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ENERGY NORM BASED ERROR ESTIMATORS FOR ADAPTIVE BEM FOR HYPERSINGULAR INTEGRAL EQUATIONS

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Abstract. For hypersingular integral equations in 2D and 3D, we analyze easy-to-implement error estimators like $(h-h/2)$-based estimators, two-level estimators, and averaging on large patches and prove their equivalence. Moreover, we introduce some ZZ-type error estimators. All of these a posteriori error estimators are analyzed within the framework of localization techniques for the energy norm.

1. Introduction

Let Ω be a polygonal resp. polyhedral, bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, with boundary $\partial\Omega$ and let $\Gamma \subseteq \partial\Omega$ be a relatively open and connected surface with Lipschitz boundary $\partial\Gamma$. Neumann screen problems on Γ yield the hypersingular integral equation

$$Wu(x) := -\frac{\partial}{\partial \boldsymbol{n}_x} \int_\Gamma \Big(\frac{\partial}{\partial \boldsymbol{n}_y} G(x,y) \Big) u(y)\, d\Gamma(y) = f(x) \quad \text{for all } x \in \Gamma \tag{1}$$

with the hypersingular integral operator W. Here $\boldsymbol{n}_x$ denotes the outer normal unit vector of Ω at some point $x \in \Gamma$, and

$$G(x,y) := \begin{cases} -\frac{1}{2\pi} \log|x-y|, & \text{for } d = 2, \\ +\frac{1}{4\pi} \frac{1}{|x-y|}, & \text{for } d = 3, \end{cases} \tag{2}$$

is the fundamental solution of the Laplacian.

For some closed subspaces $H^{1/2}_\star(\Gamma)$ of $H^{1/2}(\Gamma)$, it is known that W induces an equivalent scalar product $\langle\!\langle u\,, v\rangle\!\rangle := \langle Wu\,,\, v\rangle_\Gamma$ on $H^{1/2}_\star(\Gamma)$, where $\langle\cdot\,,\,\cdot\rangle_\Gamma$ denotes the extended $L^2(\Gamma)$-scalar product. For some given right-hand side $f \in H^{1/2}_\star(\Gamma)^*$ in its dual space, (1) can equivalently be stated as follows: Find $u \in H^{1/2}_\star(\Gamma)$ such that

$$\langle\!\langle u\,, v\rangle\!\rangle = \langle f\,,\, v\rangle_\Gamma \quad \text{for all } v \in H^{1/2}_\star(\Gamma). \tag{3}$$

In particular, the Lax-Milgram lemma applies and proves existence and uniqueness of the solution $u \in H^{1/2}_\star(\Gamma)$ of (3). (See Section 2.1–2.2 for the precise functional analytic setting.)

Based on a triangulation $\mathcal{T}_\ell$ of Γ, the Galerkin discretization employs conforming subspaces $\mathcal{S}^p_\star(\mathcal{T}_\ell) \subset H^{1/2}_\star(\Gamma)$ of $\mathcal{T}_\ell$-piecewise polynomials of degree $p \geq 1$. (See Section 2.3–2.5 for the

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precise discrete setting.) The Galerkin formulation then reads as follows: Find $U_\ell \in \mathcal{S}^p_\star(\mathcal{T}_\ell)$ such that

$$(4) \qquad \langle\!\langle U_\ell\,, V_\ell\rangle\!\rangle = \langle f\,,\, V_\ell\rangle_\Gamma \quad \text{for all } V_\ell \in \mathcal{S}^p_\star(\mathcal{T}_\ell).$$

Again the Lax-Milgram lemma applies and proves existence and uniqueness of the solution $U_\ell \in \mathcal{S}^p_\star(\mathcal{T}_\ell)$ of (4). Throughout, the index ℓ corresponds to the level of mesh-generation by an adaptive algorithm (see Algorithm 25 in Section 5 below), and the corresponding quantities are discrete and computable.

