(0) = S^0 \quad \text{in} \ \Omega,$$

for some given initial states $M^0 : \omega \rightarrow \mathbb{R}^3$ and $S^0 : \Omega \rightarrow \mathbb{R}^3$ with $|M^0| = M_s$, and homogeneous Neumann boundary conditions

$$\frac{\partial M}{\partial n} = 0 \quad \text{on} \ (0,T) \times \partial \omega \quad \text{and} \quad \frac{\partial S}{\partial n} = 0 \quad \text{on} \ (0,T) \times \partial \Omega.$$

2.2. Nondimensional form of the problem

We introduce a nondimensional form of the system (5)–(7). We perform the substitution $t' = \gamma \mu_0 M_s t$, with $t'$ being the so-called (nondimensional) reduced time, and set $T' = \gamma \mu_0 M_s T$. We rescale the spatial variable by $x' = x/L$, with $L$ being a characteristic length of the problem (measured in m), e.g., the intrinsic length scale $L = \sqrt{2A/\mu_0 M_s^2}$. However, to simplify our notation, we write $t$, $T$, $x$, $\omega$, and $\Omega$, instead of $t'$, $T'$, $x'$, $\omega/L$, and $\Omega/L$, respectively. We introduce the nondimensional vector unknowns $m = M/M_s$, so that the modulus constraint becomes $|m| = 1$, and $s = S/M_s$. Furthermore, we set $h_{\text{eff}} = H_{\text{eff}}/M_s$, $f = H_f/M_s$, $h_j = H_j/M_s$, $c = J/\mu_0$, $j = \mu_B J_c/(\gamma \mu_0 M_s^2)$, $D_0 = 2D_0/(\gamma \mu_0 M_s^2)$, $M_0^0 = M^0/M_s$ and $s^0 = S^0/M_s$. With these substitutions, the nondimensional augmented form of LLG becomes

$$\frac{\partial m}{\partial t} = -m \times (h_{\text{eff}}(m) + cs) + \alpha m \times \frac{\partial m}{\partial t} \quad \text{in} \ \omega_T,$$

where the effective field is given by

$$h_{\text{eff}}(m) = C_{\text{exch}} \Delta m - C_{\text{ani}} \nabla \phi(m) + f + h_j(m),$$  

(8)
with \( C_{\text{exch}} = 2A/(\mu_0 L^2 M^2_s) \) and \( C_{\text{ani}} = K/(\mu_0 M^2_s) \), while the diffusion equation reads
\[
\partial_t \mathbf{s} = -\nabla \cdot (\beta \mathbf{m} \otimes \mathbf{j} - D_0 (\nabla \mathbf{s} - \beta \beta' \mathbf{m} \otimes (\nabla \mathbf{s} \cdot \mathbf{m}))) - \frac{L^2 D_0}{\lambda_f} \mathbf{s} - \frac{L^2 D_0}{\lambda_f} \mathbf{s} \times \mathbf{m}
\] in \( \Omega_T \).

To simplify our notation and without loss of generality, we assume that \( L = \lambda_{sf} = \lambda_f \).

To sum up, we seek for \( \mathbf{m} : \omega_T \to \mathbb{R}^3 \) with \(|\mathbf{m}| = 1 \) and \( \mathbf{s} : \Omega_T \to \mathbb{R}^3 \) such that
\[
\begin{align*}
\partial_t \mathbf{m} &= -\mathbf{m} \times (h_{\text{eff}}(\mathbf{m}) + cs) + \alpha \mathbf{m} \times \partial_t \mathbf{m} & \text{in } \omega_T, \quad (9a) \\
\partial_t \mathbf{s} &= -\nabla \cdot (\beta \mathbf{m} \otimes \mathbf{j} - D_0 (\nabla \mathbf{s} - \beta \beta' \mathbf{m} \otimes (\nabla \mathbf{s} \cdot \mathbf{m}))) - D_0 \mathbf{s} - D_0 (\mathbf{s} \times \mathbf{m}) & \text{in } \Omega_T, \quad (9b) \\
\partial_n \mathbf{m} &= 0 & \text{on } (0, T) \times \partial \omega, \quad (9c) \\
\partial_n \mathbf{s} &= 0 & \text{on } (0, T) \times \partial \Omega, \quad (9d) \\
\mathbf{m}(0) &= \mathbf{m}^0 & \text{in } \omega, \quad (9e) \\
s(0) &= s^0 & \text{in } \Omega. \quad (9f)
\end{align*}
\]

Here, \( c, \alpha > 0 \) and \( 0 < \beta, \beta' < 1 \) are constants. For the diffusion coefficient \( D_0 \in L^\infty(\Omega) \), we assume that there exists a positive constant \( D_0 \) such that \( D_0 \geq D, \) a.e. in \( \Omega \). We also assume that \( \mathbf{f} \in L^2(\omega_T) \) and \( \mathbf{j} \in L^2(0, T; \mathbf{H}^1(\Omega)) \). Moreover, in \((9a)\) we allow a more general effective field of the form
\[
h_{\text{eff}}(\mathbf{m}) = C_{\text{exch}} \Delta \mathbf{m} + \pi(\mathbf{m}) + \mathbf{f}, \quad (10)
\]
where \( \pi : L^2(\omega) \to L^2(\omega) \) is a general time-independent field contribution. We emphasize that \((10)\) in particular covers \((8)\) with \( \pi(\mathbf{m}) = -C_{\text{ani}} \nabla \phi(\mathbf{m}) + h_s(\mathbf{m}) \).

**Remark 1.** The constraint \(|\mathbf{m}| = 1\) directly follows from the PDE formulation, provided \(|\mathbf{m}| = 1\) in \( \omega_T \). Indeed, from \((9a)\), we deduce that \( \partial_t |\mathbf{m}|^2 = 2 \mathbf{m} \cdot \partial_t \mathbf{m} = 0 \) in \( \omega_T \).

### 2.3. Weak solution of the problem

Let \( \mathbf{H}^{1'}(\Omega) = \mathbf{H}^1(\Omega)^* \) be the dual space of \( \mathbf{H}^1(\Omega) \) and denote by \( (\cdot, \cdot) \) the corresponding duality pairing, understood in the sense of the Gelfand triple \( \mathbf{H}^1(\Omega) \subset L^2(\Omega) \subset \mathbf{H}^{1'}(\Omega) \). In view of the weak formulation of \((9)\), we consider the time-dependent bilinear form \( a(t, \cdot, \cdot) : \mathbf{H}^1(\Omega) \times \mathbf{H}^{1'}(\Omega) \to \mathbb{R} \) defined by
\[
a(t, \zeta_1, \zeta_2) = (D_0 \nabla \zeta_1, \nabla \zeta_2)_\Omega - \beta \beta' (D_0 \mathbf{m}(t) \otimes (\nabla \zeta_1 \cdot \mathbf{m}(t)), \nabla \zeta_2)_\omega \\
+ (D_0 \zeta_1, \zeta_2)_\Omega + (D_0 \zeta_1, \mathbf{m}(t) \cdot \zeta_2)_\Omega,
\]
for all \( t \in [0, T] \) and \( \zeta_1, \zeta_2 \in \mathbf{H}^1(\Omega) \).

We recall from [21, Definition 1] the notion of a weak solution of the SDLLG system \((9)\), which extends the definition of weak solutions of LLG from [5].

**Definition 2.** Let \( \mathbf{m}^0 \in \mathbf{H}^1(\omega) \) with \(|\mathbf{m}^0| = 1 \) a.e. in \( \omega \) and \( s^0 \in \mathbf{H}^{1'}(\Omega) \). The tupel \( (\mathbf{m}, s) \) is called a weak solution of SDLLG if the following properties (i)–(iv) are satisfied:

(i) \( \mathbf{m} \in \mathbf{H}^1(\omega_T) \) with \(|\mathbf{m}| = 1 \) a.e. in \( \omega_T \) and \( \mathbf{m}(0) = \mathbf{m}^0 \) in the sense of traces,

(ii) \( s \in L^2(0, T; \mathbf{H}^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \partial_t s \in L^2(0, T; \mathbf{H}^{1'}(\Omega)) \) and \( s(0) = s^0 \) in the sense of traces,
C \leq \text{the piecewise linear and the two piecewise constant time-approximations defined as follows: For time interval } [0, N], \text{ it holds}

\begin{equation}
(\partial_t \mathbf{m}, \varphi)_{\omega_T} + \alpha (\partial_t \mathbf{m} \times \mathbf{m}, \varphi)_{\omega_T} = -C_{\text{ch}} (\nabla \mathbf{m} \times \mathbf{m}, \nabla \varphi)_{\omega_T} + (\pi(\mathbf{m}) \times \mathbf{m}, \varphi)_{\omega_T} + (f \times \mathbf{m}, \varphi)_{\omega_T} + c (s \times \mathbf{m}, \varphi)_{\omega_T}, \tag{11a}
\end{equation}

(iv) for almost all \( t \in [0, T] \) and all \( \zeta \in H^1(\Omega) \), it holds

\begin{equation}
\langle \partial_t s(t), \zeta \rangle + a(t, s(t), \zeta) = \beta (\mathbf{m}(t) \otimes j(t), \nabla \zeta)_{\omega_T} - \beta (j(t) \cdot n \mathbf{m}(t) \cdot \zeta)_{\partial \Omega \setminus \omega_T}. \tag{11b}
\end{equation}

Remark 3. If \((\mathbf{m}, s)\) is a weak solution of SDLLG, then it holds \( \mathbf{m} \in C(0, T; L^2(\omega)) \) and \( s \in C(0, T; L^2(\Omega)) \), cf., e.g., [16, Section 5.9.2, Theorem 2 and Theorem 3].