In practice, the accuracy of a computed Galerkin solution U_ℓ is spoilt by the singularities of the unknown solution u. One remedy is to use triangulations which are appropriately graded towards the singularities of u. These are usually obtained by adaptive mesh-refining algorithms of the type

$$\boxed{SOLVE} \quad \Longrightarrow \quad \boxed{ESTIMATE} \quad \Longrightarrow \quad \boxed{MARK} \quad \Longrightarrow \quad \boxed{REFINE}$$

which avoid any a priori knowledge of u and refine the triangulation locally, where the error appears to be large. For the marking criterion serve the local contributions of certain a posteriori error estimators which can be computed as soon as the discrete solution is known. Empirically, these adaptive algorithms lead to convergence, and even the optimal rate of convergence is regained. For hypersingular integral equations, several types of a posteriori error estimators have been proposed, see e.g. [Car97, CP07a, CP07b, EFGP13, Heu02, HMS01, MS00, FFKP13] and the references therein.

A posteriori error estimation is an important tool for reliable and efficient Galerkin boundary element computations.

In this work, we consider $(h-h/2)$-type error estimators which have been introduced for hypersingular integral equations in [EFGP13], two-level error estimators which go back to [MS00, HMS01, Heu02], averaging on large patches [CP07a, CP07b], and ZZ-type error estimators [FFKP13]. Surprisingly, the analysis of all these estimators can be done in one common analytical frame, namely localization techniques for the $H^{1/2}$-norm.

The contributions of this work can be concluded as follows: We generalize the analysis of [CP07a, CP07b, EFGP13] from the lowest-order case $p = 1$ in 2D to general $p \geq 1$ and $d = 2, 3$. In the latter works, the localization of the energy norm is done by lowest-order nodal interpolation and builds on a 2D result from [Car97]. The novel analysis allows to transfer the analysis of $(h-h/2)$-type estimators [EFGP13], averaging on large patches [CP07a, CP07b], and ZZ-type error estimators [FFKP13] to $d = 2, 3$ and $p \geq 1$. Moreover, we generalize the original analysis of [MS00] from uniform to adaptive triangulations and therefore provide an alternative proof to that of [HMS01, Heu02].

The outline of the remainder of this paper reads as follows: We first fix the notation as well as the continuous and discrete setting (Section 2). The heart of our analysis is the localization (13) of the $H^{1/2}$-norm (Section 3) which proves —under certain assumptions— the norm equivalence $\|\cdot\|_{H^{1/2}_\star(\Gamma)} \simeq \|h_\ell^{1/2}\nabla(\cdot)\|_{L^2(\Gamma)}$, i.e., the $H^{1/2}$-norm is replaced by a mesh-size weighted H^1-seminorm. Here, h_ℓ denotes the *local* mesh-size function of the underlying triangulation $\mathcal{T}_\ell$ and $\nabla(\cdot)$ denotes either the arc-length derivative for $d = 2$ resp. the surface gradient for $d = 3$. With this at hand, we then generalize the $(h-h/2)$-type estimators to $d = 2, 3$ and $p \geq 1$ (Section 4.1), give a new analysis for two-level estimators (Section 4.2), transfer the analysis for averaging on large patches (Section 4.3), and introduce new ZZ-type error estimators (Section 4.4) which generalize that of [FFKP13]. The final Section 5 provides some numerical experiments for $d = 2, 3$.

2. Preliminaries and notation

2.1. Sobolev spaces. The usual Lebesgue and Sobolev spaces on Γ are denoted by $L^2(\Gamma)$ and $H^1(\Gamma)$. Moreover, $\widetilde{H}^1(\Gamma)$ is the space of $H^1(\Gamma)$ functions which have a vanishing trace on the relative boundary $\partial\Gamma$.