Remark 4. The boundary term in (11b) is missing in [27]. This error has recently been noticed and corrected, so that the overall result of [27] remains valid [18]. The present analysis provides an alternate proof for the existence of solutions and hence validity of the results of [27] [18].

The following lemma highlights the parabolic nature of equation (9b).

Lemma 5. The time-dependent bilinear form \( a(t, \cdot, \cdot) \) is continuous and positive definite. Indeed, it holds

\begin{equation}
a(t, \zeta, \xi) \geq (1 - \beta \beta^*) D_0 \| \xi \|^2_{H^1(\Omega)} \tag{12}
\end{equation}

for almost all \( t \in [0, T] \).

Proof. The continuity directly follows from the regularity assumptions on the data, as \(|\mathbf{m}| = 1\) a.e. in \( \Omega_T \). As for the positive definiteness, we note

\[
|\mathbf{m}(t) \otimes (\nabla \zeta \cdot \mathbf{m}(t)) \cdot \nabla \xi| \leq |\nabla \xi|^2 \quad \text{for all } \zeta \in H^1(\Omega).
\]

As a consequence, since \( D_0 \geq D_* \) and \( 0 < \beta \beta^* < 1 \), we get

\[
a(t, \zeta, \xi) = (D_0 \nabla \zeta, \nabla \xi)_{\Omega_T} - \beta \beta^* (D_0 \mathbf{m}(t) \otimes (\nabla \zeta \cdot \mathbf{m}(t)) \cdot \nabla \xi)_{\omega_T} + (D_0 \zeta, \xi)_{\Omega_T}
\geq (D_0 \nabla \zeta, \nabla \xi)_{\Omega_T} - \beta \beta^* (D_0 \mathbf{m}(t) \otimes (\nabla \zeta \cdot \mathbf{m}(t))) \cdot \nabla \xi)_{\omega_T} + (D_0 \zeta, \xi)_{\Omega_T}
\geq (1 - \beta \beta^*) D_0 \| \nabla \xi \|^2_{L^2(\Omega_T)} + D_* \| \xi \|^2_{L^2(\Omega_T)}.
\]

This establishes (12) and concludes the proof. \( \square \)

3. Numerical algorithm

For the time discretization, we consider a uniform partition \( 0 = t_0 < t_1 < \cdots < t_N = T \) of the time interval \([0, T]\) with time-step size \( k = T/N \), i.e., \( t_j = jk \) for \( 0 \leq j \leq N \).

Given a sequence of functions \( \{ \varphi^j \}_{0 \leq j \leq N} \), such that any \( \varphi^j \) is associated with the time-step \( t_j \), for \( 0 \leq j \leq N - 1 \) we define the difference quotient \( d_j \varphi^{j+1} := (\varphi^{j+1} - \varphi^j)/k \). We consider the piecewise linear and the two piecewise constant time-approximations defined as follows: For \( 0 \leq j \leq N - 1 \) and \( t \in [t_j, t_{j+1}) \), we have

\begin{equation}
\varphi^j(t) := \frac{t - t_j}{k} \varphi^{j+1} + \frac{t_{j+1} - t}{k} \varphi^j, \quad \varphi^j(t) := \varphi^j, \quad \varphi^j(t) := \varphi^{j+1}. \tag{13}
\end{equation}
Obviously, it holds $\partial_t \phi_h(t) = d_1 \phi^{t+1}$ for all $t \in [t_j, t_{j+1})$.

For the spatial discretization, let $\{T_h^\Omega\}_{h > 0}$ be a shape-regular and (globally) quasi-uniform family of regular tetrahedral triangulations of $\Omega$, parameterized by the meshsize $h = \max h_K$, where $h_K = \text{diam}(K)$ for all $K \in T_h^\Omega$. By $T_h^\omega$, we denote the restriction of $T_h^\Omega$ to $\omega$. We assume that $\omega$ is resolved, i.e.,

$$T_h^\omega = T_h^\Omega |_{\omega} = \{ K \in T_h^\Omega : K \cap \omega \neq \emptyset \}$$

Let us denote by $S^1(T_h^\omega)^3$ the standard finite element space of globally continuous and piecewise affine functions from $\Omega$ to $\mathbb{R}^3$. Correspondingly, we also consider $S^1(T_h^\omega)^3$. By $I_h^\Omega : C(\overline{\Omega}) \to S^1(T_h^\Omega)^3$ and $I_h^\omega : C(\overline{\omega}) \to S^1(T_h^\omega)^3$, we denote the nodal interpolation operators onto these spaces. Since $\omega$ is resolved, these operators coincide on $\omega$, i.e., $I_h^\Omega(\phi)|_\omega = I_h^\omega(\phi)|_\omega$ for all $\phi \in C(\overline{\Omega})$. In particular, there is no ambiguity, if we denote both operators by $I_h$. The set of nodes of the triangulation $T_h^\omega$ is denoted by $N_h^\omega$.

We recall that, under the constraint $[m] = 1$, the strong form of (9a) can equivalently be stated as

$$a_0 \partial_t m + m \times \partial_t m = h_{\text{eff}}(m) + cs - ((h_{\text{eff}}(m) + cs) \cdot m) m,$$

This formulation is used to construct the upcoming numerical scheme. Since (13) is linear in $\partial_t m$, the main idea is to introduce an additional free variable $v = \partial_t m$. To discretize $v$, we introduce the discrete tangent space defined by

$$\mathcal{K}_h : = \{ \psi_h \in S^1(T_h^\omega)^3 : \psi_h(z) \cdot \phi_h(z) = 0 \text{ for all } z \in N_h^\omega \}$$

for any $\phi_h \in S^1(T_h^\omega)^3$. Moreover, we consider the set

$$\mathcal{M}_h : = \{ \phi_h \in S^1(T_h^\omega)^3 : \phi_h(z) = 1 \text{ for all } z \in N_h^\omega \}.$$

These sets reflect two main properties of $m$ and $v$, namely the orthogonality $m \cdot v = 0$ and the unit-length constraint $[m] = 1$.

Let $U_h = \{ \phi_h \in S^1(T_h^\omega)^3 : \phi_h(z) \geq 1 \text{ for all } z \in N_h^\omega \}$. We consider the nodal projection map $\Pi_h : U_h \to \mathcal{M}_h$ defined by $\Pi_h \phi_h(z) = \phi_h(z) / \| \phi_h(z) \|$ for all $z \in N_h^\omega$ and $\phi_h \in U_h$. A simple argument based on the elementwise use of barycentric coordinates shows that $\| \Pi_h \phi_h \|_{L^2(\omega)} = 1$ for all $\phi_h \in U_h$. Moreover, we have the estimate

$$\| \nabla \Pi_h \phi_h \|_{L^2(\omega)} \leq c_{11} \| \nabla \phi_h \|_{L^2(\omega)} \text{ for all } \phi_h \in U_h,$$

where the constant $c_{11} > 0$ depends only on the shape-regularity of the triangulation, cf., e.g., [11], Lemma 2.2. With an additional angle condition on $T_h^\omega$, it is well known that (15) holds even with $c_{11} = 1$, cf. [10].

Let $m_0^h \in \mathcal{M}_h$ and $s_0^h \in S^1(T_h^\omega)^3$ be suitable approximations of the initial conditions. Moreover, we consider a numerical realization $\pi_h : L^2(\omega) \to L^2(\omega)$ of $\pi$, which is assumed to fulfill a certain set of properties, see (H2)-(H3) below. This allows us to include the approximation errors, e.g., those which arise from the numerical computation of the stray field, into the overall convergence analysis. For ease of presentation, we assume that $f$ and $j$ are continuous in time, i.e., $f \in C(0, T; L^2(\omega))$ and $j \in C(0, T; H^1(\Omega))$, so that the expressions $f^t = f(t_j)$ and $j^t = j(t_j)$
are meaningful for all $0 \leq j \leq N$. It is even possible to replace $f^j$ and $j^j$ by some numerical approximation $f^\ell_j$ and $j^\ell_j$ as long as some weak convergence properties are fulfilled, cf. \[15\].

Analogously to what we have done in Section 7 for the continuous problem, for $0 \leq i \leq N - 1$ we define the bilinear form $a_{i\cdot} : S^1(T_h)^3 \times S^1(T_h)^3 \to \mathbb{R}$ by

$$a_{i\cdot}(\xi_1, \xi_2) = (D_0 \nabla \xi_1, \nabla \xi_2)_\Omega - \beta \delta (D_0 \Pi_h m_i^{i+1} \otimes (\nabla \xi_1 \cdot \Pi_h m_i^{i+1}), \nabla \xi_2)_\omega + (D_0 \xi_1, \xi_2)_\Omega + (D_0 (\nabla \xi_1 \cdot \Pi_h m_i^{i+1}) \cdot \xi_2)_\omega,$$  

for all $\xi_1, \xi_2 \in S^1(T_h)^3$. For the numerical integration of the SDLLG system (9), we propose the following algorithm.

**Algorithm 6.** Input: $m^0_h, s^0_h, \{f^i\}_{0 \leq i \leq N-1}$, $\{j^{i+1}\}_{0 \leq i \leq N-1}$, parameter $0 \leq \theta \leq 1$.

For all $0 \leq i \leq N-1$ iterate:

(i) compute $v^i_h \in K_m$ such that

$$a_{i\cdot}(v^i_h, \phi_h) + \left( m_i^i \times v^i_h, \phi_h \right)_\omega + C_{\text{exch}} \delta h \left( \nabla v^i_h, \nabla \phi_h \right)_\omega$$  

$$= - C_{\text{exch}} \left( \nabla m_i^i, \nabla \phi_h \right)_\omega + \left( \Pi_3 m_i^i, \phi_h \right)_\omega + \left( f^i, \phi_h \right)_\omega + \beta \left( \nabla (s^{i+1}_h), \phi_h \right)_\omega$$  

(16a)

for all $\phi_h \in K_m$.

(ii) define $m^{i+1}_h$ in $S^1(T_h)^3$ by

$$m^{i+1}_h = m^i_h + k v^i_h.$$  

(16b)

(iii) compute $s^{i+1}_h \in S^1(T_h)^3$ such that

$$a_{i\cdot}(s^{i+1}_h, \xi_h) + a_{i\cdot}(s^{i+1}_h, \xi_h) = \beta \left( \Pi_3 m^{i+1}_h \otimes \nabla \xi_h \right)_\omega - \beta \left( j^{i+1} \cdot n, \Pi_3 m^{i+1}_h \cdot \xi_h \right)_{\partial \Omega \cap T_h}$$  

(16c)

for all $\xi_h \in S^1(T_h)^3$.

Output: Sequence of discrete functions $\{v^i_h, m^{i+1}_h, s^{i+1}_h\}_{0 \leq i \leq N-1}$.

The overall system (9) is a non-linearly coupled system of a linear diffusion equation for $s$ with the non-linear LLG equation for $m$. However, our scheme only requires the solution of two linear systems per time-step, since the treatment of the micromagnetic part and the spin diffusion part is completely decoupled for the time-integration. This greatly simplifies an actual numerical implementation as well as the possible preconditioning of iterative solvers.

**Remark 7.** Unlike this work, earlier results on the tangent plane integrator [2] [4] [7] [8] [24] define $m^{i+1}_h := \Pi_3 (m^i_h + k v^i_h)$ in (16b). Unconditional convergence in the sense of Theorem 12 can then be achieved with an additional angle condition on the triangulation $T_h^\omega$, which ensures (15) with $c_\Omega = 1$. This assumption is avoided in the present work.

The following result follows from standard scaling arguments.

**Lemma 8.** Let $(T_h)_{h>0}$ be a quasi-uniform family of triangulations of $\Omega$ and $r \in [1, \infty)$. Then,

$$C^{-1} \|w_h\|_{U'((\Omega)} \leq h^3 \sum_{z \in N_h} |w_h(z)|', \leq C \|w_h\|_{U'((\Omega)} \quad \text{for all } w_h \in S^1(T_h).$$

The constant $C > 0$ depends only on $r$, but is independent of the meshsize $h$. $\Box$
The following proposition states that the above algorithm is well defined, cf. [11] Proposition 3.1 and Proposition 4.1] for corresponding results in the frame of harmonic maps and the harmonic flow heat flow.

**Proposition 9.** Algorithm \([\text{Algorithm 6}]\) is well defined in the following sense: For each time-step \(0 \leq i \leq N - 1\), there exists a unique solution \((v_i^+, m_i^{+1}, s_i^{+1})\). Moreover, it holds

\[
|v_i^{+1}(z)|^2 = 1 + k^2 \sum_{l=0}^{i} |v_l^+(z)|^2 \geq 1 \quad \text{for all } z \in \mathcal{N}_h^\omega,
\]

as well as

\[
C_*^{-1} \|m_i^{+1}\|_{L^2(\omega)}^2 \leq 1 + k^2 \sum_{l=0}^{i} \|v_l^+(z)\|^2,
\]

where the constant \(C_* > 0\) depends only on the shape-regularity of \(\{T_h^\omega\}_{h>0}\) but is independent of \(h\) and \(k\).

**Proof.** Let \(0 \leq i \leq N - 1\). For step (i) of the algorithm, it is straightforward to show that problem \([\text{Algorithm 6a}]\) is characterized by a positive definite bilinear form. Unique solvability thus follows from linearity and finite space dimension. Step (ii) is clearly well defined. For all \(h \in \mathcal{N}_h\), the nodewise orthogonality from \(\text{Algorithm 6b}^\dagger\) proves

\[
|m_i^{+1}(z)|^2 = |m_i^+(z)|^2 + k^2 \sum_{l=0}^{i} |v_l^+(z)|^2 \geq 1.
\]

Since \(m_i^0 \in M_h\), mathematical induction proves

\[
|m_i^{+1}(z)|^2 = |m_i^+(z)|^2 + k^2 \sum_{l=0}^{i} |v_l^+(z)|^2 = 1 + k^2 \sum_{l=0}^{i} |v_l^+(z)|^2 \geq 1.
\]

This proves \([\text{17}]\). The norm equivalence from Lemma \([\text{Algorithm 6b}]\) in the case \(r = 2\) yields

\[
\|m_i^{+1}\|_{L^2(\omega)}^2 \lesssim h^3 \sum_{z \in \mathcal{N}_h^\omega} |m_i^{+1}(z)|^2 = h^3 \sum_{z \in \mathcal{N}_h^\omega} \left(1 + k^2 \sum_{l=0}^{i} |v_l^+(z)|^2\right)
\]

\[
= h^3 (\# \mathcal{N}_h^\omega) + k^2 \sum_{l=0}^{i} h^3 \sum_{z \in \mathcal{N}_h^\omega} |v_l^+(z)|^2 \lesssim 1 + k^2 \sum_{l=0}^{i} \|v_l^+(z)\|^2_{L^2(\omega)}.
\]

This establishes \([\text{13}]\). For step (iii), we use the same argument as for step (i). Due to \([\text{17}]\), the nodewise projections in \([\text{Algorithm 6c}]\) are well defined. Let \(b'_h : S^i(T_h^\omega) \times S^i(T_h^\omega) \to \mathbb{R}\) be the bilinear form associated to problem \([\text{Algorithm 6c}]\), i.e.,

\[
b'_h(\xi, \zeta) = \frac{1}{k} (\xi, \zeta) + (D_0 \nabla \xi, \nabla \zeta)_{\Omega} - \beta \beta' (D_0 \Pi_h m_i^{+1} \otimes (\nabla \xi_1 \cdot \Pi_h m_i^{+1}), \nabla \zeta_2)_{\Omega}
\]

\[
+ (D_0 \Pi_h \xi, \zeta_2)_{\Omega} + (D_0 (\xi_1 \times \Pi_h m_i^{+1}), \zeta_2)_{\Omega}.
\]

Since \(\|\Pi_h m_i^{+1}\|_{L^2(\omega)} = 1\), we see

\[
(D_0 \Pi_h m_i^{+1} \otimes (\nabla \xi_1 \cdot \Pi_h m_i^{+1}), \nabla \zeta_2)_{\Omega} \lesssim (D_0 |\nabla \xi_1|, |\nabla \zeta_2|)_{\Omega}.
\]
It follows that
\[
\frac{b_h^\prime(\xi, \xi)}{h} \geq \frac{1}{k} \|\xi\|^2_{L^2(\Omega)} + (1 - \beta\beta') (D_\Omega \nabla \xi, \nabla \xi) + (D_\Omega \xi, \xi)_{\Omega} \\
\geq \frac{1 + kD_\Omega}{k} \|\xi\|^2_{L^2(\Omega)} + D_\Omega (1 - \beta\beta') \|\nabla \xi\|^2_{L^2(\Omega)}.
\]

As \(0 < \beta\beta' < 1\) and \(D_\Omega > 0\), \(b_h^\prime(\cdot, \cdot)\) is positive definite and problem (16c) is thus well posed.

4. Convergence analysis

In this section, we consider the convergence properties of Algorithm 6 and show that it is indeed unconditionally convergent towards a weak solution of SDLLG in the sense of Definition 2. We emphasize that the proof is constructive in the sense that it even shows existence of weak solutions. We start by collecting some general assumptions:

(H1) The discrete initial data \(m_0^h \in M_h\) and \(s_0^h \in S^1(\Omega_h^\omega)\) satisfy
\[
\lim_{h \to 0} m_0^h \rightarrow m_0 \text{ in } H^1(\omega) \quad \text{and} \quad \lim_{h \to 0} s_0^h \rightarrow s_0 \text{ in } L^2(\Omega).
\]

(H2) The general field contribution \(\pi_h^\prime\) is bounded, i.e.,
\[
\|\pi_h^\prime(w)\|_{L^2(\omega)} \leq C\pi \|w\|_{L^2(\omega)} \quad \text{for all } w \in L^2(\omega),
\]
with a constant \(C\pi > 0\) which depends only on \(|\omega|\).

(H3) It holds
\[
\pi_h^\prime(w_h^k) \rightarrow \pi(w) \text{ in } L^2(\omega_T) \quad \text{as } (h, k) \to 0
\]
for any sequence \(w_h^k \rightarrow w\) in \(L^2(\omega_T)\).

Remark 10. Usual stray field discretizations by hybrid FEM-BEM methods, e.g., the Fredkin-Koehler approach from [17], or FEM-BEM coupling methods satisfy (H2)–(H3), see [15].