Sobolev spaces of fractional order $0 < s < 1$ are defined by the K-method of interpolation [McL00, Appendix B]: For $0 < s < 1$, we let $H^s(\Gamma) := [L^2(\Gamma), H^1(\Gamma)]_s$ and $\widetilde{H}^s(\Gamma) := [L^2(\Gamma), \widetilde{H}^1(\Gamma)]_s$. For $0 < s \leq 1$, Sobolev spaces of negative order are defined by duality $H^{-s}(\Gamma) := \widetilde{H}^s(\Gamma)^*$ and $\widetilde{H}^s(\Gamma) := H^s(\Gamma)^*$, where duality is understood with respect to the extended $L^2(\Gamma)$-scalar product $\langle \cdot\,, \cdot\rangle_\Gamma$. In general, there holds the continuous inclusion $\widetilde{H}^{\pm s}(\Gamma) \subseteq H^{\pm s}(\Gamma)$ with $\|v\|_{H^{\pm s}(\Gamma)} \leq \|v\|_{\widetilde{H}^{\pm s}(\Gamma)}$. We note that $\widetilde{H}^{\pm s}(\Gamma) = H^{\pm s}(\Gamma)$ for $0 < s < 1/2$ with equivalent norms. Moreover, it holds that $\widetilde{H}^{\pm s}(\partial\Omega) = H^{\pm s}(\partial\Omega)$ even with equal norms for all $0 < s \leq 1$. Finally, the treatment of the closed boundary $\Gamma = \partial\Omega$ requires the definition of $H_0^{\pm s}(\partial\Omega) = \big\{v \in H^{\pm s}(\partial\Omega) \,:\, \langle v\,, 1\rangle_{\partial\Omega} = 0\big\}$ for all $0 \leq s \leq 1$, where $H^0(\partial\Omega) := L^2(\Gamma)$.

All details and equivalent definitions of the Sobolev spaces are, for instance, found in the monographs [HW08, McL00, SS11, Ste08a].

2.2. Hypersingular integral equation. For $0 \leq s \leq 1$, the hypersingular integral operator $W : \widetilde{H}^s(\Gamma) \to H^{s-1}(\Gamma)$ is well-defined, linear, and continuous. For $s = 1/2$, W is symmetric and elliptic up to constant functions.

For $\Gamma \subsetneqq \partial\Omega$ and $s \geq 1/2$, the space $\widetilde{H}^s(\Gamma)$ does not contain non-trivial constant functions and hence $W : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is elliptic [SS11, Theorems 3.5.9]. In particular, $\langle\!\langle u\,, v\rangle\!\rangle := \langle Wu\,, v\rangle_\Gamma$ defines a scalar product on $\widetilde{H}^{1/2}(\Gamma) =: H_\star^{1/2}(\Gamma)$.

For $\Gamma = \partial\Omega$, the constant functions have to be factored out, and $W : H_0^{1/2}(\partial\Omega) \to H_0^{-1/2}(\partial\Omega)$ is elliptic [SS11, Theorem 3.5.3]. In particular, the definition $\langle\!\langle u\,, v\rangle\!\rangle := \langle Wu\,, v\rangle_{\partial\Omega}$ $+\langle u\,, 1\rangle_{\partial\Omega}\langle v\,, 1\rangle_{\partial\Omega}$ provides a scalar product on $H^{1/2}(\partial\Omega)$ which reduces to $\langle Wu\,, v\rangle_{\partial\Omega}$ if either u or v belong to $H_0^{1/2}(\partial\Omega) =: H_\star^{1/2}(\Gamma)$.

With this notation and provided that $f \in H_0^{-1/2}(\Gamma)$ in case of $\Gamma = \partial\Omega$, the variational formulation (3) is equivalently stated by

$$\langle\!\langle u\,, v\rangle\!\rangle = \langle f\,, v\rangle_\Gamma \quad \text{for all } v \in \widetilde{H}^{1/2}(\Gamma). \tag{5}$$

Moreover, the scalar product $\langle\!\langle \cdot\,, \cdot\rangle\!\rangle$ induces an equivalent norm on $\widetilde{H}^{1/2}(\Gamma)$. This variational form is, in fact, an equivalent formulation of the hypersingular integral equation (1) and fits in the frame of the Lax-Milgram lemma.

2.3. Triangulation and reference elements. For $d = 2$, let $\mathcal{T}_\ell = \{T_1, \ldots, T_N\}$ be a triangulation of Γ into affine line segments, i.e., each element $T \in \mathcal{T}_\ell$ is the affine image of the reference interval $T_{\rm ref} = [0, 1]$. By $\mathcal{P}^p(T_{\rm ref}) = \operatorname{span}\big\{x^k \,:\, k = 0, \ldots, p\big\}$, we denote the space of polynomials of degree at most p on $T_{\rm ref}$.

For $d = 3$, let $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ be a regular triangulation of Γ into affine surface triangles, i.e., each element $T \in \mathcal{T}_\ell$ is the affine image of the reference triangle $T_{\rm ref} = {\rm conv}\{(0,0),(0,1),(1,0)\}$. By $\mathcal{P}^p(T_{\rm ref}) = {\rm span}\,\{x^iy^k \,|\, 0 \le i+k \le p\}$, we denote the space of polynomials of total degree at most p on $T_{\rm ref}$.

Given $\mathcal{T}_\ell$, we define the local mesh-width function $h_\ell \in L^\infty(\Gamma)$ on $\mathcal{T}_\ell$ by $h_\ell|_T := h_\ell(T) := {\rm diam}(T)$. For $d = 2$, the triangulation $\mathcal{T}_\ell$ is called γ-shape regular, if $h_\ell(T) \le \gamma\, h_\ell(T')$ for all $T, T' \in \mathcal{T}_\ell$ with $T \cap T' \neq \emptyset$. For $d = 3$, $\mathcal{T}_\ell$ is called γ-shape regular, if $h_\ell(T) \le \gamma|T|^{1/2}$ for all $T \in \mathcal{T}_\ell$.

By $\mathcal{N}_\ell$, we denote the set of all vertices of $\mathcal{T}_\ell$. Moreover, for each $T \in \mathcal{T}_\ell$ the element patch is defined by $\omega_\ell(T) = \bigcup \big\{T' \in \mathcal{T}_\ell \,:\, T' \cap T \neq \emptyset\big\}$, i.e., $\omega_\ell(T)$ is the union of all elements which touch T.

2.4. Discrete spaces. In either case $d = 2, 3$ and for each element $T \in \mathcal{T}_\ell$, we fix an affine parametrization $F_T : T_{\rm ref} \to T$ of T. Spaces of discontinuous resp. continuous $\mathcal{T}_\ell$-piecewise polynomials on Γ are then defined by

$$\mathcal{P}^p(\mathcal{T}_\ell) = \big\{v \in L^\infty(\Gamma) \,:\, v \circ F_T \in \mathcal{P}^p(T_{\rm ref}) \quad \text{for all } T \in \mathcal{T}_\ell\big\}, \tag{6}$$

$$\mathcal{S}^p(\mathcal{T}_\ell) = \mathcal{P}^p(\mathcal{T}_\ell) \cap C(\Gamma). \tag{7}$$

Moreover, we define

$$\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell) = \mathcal{S}^p(\mathcal{T}_\ell) \cap \widetilde{H}^{1/2}(\Gamma) = \begin{cases} \big\{v \in \mathcal{S}^p(\mathcal{T}_\ell) \,:\, v|_{\partial\Gamma} = 0\big\}, & \text{for } \Gamma \subsetneqq \partial\Omega, \\ \mathcal{S}^p(\mathcal{T}_\ell), & \text{for } \Gamma = \partial\Omega. \end{cases} \tag{8}$$

2.5. Galerkin discretization. The Galerkin formulation of (5) reads

$$\langle\!\langle U_\ell \,, V_\ell \rangle\!\rangle = \langle f \,,\, V_\ell \rangle_\Gamma \quad \text{for all } V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell) \tag{9}$$

and admits a unique solution. The Galerkin projection $\mathbb{G}_\ell : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ is defined by

$$\langle\!\langle \mathbb{G}_\ell v \,, V_\ell \rangle\!\rangle = \langle\!\langle v \,, V_\ell \rangle\!\rangle \quad \text{for all } V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell). \tag{10}$$