Remark 11. For a discrete operator \(\pi_h^\prime : H^1(\omega) \rightarrow L^2(\omega)\), assumption (H2) can be relaxed to
\[
\|\pi_h^\prime(w)\|_{L^2(\omega)} \leq C\pi \|w\|_{H^1(\omega)} \quad \text{for all } w \in H^1(\omega).
\]
Within this setting, and with an appropriate modification of assumption (H3), the hybrid FEM-BEM method from [15] for the computation of the stray field can also be included into our analysis. Then, the proof of Proposition 19 below becomes more technical, but the assertion remains true. We refer to the argument of [15] which can be adapted accordingly.

From now on, we consider the time-approximations \(m_h^k, m_h^\pm, s_h^k, s_h^\pm\) defined by (13). The next theorem is the main result of this work.

Theorem 12. Let \(\{(\Omega_h^\omega)\}_{h>0}\) be a shape-regular and quasi-uniform family of triangulations.

(a) Suppose \(1/2 < \theta \leq 1\) and that assumptions (H1)–(H2) are satisfied.

Then, there exist \(m \in L^2(\omega_T)\) and \(s \in L^2(0, T; H^1(\Omega))\) such that
\[
\lim_{h \to 0} m_h^k \rightarrow m \text{ in } L^2(\omega_T) \quad \text{and} \quad \lim_{h \to 0} s_h^k \rightarrow s \text{ in } L^2(0, T; H^1(\Omega)).
\]
(b) In addition to the above, let assumption (H3) be satisfied. Then, it holds

\[(m_{0i}, s_{0i}) \rightarrow (m, s) \text{ in } H^1(\omega_T) \times \left[ L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \right], \]

where \((m, s)\) is a weak solution of SDLLG.

**Remark 13.** In particular, Theorem 12 yields existence of weak solutions, and each accumulation point of \((m_{0i}, s_{0i})\) is a weak solution of SDLLG in the sense of Definition 6.

The proof of Theorem 12 will roughly be done in three steps, namely

(i) boundedness of the discrete quantities and energies,
(ii) existence of weakly convergent subsequences via compactness,
(iii) identification of the limits with weak solutions of SDLLG.

For the sake of readability, we split our argument into several lemmata.

To start with, we recall the following result, which states a well-known and simple algebraic trick which often simplifies the computation and the estimation of sums.

**Lemma 14** (Abel’s summation by parts). Let \(X\) be a vector space endowed with a symmetric bilinear form \((\cdot, \cdot)\). Given an integer \(j \geq 1\), let \(\{v_i\}_{0 \leq i < j} \subset X\). Then, it holds

\[
\sum_{i=0}^{j-1} (v_{i+1} - v_i, v_{i+1}) = \frac{1}{2} (v_j, v_j) - \frac{1}{2} (v_0, v_0) + \frac{1}{2} \sum_{i=0}^{j-1} (v_{i+1} - v_i, v_{i+1} - v_i). \quad \Box
\]

The first ingredient for step (i) is the following proposition.

**Proposition 15.** Let \(1 \leq j \leq N\) and suppose that the assumptions of Theorem 12(a) are satisfied. Then, the discrete functions \(\left\{ s_{0i}^{(1)} \right\}_{0 \leq i < j}\) obtained through Algorithm 6 fulfill

\[
\|s_{0i}^{(1)}\|_{L^2(\Omega)}^2 + k \sum_{i=0}^{j-1} \|s_{0i}^{(1)}\|_{H^1(\Omega)}^2 + \sum_{i=0}^{j-1} \|s_{0i}^{(1)} - s_{0i}^{(1)}\|_{L^2(\Omega)}^2 \leq C. \quad (19)
\]

The constant \(C > 0\) depends only on the data, but is in particular independent of the discretization parameters \(h\) and \(k\).

**Proof.** Let \(0 \leq i \leq j - 1\). For \((16\text{c})\), we choose \(\xi_h = s_{0i}^{(1)}\) as test function. After multiplication by \(k\), we obtain

\[
\left( s_{0i+1} - s_{0i}^{(1)} \right) + k \left( D_0 \nabla s_{0i+1}^{(1)}, \nabla s_{0i+1}^{(1)} \right) + k \beta \left( D_0 \Pi_h m_{0i}^{(1)} \otimes \left( \nabla s_{0i+1}^{(1)} \cdot \Pi_h m_{0i}^{(1)} \right), \nabla s_{0i+1}^{(1)} \right)_\Omega \\
+ k \left( D_0 s_{0i+1}, s_{0i+1}^{(1)} \right) = k \beta \left( \Pi_h m_{0i}^{(1)} \otimes j^{(1)}, \nabla s_{0i+1}^{(1)} \right) - k \beta \left( j^{(1)} \cdot n, \Pi_h m_{0i}^{(1)} \cdot s_{0i+1}^{(1)} \right)_{\partial \Omega \cap \partial \omega}.
\]

Since \(D_0 \geq D\), and \(\|\Pi_h m_{0i}^{(1)}\|_{L^\infty(\omega)} = 1\), it follows that

\[
\left( s_{0i+1} - s_{0i}^{(1)} \right) + k D_0 \left( 1 - \beta \delta \right) \left( \nabla s_{0i+1}^{(1)}, \nabla s_{0i+1}^{(1)} \right) + k D_0 \left( s_{0i+1}^{(1)}, s_{0i+1}^{(1)} \right) \Omega \\
\leq k \beta \left( \Pi_h m_{0i}^{(1)} \otimes j^{(1)}, \nabla s_{0i}^{(1)} \right) - k \beta \left( j^{(1)} \cdot n, \Pi_h m_{0i}^{(1)} \cdot s_{0i+1}^{(1)} \right)_{\partial \Omega \cap \partial \omega},
\]

where \((m, s)\) is a weak solution of SDLLG.
Summing up over \(i = 0, \ldots, j - 1\), and exploiting Abél's summation by parts from Lemma 14 for the term \(\sum_{i=0}^{j-1} (s_h^{i+1} - s_h^i, s_h^i)_{\Omega} \setminus \Omega\), we get
\[
\frac{1}{2} \|s_h^j\|^2_{L^2(\Omega)} + \frac{1}{2} \sum_{i=0}^{j-1} \|s_h^{i+1} - s_h^i\|^2_{L^2(\Omega)} + kD_\varepsilon (1 - \beta \varepsilon') \sum_{i=0}^{j-1} \|\nabla s_h^{i+1}\|^2_{L^2(\Omega)} + kD_\varepsilon \sum_{i=0}^{j-1} \|s_h^{i+1}\|^2_{L^2(\Omega)}
\]
\[
\leq \frac{1}{2} \|s_h^0\|^2_{L^2(\Omega)} + k\beta \sum_{i=0}^{j-1} \left[ (\Pi_h m_h^{i+1} \otimes j^{i+1}, \nabla j^{i+1})_{\omega} - (j^{i+1} \cdot n, \Pi_h m_h^{i+1} \cdot s_h^{i+1})_{\partial\Omega \setminus \partial\omega} \right].
\]

Exploiting \(0 < 1 - \beta \varepsilon' < 1\) on the left-hand side, the Cauchy-Schwarz inequality and the Young inequality on the right-hand side, we obtain, for any choice of \(\varepsilon > 0\),
\[
\frac{1}{2} \|s_h^j\|^2_{L^2(\Omega)} + \frac{1}{2} \sum_{i=0}^{j-1} \|s_h^{i+1} - s_h^i\|^2_{L^2(\Omega)} + kD_\varepsilon (1 - \beta \varepsilon') \sum_{i=0}^{j-1} \|\nabla s_h^{i+1}\|^2_{L^2(\Omega)} + kD_\varepsilon \sum_{i=0}^{j-1} \|s_h^{i+1}\|^2_{L^2(\Omega)}
\]
\[
\leq \frac{1}{2} \|s_h^0\|^2_{L^2(\Omega)} + \frac{Ck\beta}{2\varepsilon} \sum_{i=0}^{j-1} \|j^{i+1}\|^2_{H^1(\Omega)} + \frac{C\beta \varepsilon}{2} \sum_{i=0}^{j-1} \|s_h^{i+1}\|^2_{H^1(\Omega)}.
\]

Here the constant \(C > 0\) is the stability constant of the trace operator. It follows that
\[
\frac{1}{2} \|s_h^j\|^2_{L^2(\Omega)} + \frac{1}{2} \sum_{i=0}^{j-1} \|s_h^{i+1} - s_h^i\|^2_{L^2(\Omega)} + kD_\varepsilon (1 - \beta \varepsilon') \sum_{i=0}^{j-1} \|\nabla s_h^{i+1}\|^2_{L^2(\Omega)} + kD_\varepsilon \sum_{i=0}^{j-1} \|s_h^{i+1}\|^2_{L^2(\Omega)}
\]
\[
\leq \frac{1}{2} \|s_h^0\|^2_{L^2(\Omega)} + \frac{Ck\beta}{2\varepsilon} \sum_{i=0}^{j-1} \|j^{i+1}\|^2_{H^1(\Omega)} + \frac{C\beta \varepsilon}{2} \sum_{i=0}^{j-1} \|s_h^{i+1}\|^2_{H^1(\Omega)}.
\]

If we choose \(\varepsilon < 2D_\varepsilon (1 - \beta \varepsilon') \varepsilon \beta\), then all the coefficients on the left-hand side are positive.

From (H1) and the regularity of \(j\), we know that the right-hand side is uniformly bounded with respect to \(h\) and \(k\). This yields the estimate (19). \(\Box\)

**Corollary 16.** Under the assumptions of Proposition 15, the sequences \([s_h]_k\) and \(\{s_h^k\}_{k}^{n}\) are uniformly bounded in \(L^2(0, T; H^1(\Omega))\) and in \(L^\infty(0, T; L^2(\Omega))\), i.e.,
\[
\|s_h_k\|_{L^2(0, T; H^1(\Omega))} + \|s_h^k\|_{L^2(0, T; H^1(\Omega))} + \|s_h_k\|_{L^\infty(0, T; L^2(\Omega))} + \|s_h^k\|_{L^2(0, T; L^2(\Omega))} \leq C,
\]
where the constant \(C > 0\) depends only on the data, but is in particular independent of the discretization parameters \(h\) and \(k\).

**Proof.** The result follows from the boundedness of the discrete functions \(\{s_h^1\}_{0 \leq i \leq N-1}\) from Proposition 15. \(\Box\)

Let \(\mathcal{P}_h : L^2(\Omega) \to S^1(T_h^\Omega)^3\) be the \(L^2(\Omega)\)-orthogonal projection onto \(S^1(T_h^\Omega)^3\), i.e.,
\[
(\mathcal{P}_h \eta - \eta, \eta)_{\Omega} = 0 \quad \text{for all } \eta \in L^2(\Omega), \quad \eta_h \in S^1(T_h^\Omega)^3.
\]

Since \(\{T_h^\Omega\}_{h, \varepsilon}\) is quasi-uniform, it is well known that \(\mathcal{P}_h\) is stable in \(H^1(\Omega)\), i.e.,
\[
\|\mathcal{P}_h \eta\|_{H^1(\Omega)} \leq \|\eta\|_{H^1(\Omega)} \quad \text{for all } \eta \in H^1(\Omega).
\]

We also refer to [9] [23] for recent results on \(H^1\)-stability on locally refined meshes. With this, we obtain uniform boundedness of \(\partial_t s_h\).
Proposition 17. The sequence \( \{ \partial_t s_h \} \) is uniformly bounded in \( L^2(0, T; \tilde{H}^{-1}(\Omega)) \), i.e.,

\[
|\partial_t s_h|_{L^2(0, T; \tilde{H}^{-1}(\Omega))} \leq C,
\]

where the constant \( C > 0 \) depends only on the data, but is in particular independent of the discretization parameters \( h \) and \( k \).

Proof. Let \( w \in H^1(\Omega) \setminus \{ 0 \} \), \( 0 \leq i \leq N - 1 \), and \( t \in [t_i, t_{i+1}) \). From (16c) and the \( H^1 \)-stability (20) of \( p_h \), we get

\[
(\partial_t s_h(t), w)_\Omega = (\partial_t s_h(t), w)_\Omega = (d s_h^{i+1}, w)_\Omega = (d s_h^{i+1}, p_h w)_\Omega
\]

\[
= \beta (\Pi_h m_h^{i+1} \otimes j^{i+1}, \nabla p_h w)_\Omega - \beta (j^{i+1} \cdot n, \Pi_h m_h^{i+1} \cdot p_h w)_{\partial \Omega} = a_h^{i+1}(s_h^{i+1}, p_h w)
\]

\[
\lesssim \left( \|j^{i+1}\|_{H^1(\Omega)} + \|s_h^{i+1}\|_{H^1(\Omega)} \right) \|P_h w\|_{H^1(\Omega)} \lesssim \left( \|j^{i+1}\|_{H^1(\Omega)} + \|s_h^{i+1}\|_{H^1(\Omega)} \right) \|w\|_{H^1(\Omega)}.
\]

Dividing by \( \|w\|_{H^1(\Omega)} \) and taking the supremum over \( w \in H^1(\Omega) \setminus \{ 0 \} \), we obtain

\[
|\partial_t s_h(t)|_{H^{-1}(\Omega)} \lesssim \|j^{i+1}\|_{H^1(\Omega)} + \|s_h^{i+1}\|_{H^1(\Omega)}.
\]

Squaring, integrating over \( (t_i, t_{i+1}) \), and summing up over \( 0 \leq i \leq N - 1 \), we get

\[
|\partial_t s_h|_{L^2(0, T; \tilde{H}^{-1}(\Omega))}^2 \lesssim k \sum_{i=0}^{N-1} |j^{i+1}|_{H^1(\Omega)}^2 + k \sum_{i=0}^{N-1} \|s_h^{i+1}\|_{H^1(\Omega)}^2.
\]

The boundedness from Proposition 15 thus yields (21).

We derive the corresponding estimates for the discrete quantities \( \{(v_h^i, m_h^{i+1})\}_{0 \leq i \leq N-1} \).

Lemma 18. Let \( 0 \leq i \leq N - 1 \). The discrete functions \( \{(v_h^i, m_h^{i+1})\} \) obtained through Algorithm 6 fulfill

\[
\alpha \|v_h^i\|_{L^2(\omega)}^2 + \frac{C_{\text{exc}}}{2k} \left( \|\nabla m_h^{i+1}\|_{L^2(\omega)}^2 - \|\nabla m_h^i\|_{L^2(\omega)}^2 \right) + C_{\text{exc}} \left( \theta - \frac{1}{2} \right) \|\nabla v_h^i\|_{L^2(\omega)}^2
\]

\[
= (\pi_h(m_h^i), v_h^i) + (f_h, v_h^i) + c (s_h^i, v_h^i)_\omega.
\]

Proof. We test (16a) with \( \phi_h = v_h^i \in K_{m_h^i} \) to get

\[
\alpha \|v_h^i\|_{L^2(\omega)}^2 + \frac{C_{\text{exc}}}{k} \left( \|\nabla m_h^i + k \nabla v_h^i\|_{L^2(\omega)}^2 \right) + C_{\text{exc}} \left( \theta - 1 \right) \|\nabla v_h^i\|_{L^2(\omega)}^2
\]

\[
= (\pi_h(m_h^i), v_h^i) + (f_h, v_h^i) + c (s_h^i, v_h^i)_\omega.
\]

Exploiting the vector identity

\[
2 (a + b) \cdot a = |a|^2 + |a + b|^2 - |b|^2 \quad \text{for all } a, b \in \mathbb{R}^3
\]

with the choice \( a = k \nabla v_h^i \) and \( b = \nabla m_h^i \), and taking into account (16b), we obtain (22).
Proposition 19. Suppose that the assumptions of Theorem 12(a) are satisfied. Then, there exists \( k_0 > 0 \) such that for all time-step sizes \( 0 < k < k_0 \) and \( 1 \leq j \leq N \) the discrete functions \( \{ (v_k^j, m_k^j) \}_{0 \leq j \leq 1} \) obtained through Algorithm 6 fulfill

\[
\left\| \nabla m_k^{j+1} \right\|^2_{L^2} + k \sum_{i=0}^{j-1} \left\| m_i^j \right\|^2_{L^2} + \left( \theta - \frac{1}{2} \right) k^2 \sum_{i=0}^{j-1} \left\| \nabla v_i^j \right\|^2_{L^2} \leq C. \tag{23}
\]

The constant \( C > 0 \) depends only on the data and \( k_0 \), but is otherwise independent of the discretization parameters \( h \) and \( k \).

Proof. Let \( 1 \leq j \leq N \). From Lemma 13, multiplying \((22)\) by \( k/C_{\text{exch}} \), summing up over \( 0 \leq i \leq j - 1 \) and exploiting the telescopic sum, we obtain

\[
\frac{1}{2} \left\| \nabla m_i^j \right\|^2_{L^2} + \frac{ak}{C_{\text{exch}}} \sum_{i=0}^{j-1} \left\| m_i^j \right\|^2_{L^2} + k^2 \left( \theta - \frac{1}{2} \right) \sum_{i=0}^{j-1} \left\| \nabla v_i^j \right\|^2_{L^2} = \frac{1}{2} \left\| \nabla m_i^j \right\|^2_{L^2} + \frac{k}{C_{\text{exch}}} \sum_{i=0}^{j-1} \left[ \left( \pi_h(m_i^j), v_i^j \right)_\omega + \left( f, v_i^j \right)_\omega + c \left( s_h, v_i^j \right)_\omega \right].
\]

The Cauchy-Schwarz inequality and the Young inequality, together with assumption (H2), yield for any \( \varepsilon > 0 \)

\[
\frac{1}{2} \left\| \nabla m_i^j \right\|^2_{L^2} + \frac{k}{C_{\text{exch}}} \left( \alpha - \frac{2 + c}{2} \varepsilon \right) \sum_{i=0}^{j-1} \left\| m_i^j \right\|^2_{L^2} + k^2 \left( \theta - \frac{1}{2} \right) \sum_{i=0}^{j-1} \left\| \nabla v_i^j \right\|^2_{L^2} \leq \frac{1}{2} \left\| \nabla m_i^j \right\|^2_{L^2} + \frac{k}{2\varepsilon C_{\text{exch}}} \sum_{i=0}^{j-1} \left[ C_m \left\| m_i^j \right\|^2_{L^2} + \left\| f \right\|^2_{L^2} + c \left\| s_h \right\|^2_{L^2} \right].
\]

From Proposition 15 we deduce

\[
k \sum_{i=0}^{j-1} \left\| m_i^j \right\|^2_{L^2} \leq C' \left( 1 + k^2 \sum_{i=0}^{j-1} \left\| v_i^j \right\|^2_{L^2} \right),
\]

where the constant \( C' > 0 \) depends only on \( |\omega|, T \) and \( C_m \). We thus obtain

\[
\frac{1}{2} \left\| \nabla m_i^j \right\|^2_{L^2} + \frac{k}{C_{\text{exch}}} \left( \alpha - \frac{2 + c}{2} \varepsilon - \frac{kC_m^2 C'}{2\varepsilon} \right) \sum_{i=0}^{j-1} \left\| m_i^j \right\|^2_{L^2} + k^2 \left( \theta - \frac{1}{2} \right) \sum_{i=0}^{j-1} \left\| \nabla v_i^j \right\|^2_{L^2} \leq \frac{1}{2} \left\| \nabla m_i^j \right\|^2_{L^2} + \frac{C'C_m^2}{2\varepsilon C_{\text{exch}}} + \frac{k}{2\varepsilon C_{\text{exch}}} \sum_{i=0}^{j-1} \left[ \left\| f \right\|^2_{L^2} + c \left\| s_h \right\|^2_{L^2} \right].
\]

Note that \( \theta > 1/2 \). If we choose \( \varepsilon < 2\alpha/(2 + c) \), for \( k < k_0 := \varepsilon (2\alpha - (2 + c)\varepsilon) / (C_m^2 C') \) all the coefficients on the left-hand side are positive. From the regularity of \( f \), assumption (H1), and the boundedness from Proposition 15, we know that the right-hand side is uniformly bounded. This yields the estimate \((23)\). \( \square \)
Corollary 20. Under the assumptions of Proposition 19 and if \( k < k_0 \), the sequences \( \{m_{hk}\} \), \( \{\Pi_h m_{hk}\} \) and \( \{v_{hk}\} \) are uniformly bounded. In particular, it holds
\[
\|m_{hk}\|_{H^1(\omega_T)} + \|m_{hk}\|_{L^2(0,T;H^1(\omega))} + \|\Pi_h m_{hk}\|_{L^2(0,T;H^1(\omega))} + \|v_{hk}\|_{L^2(\omega_T)} \leq C,
\]
where the constant \( C > 0 \) depends only on the data and \( k_0 \), but is independent of the discretization parameters \( h \) and \( k \).

**Proof.** The result follows from the boundedness of the discrete functions \( \{(v'_h, m^{i+1}_h)\}_{0 \leq i \leq N-1} \) from Proposition 9 and Proposition 19 and from (15).

We can now proceed with step (ii) of the proof and conclude the existence of weakly convergent subsequences.

**Proposition 21.** Suppose that the assumptions of Theorem 22 (a) are satisfied. Then, there exist \( m \in H^1(\omega_T) \cap L^\infty(0,T;H^1(\omega)) \) and \( s \in L^2(0,T;H^1(\omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;\tilde{H}^{-1}(\Omega)) \), with \( |m| = 1 \) a.e. in \( \omega_T \), such that there holds
\[
\begin{align*}
\text{(24a)} & \quad m_{hk} \xrightarrow{\text{sub}} m \text{ in } H^1(\omega_T), \\
\text{(24b)} & \quad m_{hk}, m_{hk}^+; \Pi_h m_{hk}^+ \xrightarrow{\text{sub}} m \text{ in } L^2(0,T;H^1(\omega)), \\
\text{(24c)} & \quad m_{hk}, m_{hk}^+; \Pi_h m_{hk}^+ \xrightarrow{\text{sub}} m \text{ in } L^2(\omega_T), \\
\text{(24d)} & \quad v_{hk} \xrightarrow{\text{sub}} \partial_t m \text{ in } L^2(\omega_T), \\
\text{(24e)} & \quad s_{hk}, s_{hk}^± \xrightarrow{\text{sub}} s \text{ in } L^2(0,T;H^1(\omega)), \\
\text{(24f)} & \quad \partial s_{hk} \xrightarrow{\text{sub}} \partial s \text{ in } L^2(0,T;\tilde{H}^{-1}(\Omega))
\end{align*}
\]
for \( (h,k) \to (0,0) \). Moreover, there exists one subsequence for which (24) holds simultaneously.

**Proof.** The boundedness results from Corollary 20 in combination with the Eberlein-Smulian theorem, allowing us to extract weakly convergent subsequences of \( \{m_{hk}\} \), \( \{m_{hk}^+\} \), \( \{\Pi_h m_{hk}^+\} \) and \( \{v_{hk}\} \).

Let \( m \in H^1(\omega_T) \) be such that \( m_{hk} \xrightarrow{\text{sub}} m \text{ in } H^1(\omega_T) \). From the continuous inclusions \( H^1(\omega_T) \subset L^2(0,T;H^1(\omega)) \subset L^2(\omega_T) \) and the compact embedding \( H^1(\omega_T) \subset L^2(\omega_T) \), we deduce
\[
\begin{align*}
\text{(25)} & \quad \text{m}_{hk} \xrightarrow{\text{sub}} m \text{ in } L^2(0,T;H^1(\omega)) \quad \text{and} \quad \text{m}_{hk} \xrightarrow{\text{sub}} m \text{ in } L^2(\omega_T).
\end{align*}
\]
With \( \|m_{hk} - m_{hk}^+\|_{L^2(\omega_T)} \leq k \|v_{hk}\|_{L^2(\omega_T)} \), we can identify the limits of the subsequences of \( \{m_{hk}\} \) and \( \{m_{hk}^+\} \). As \( \partial_t m_{hk} = v_{hk} \), it clearly holds that \( v = \partial_t m \) a.e. in \( \omega_T \).

We now prove that the limiting function \( m \) satisfies the unit-length constraint. First, we observe that
\[
\begin{align*}
\|m^2 - m_{hk}^2\|_{L^2(\omega_T)} \leq \|m + m_{hk}^+\|_{L^2(\omega_T)} \|m - m_{hk}^+\|_{L^2(\omega_T)} \xrightarrow{\text{sub}} 0
\end{align*}
\]
for \( (h,k) \to 0 \). For \( 0 \leq i \leq N-1 \) and \( K \in T_h^\omega \), a standard interpolation estimate for the piecewise linear function \( m_h^{i+1} \in S^1(T_h^{\omega,i}) \) yields
\[
\|m_h^{i+1}\|^2 - J_h \left( m_h^{i+1}\right) ^2 \|_{L^2(K)} \lesssim h_K^2 \|\nabla m_h^{i+1}\|^2 \|_{L^2(K)} \lesssim h_K^2 \|\nabla m_h^{i+1}\|^2 \|_{L^2(K)} \lesssim h_k^{1/2} \|\nabla m_h^{i+1}\|^2 \|_{L^2(K)}.
\]
From Proposition 19 we obtain
\[
\left\| m^*_t h^k - I_h \left( \left| m^*_t h^k \right|^2 \right) \right\|_{L^2(\omega_T)} \lesssim h^{1/2},
\] (26)

For all \( 1 \leq j \leq N \), Proposition 17 and the discrete norm equivalence of Lemma 8 with \( r = 2 \) yield
\[
\left\| I_h \left( \left| m^*_t h^k \right|^2 \right) - 1 \right\|_{L^1(\omega_T)} \lesssim \sum_{z \in N} h^3 \left\| m^*_t h^k (z) \right\|^2 - 1 \right\| \leq k^2 \sum_{z \in N} h^3 \sum_{z \in N} |v_{h}^j(z)|^2 \lesssim k^2 \sum_{i=0}^{r-1} \left\| v_{h}^j \right\|_{L^2(\omega_T)}^2.
\]

Then, from Proposition 19 we deduce
\[
\left\| I_h \left( \left| m^*_t h^k \right|^2 \right) - 1 \right\|_{L^1(\omega_T)} \lesssim k.
\] (27)

Combining (26)–(27), the triangle inequality thus yields that \( \left| m^*_t h^k \right|^2 \) \( \rightarrow \) 1 in \( L^1(\omega_T) \) for \( (h, k) \rightarrow 0 \), whence \( \left| m \right| = 1 \) a.e. in \( \omega_T \) follows from (25).

For \( x \in \mathbb{R}^3 \) with \( |x| \geq 1 \), it holds that
\[
\left| \frac{x - x}{|x|} \right| = |x| - 1 = \frac{|x|^2 - 1}{|x| + 1} \leq \frac{1}{2} \left( |x|^2 - 1 \right).
\]

Due to (17), this yields for all \( 1 \leq j \leq N \)
\[
\left| m^*_t h^k - \Pi_h m^*_t h^k \right| \leq \frac{1}{2} \left( \left| m^*_t h^k (z) \right|^2 - 1 \right) = \frac{1}{2} k^2 \sum_{i=0}^{r-1} \left\| v_{h}^j(z) \right\|^2,
\]
whence by virtue of Proposition 19
\[
\left\| m^*_t h^k - \Pi_h m^*_t h^k \right\|_{L^1(\omega_T)} \lesssim k^2 \sum_{i=0}^{r-1} \left\| v_{h}^j \right\|_{L^2(\omega_T)}^2 \lesssim k.
\]

This implies \( \Pi_h m^*_t h^k \) \( \rightarrow \) \( m \) in \( L^1(\omega_T) \) as \( (h, k) \rightarrow (0, 0) \). Since \( \left\| \Pi_h m^*_t h^k \right\|_{L^\infty(\omega_T)} + \left\| m \right\|_{L^\infty(\omega_T)} = 2 \),
we have \( \Pi_h m^*_t h^k \) \( \rightarrow \) \( m \) even in \( L^2(\omega_T) \) as well as \( \Pi_h m^*_t h^k \) \( \rightarrow \) \( m \) in \( L^2(0, T; H^1(\omega)) \).

From Corollary 16 we similarly deduce the existence of weakly convergent subsequences of \( \{s_{h,i}\} \) and \( \{s_{h,i}^*\} \). Due to Proposition 15, the quantity \( \sum_{i=0}^{N-1} \left\| s_{h,i}^* - s_{h,i} \right\|_{L^2(\Omega)}^2 \) is bounded. This allows to identify the weak limits, since
\[
\left\| s_{h,i} - s_{h,i}^* \right\|_{L^2(\Omega)} \lesssim k \sum_{i=0}^{N-1} \left\| s_{h,i}^* - s_{h,i} \right\|_{L^2(\Omega)} \rightarrow 0 \quad \text{for } (h, k) \rightarrow 0.
\]

Finally, from Proposition 17 we deduce the existence of a weakly convergent subsequence of \( \{\partial s_{h,i}\} \), and it is easy to see that its limit is precisely \( \partial s \), cf. [16] Section 7.1.2, Theorem 3. This establishes (24e)–(24f) and thus concludes the proof.

\[ \square \]

**Remark 22.** As the constants which guarantee the boundedness of Proposition 15 Corollary 16 and Proposition 17 are independent of \( T \), we deduce that \( s \in L^2(\mathbb{R}^+; H^1(\Omega)) \cap L^2(\mathbb{R}^+; L^2(\Omega)) \cap H^1(\mathbb{R}^+; H^{-1}(\Omega)) \).
We have collected all the ingredients for the proof of our main theorem.

**Proof of Theorem**[72] The result of part (a) follows directly from Proposition[21]. To conclude the proof of part (b), it remains to identify the limiting functions \((m, s)\) with a weak solution of SDLLG in the sense of Definition[2].

To check (11a), we essentially proceed as in[2]. Let \(\phi \in C^\infty(\Omega_T)\). For \(0 \leq i \leq N - 1\), we test (16a) with respect to \(\phi_h = I_h \left( (m_{\text{hi}} \times \phi) (t_i) \right) \in K_{m_{\text{hi}}} \), with \(I_h\) being the nodal interpolation operator onto \(S^1(T_h^{\infty})^3\). Multiplication with \(k\) and summation over \(0 \leq i \leq N - 1\) yield

\[
\left( av_{hk} + m_{hk} \times v_{hk}, I_h \left( m_{hk} \times \phi_k \right) \right)_{\text{dwr}} = -C_{\text{exch}} \left( \nabla \left( m_{hk} + \theta k \phi \right), \nabla \left( m_{hk} \times \phi_k \right) \right)_{\text{dwr}} + \left( \pi_h(m_{hk}), I_h \left( m_{hk} \times \phi_k \right) \right)_{\text{dwr}} + \left( f'_h, I_h \left( m_{hk} \times \phi_k \right) \right)_{\text{dwr}} + c \left( s_{hk}, I_h \left( m_{hk} \times \phi_k \right) \right)_{\text{dwr}},
\]

where \(I_h \left( m_{hk} \times \phi_k \right) (t) = I_h \left( (m_{hk} \times \phi) (t) \right)\) for all \(t \in [t_i, t_{i+1})\). From the well-known approximation properties of \(I_h\) and the boundedness of \(\sqrt{k} \left\| \nabla v_{hk} \right\|_{L^2(\Omega_T)}\) from Proposition[19] for \(\theta \in (1/2, 1]\), we deduce

\[
\left( av_{hk} + m_{hk} \times v_{hk}, m_{hk} \times \phi_k \right)_{\text{dwr}} + C_{\text{exch}} \left( \nabla \left( m_{hk} + \theta k \phi \right), \nabla \left( m_{hk} \times \phi_k \right) \right)_{\text{dwr}} - \left( \pi_h(m_{hk}), m_{hk} \times \phi_k \right)_{\text{dwr}} - \left( f'_h, m_{hk} \times \phi_k \right)_{\text{dwr}} - c \left( s_{hk}, m_{hk} \times \phi_k \right)_{\text{dwr}} = O(h).
\]

Passing to the limit for \((h, k) \to (0, 0)\), we obtain

\[
\left( a \partial_t m + m \times \partial_t m, m \times \phi \right)_{\text{dwr}} = -C_{\text{exch}} \left( \nabla m, \nabla (m \times \phi) \right)_{\text{dwr}} + \left( \pi(m) + f \circ s, m \times \phi \right)_{\text{dwr}}.
\]

In the latter, we have used the convergence properties from Proposition[21] assumption (H3) for the general field contribution, as well as \(f'_h \to f\) and \(m_{hk} \times \phi_k \xrightarrow{\text{sub}} m \times \phi\) in \(L^2(\Omega_T)\).

Direct calculations and standard properties of the cross product yield the identities

\[
\left( \nabla m, \nabla (m \times \phi) \right)_{\text{dwr}} = \left( \nabla m \times m, \nabla \phi \right)_{\text{dwr}}, \quad \left( \partial_t m, m \times \phi \right)_{\text{dwr}} = \left( \partial_t (m \times \phi) \right)_{\text{dwr}},
\]

\[
\left( m \times \partial_t m, m \times \phi \right)_{\text{dwr}} = \left( \partial_t m, m \times \phi \right)_{\text{dwr}}, \quad \left( \pi(m), m \times \phi \right)_{\text{dwr}} = \left( \pi(m \times m), m \times \phi \right)_{\text{dwr}},
\]

\[
\left( f, m \times \phi \right)_{\text{dwr}} = \left( f \times m, m \times \phi \right)_{\text{dwr}}, \quad \left( s, m \times \phi \right)_{\text{dwr}} = \left( s \times m, m \times \phi \right)_{\text{dwr}},
\]

from which, by density, we deduce (11a).

To check (11b), let \(\phi \in C^\infty(0, T; C^0(\bar{\Omega}))\). Given \(0 \leq i \leq N - 1\), let \(t \in [t_i, t_{i+1})\). In (16c) we choose the test function \(\zeta_h = I_h(\phi(t)) \in S^1(T_h^{\infty})^3\). Integration in time over \([t_i, t_{i+1})\) and summation over \(0 \leq i \leq N - 1\) yield

\[
\left( \partial_s s_h, I_h \phi \right)_{\text{dwr}} + \left( D_h \nabla s_h, \nabla I_h \phi \right)_{\text{dwr}} - \beta \beta' \left( D_h \Pi_h g_{hk}, \nabla I_h \phi \right)_{\text{dwr}} + \left( D_h s_h, I_h \phi \right)_{\text{dwr}} + \left( D_h \left( s_{hk} \times \Pi_h m_{hk} \right), I_h \phi \right)_{\text{dwr}} \right.
\]

\[
\left. = \beta \left( \Pi_h m_{hk} \times j_h, \nabla I_h \phi \right)_{\text{dwr}} - \beta \left( j_h \times s_{hk}, I_h \phi \right)_{(0,T) \times (\partial T \cup \partial \Omega)} \right),
\]

where \(I_h \phi(t) = I_h(\phi(t))\) for all \(t \in (0, T)\). Passing (28) to the limit for \((h, k) \to (0, 0)\), due to the convergence properties stated in Proposition[21] in combination with the standard approximation...
properties of $I_h$, we deduce
\[
\int_0^T \left( \partial_t s(t), \varphi(t) \right) + (D_t \nabla s, \nabla \varphi)_{\Omega_T} - \beta \beta' (D_t m \otimes (\nabla s \cdot m), \nabla \varphi)_{\omega_T} \\
+ (D_t(s, \varphi)_{\Omega_T} + (D_0 (s \times m), \varphi)_{\omega_T} = \beta (m \otimes j, \nabla \varphi)_{\omega_T} - \beta (j \cdot n, m \cdot \varphi)_{(0, T) \times (\partial \Omega_T \cap \partial \omega)}).
\]
By density, this is also true for all $\varphi \in C^\infty(0, T; H^1(\Omega))$. Hence in particular for each $\zeta \in H^1(\Omega)$ and a.e. $t \in (0, T)$ we have [111].

Since $m_{\text{sub}} \xrightarrow{sub} m$ in $H^1(\omega_T)$ with $m_{\text{sub}}(0) = m_0^h$, and $s_{\text{sub}} \xrightarrow{sub} s$ in $H^1(0, T; \tilde{H}^{-1}(\Omega))$ with $s_{\text{sub}}(0) = s_0^h$, assumption (H1) allows to deduce $m(0) = m^0$ and $s(0) = s^0$ in the sense of traces. □

5. Energy estimate

In this section, we exploit our constructive convergence proof to derive an energy estimate for weak solutions of SDLLG, which is also meaningful from a physical point of view. The total magnetic Gibbs free energy from [3] is strongly related to the standard form [1] of LLG and does not take into account the interaction between the magnetization and spin accumulation. As we are dealing with the augmented form (7) of LLG, we extend (3) and define the free energy of the system by
\[
E(M, S) = \frac{A}{M_s} \int_\omega |\nabla M|^2 + K \int_\omega \phi(M/M_s) - \mu_0 \int_\omega H_e \cdot M - \frac{\mu_0}{2} \int_\omega H_e(M) \cdot M - J \int_\omega S \cdot M. \tag{29}
\]
This definition is in agreement with (7), since in this case it is easy to see that
\[
-\frac{\delta E(M, S)}{\delta M} = \mu_0 H_{\text{eff}}(M) + JS,
\]
where the effective field is given by [4]. A simple formal computation shows that strong solutions to (7) satisfy
\[
\frac{dE}{dt} = -\frac{\alpha}{\gamma M_s} \int_\omega \left| \frac{\partial M}{\partial t} \right|^2 - \mu_0 \int_\omega \frac{\partial H_e}{\partial t} \cdot M - J \int_\omega \frac{\partial S}{\partial t} \cdot M. \tag{30}
\]
Neglecting the spin accumulation term and assuming that $H_e$ is constant in time, equation (30) reduces to
\[
\frac{dE}{dt} = -\frac{\alpha}{\gamma M_s} \int_\omega \left| \frac{\partial M}{\partial t} \right|^2 \leq 0,
\]
which reveals the well-known dissipative behavior of solutions to the standard form (1) of LLG.

The main aim of this section is to prove a property corresponding to (30) in the context of weak solutions. To this end, we move to the nondimensional framework introduced in Section 2 and consider the following assumptions:

(H4) the operator $\pi : L^2(\omega) \to L^2(\omega)$ from [10] is linear, self-adjoint, and bounded;

(H5) it holds
\[
\pi_h(w_{hk}) \to \pi(w) \text{ in } L^2(\omega_T) \quad \text{as } (h, k) \to 0
\]
for any sequence $w_{hk} \to w$ in $L^2(\omega_T)$, which is slightly stronger than (H3);

(H6) the applied field $\mathbf{f}$ belongs to $H^1(0, T; L^2(\omega))$. 19
Remark 23. For some fixed easy axis $e \in S^2$ and the corresponding uniaxial anisotropy density function $\phi(m) = 1 - (m \cdot e)^2$, the operator

$$\pi(m) = h_e(m) - C_{ani}\nabla\phi(m) = h_e(m) + 2C_{ani}(e \cdot m)m$$

(31)

satisfies (H4). Moreover, all the stray field discretizations mentioned in Remark 70 satisfy (H5), see [15]. The operator $\pi$ from (31) is even well defined and bounded as operator $\pi : L^p(\omega) \rightarrow L^q(\omega)$ for all $1 < p < \infty$, see [28]. Unlike [4], the proof of our energy estimate, see Theorem 24 below, avoids this additional regularity, but only relies on the energy setting $p = 2$.

Up to an additive constant, the nondimensional counterpart of (29) reads

$$E(m, s) = \frac{1}{2} C_{exch} \int_\omega |\nabla m|^2 - \int_\omega f \cdot m - \frac{1}{2} \int_\omega \pi(m) \cdot m - c \int_\omega s \cdot m.$$  

(32)

The following theorem proves an energy estimate which generalizes (30) to weak solutions.

Theorem 24. Suppose that assumptions (H1)–(H2) and (H4)–(H6) are satisfied. Let $(m, s)$ be a weak solution of SDLLG obtained as a weak limit of the finite element solutions from Algorithm 6 for $1/2 < \theta \leq 1$. Then, the energy functional from (32) satisfies

$$E(m(t), s(t)) + \alpha \int_{t_0}^t \int_\omega \|\partial_t m(t')\|^2_{L^2(\omega)} + \int_{t_0}^t \int_\omega \|\partial_t f(t')\|^2_{L^2(\omega)} + c \int_{t_0}^t \|\partial_t s(t')\|^2_{L^2(\omega)}$$

$$\leq E(m_0, s_0)$$

(33)

for almost all $t \in (0, T)$.

Proof. Given $t \in (0, T)$, let $0 \leq j \leq N - 1$ such that $t \in [t_j, t_{j+1})$. Let $0 \leq i \leq j$. From Lemma 18 we get

$$E(m_h^{i+1}, s_h^{i+1}) - E(m_h^i, s_h^i)$$

$$= \frac{1}{2} C_{exch} \left( \|\nabla m_h^{i+1}\|^2_{L^2(\omega)} - \|\nabla m_h^i\|^2_{L^2(\omega)} - \|f_{i+1}\|^2_{L^2(\omega)} - \|f_i\|^2_{L^2(\omega)} \right)$$

$$- \frac{1}{2} \left( \pi(m_h^{i+1}, m_h^i) \right)_{\omega} + \frac{1}{2} \left( \pi(m_h^i, m_h^i) \right)_{\omega} - c \left( s_h^{i+1}, m_h^{i+1} \right)_{\omega} + \frac{1}{2} \left( s_h^i, m_h^i \right)_{\omega}$$

$$= -\alpha k \left\| v_h^i \right\|^2_{L^2(\omega)} - C_{exch} k^2 \left( \theta - \frac{1}{2} \right) \left\| \nabla v_h^i \right\|^2_{L^2(\omega)} - \left( f_{i+1}, m_h^{i+1} \right)_{\omega} + \frac{1}{2} \left( f_i, m_h^i + k v_h^i \right)_{\omega}$$

$$+ k \left( \pi_s(m_h^i, v_h^i) - \frac{1}{2} \left( \pi(m_h^i, m_h^i) \right)_{\omega} + \frac{1}{2} \left( \pi(m_h^i, m_h^i) \right)_{\omega} - c \left( s_h^{i+1}, m_h^{i+1} \right)_{\omega} + \frac{1}{2} \left( s_h^i, m_h^i + k v_h^i \right)_{\omega} \right)$$

$$= T_1 - T_2 - T_3.$$  

By definition (16b), it holds $m_h^{i+1} = m_h^i + k v_h^i$. We thus obtain

$$T_1 = -\left( f_{i+1} - f_i, m_h^{i+1} \right)_{\omega} - \left( f_i, m_h^{i+1} - m_h^i - k v_h^i \right)_{\omega} = -k \left( d f_{i+1}, m_h^{i+1} \right)_{\omega}.$$  

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Analogously, we see that $T_3 = -ck\left(d_s^{i+1}, m_h^{i+1}\right)_\omega$. Since $\pi$ is linear and self-adjoint, (16b) also reveals

$$T_2 = -k \left(\pi(m_h^i) - \pi_h(m_h^{i})_\omega, v_h^i\right)_\omega + \frac{1}{2} \left(\pi(m_h^{i+1}) - \pi(m_h^{i+1}), m_h^{i+1}\right)_\omega - \frac{1}{2} \left(\pi(m_h^{i+1}) - \pi(m_h^{i+1}), m_h^{i+1}\right)_\omega$$

$$= -k \left(\pi(m_h^i) - \pi_h(m_h^{i})_\omega, v_h^i\right)_\omega + \frac{1}{2} k \left(\pi(m_h^i), v_h^i\right)_\omega - \frac{1}{2} k^2 \left(\pi(v_h^i), v_h^i\right)_\omega.$$

Altogether, we thus obtain

$$E(m_h^{i+1}, s_h^{i+1}) - E(m_h^i, s_h^i) + \alpha k \|v_h^i\|_{L^2(\Omega)}^2 + k \left(d_s^{i+1}, m_h^{i+1}\right)_\omega + c k \left(d_s^{i+1}, m_h^{i+1}\right)_\omega$$

$$= -C_{\text{exch}} k^2 \left(\theta - \frac{1}{2}\right) \|v_h^i\|_{L^2(\Omega)}^2 - k \left(\pi(m_h^i) - \pi_h(m_h^{i})_\omega, v_h^i\right)_\omega - \frac{1}{2} k^2 \left(\pi(v_h^i), v_h^i\right)_\omega.$$

Since $\pi$ is a bounded operator, it follows that

$$E(m_h^{i+1}, s_h^{i+1}) - E(m_h^i, s_h^i) + \alpha k \|v_h^i\|_{L^2(\Omega)}^2 + k \left(d_s^{i+1}, m_h^{i+1}\right)_\omega$$

$$+ c k \left(d_s^{i+1}, m_h^{i+1}\right)_\omega + k \left(\pi(m_h^i) - \pi_h(m_h^{i})_\omega, v_h^i\right)_\omega \lesssim k^2 \|v_h^i\|_{L^2(\Omega)}^2.$$ 

Summation over $0 \leq i \leq j$ and the boundedness from Proposition 19 yield

$$E(m_h^1(t_{j+1}), s_h^1(t_{j+1})) - E(m_h^0(t_j), s_h^0(t_j)) + \alpha \int_{t_j}^{t_{j+1}} \|v_h(t')\|_{L^2(\Omega)}^2 + \int_{t_j}^{t_{j+1}} \left(\partial_t f_h(t'), m_h(t')\right)_\omega$$

$$+ c \int_{t_j}^{t_{j+1}} \left(\partial_t s_h(t'), m_h(t')\right)_\omega + \int_{t_j}^{t_{j+1}} \left(\pi(m_h(t')) - \pi_h(m_h(t')_\omega, v_h(t'))\right)_\omega$$

$$\lesssim k \int_{t_j}^{t_{j+1}} \|v_h(t')\|_{L^2(\Omega)}^2 \lesssim k.$$ 

The available convergence results on $m_h^j, s_h^j, s_h^j, v_h^j, f_h$, as well as assumption (H5), allow us to employ standard arguments with weakly lower semicontinuity for the limit $(h,k) \to 0$. This concludes the proof of (33). □

Acknowledgments

The research of GH, MP, DP, and DS is supported through the project MA09-029 funded by Vienna Science and Technology Fund (WWTF). The research of DP and MR is supported through the doctoral school Dissipation and Dispersion in Nonlinear PDEs, funded by the Austrian Science Fund (FWF) under grant W1245, and through the innovative projects initiative of Vienna University of Technology. Additionally, CA and DS acknowledge financial support of the Austrian Federal Ministry of Economy, Family and Youth and the National Foundation for Research, Technology and Development (Christian Doppler Laboratory Advanced Magnetic Sensing and Materials). GH would like to thank the EPSRC (grant EP/K008412/1) and the Royal Society (UF080837) for financial support.
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