Clearly, $\mathbb{G}_\ell V_\ell = V_\ell$ for all $V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$. Moreover, $\mathbb{G}_\ell$ is the orthogonal projection onto $\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ with respect to the energy norm $|\!|\!|\cdot|\!|\!|$, and it thus holds

$$|\!|\!|(1-\mathbb{G}_\ell)v|\!|\!| = \min_{V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)} |\!|\!|v - V_\ell|\!|\!| \quad \text{as well as} \quad |\!|\!|\mathbb{G}_\ell v|\!|\!|^2 + |\!|\!|(1-\mathbb{G}_\ell)v|\!|\!|^2 = |\!|\!|v|\!|\!|^2 \tag{11}$$

for all $v \in \widetilde{H}^{1/2}(\Gamma)$. Finally, in the case of $\Gamma = \partial\Omega$ and $v \in H^{1/2}_0(\partial\Omega)$, we note that $\mathbb{G}_\ell v \in \mathcal{S}^p(\mathcal{T}_\ell)$ also satisfies $\langle \mathbb{G}_\ell v \,,\, 1\rangle = 0$ which immediately follows from (10) for $V_\ell = 1$. In particular, the Galerkin solutions of (4) and (9) coincide.

3. Localization of energy norm

The error estimators which will be considered in Section 4, rely on the fact that, for certain discrete functions, one may replace the energy norm $|\!|\!|\cdot|\!|\!| \simeq \|\cdot\|_{\widetilde{H}^{1/2}(\Gamma)}$ by some h_ℓ-weighted H^1-seminorm. To be more precise, let $\widehat{\mathcal{T}}_\ell$ be the uniform refinement of $\mathcal{T}_\ell$, i.e.,

$$\mathcal{P}^0(\mathcal{T}_\ell) \subset \mathcal{P}^0(\mathcal{T}_{\ell+1}) \quad \text{and} \quad \widehat{h}_\ell = h_\ell/2 \text{ for the corresponding mesh-sizes.} \tag{12}$$

Let $p, q \geq 1$ be fixed polynomial degrees. We prove, for all $\widehat{V}_\ell \in \widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell)$, that

$$C_{\rm apx}^{-1} \, |||(1 - P_\ell)\widehat{V}_\ell||| \leq \min_{V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)} \|h_\ell^{1/2}\nabla(\widehat{V}_\ell - V_\ell)\|_{L^2(\Gamma)} \leq \|h_\ell^{1/2}\nabla(1 - P_\ell)\widehat{V}_\ell\|_{L^2(\Gamma)}, \tag{13a}$$

$$C_{\rm inv}^{-1} \, \|h_\ell^{1/2}\nabla(1 - P_\ell)\widehat{V}_\ell\|_{L^2(\Gamma)} \leq \min_{V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)} |||\widehat{V}_\ell - V_\ell||| \leq |||(1 - P_\ell)\widehat{V}_\ell|||, \tag{13b}$$

where $P_\ell : \widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ is an appropriate operator.

Possible choices for P_ℓ include the Galerkin projection $P_\ell = \mathbb{G}_\ell$, the usual Lagrange interpolation operator $P_\ell = I_\ell$ (see Section 3.5), as well as an appropriate Scott-Zhang projection $P_\ell = J_\ell$ (see Section 3.4). The original construction of J_ℓ in [SZ90], however, does not allow to employ J_ℓ for functions in $\widetilde{H}^{1/2}(\Gamma)$, and we therefore generalize the definition of [SZ90] in Section 3.2.

The estimate (13b) is an inverse estimate, since a locally weighted H^1-seminorm is bounded by the weaker $H^{1/2}$-norm. It will be proved in Section 3.3 for all $V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)$. The estimate (13a) is an approximation estimate with the fractional-order Sobolev space $\widetilde{H}^{1/2}(\Gamma)$ on the left-hand side and a locally weighted $H^1(\Gamma)$-seminorm on the right-hand side. Note that the minima in (13) are taken over different spaces $\mathcal{S}^p(\mathcal{T}_\ell) \supseteq \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ which coincide for $\Gamma = \partial\Omega$.

3.1. Interpolation spaces. In this short section, we collect the results on interpolation spaces used in the subsequent analysis. The reader is referred to, e.g., the monograph [Tar07] for further results and different interpolation techniques. We stress that different interpolation techniques lead to the same interpolation spaces, but equivalent norms only. We assume that one interpolation method, e.g. real interpolation with the K-functional, is used throughout.

The following result is known as interpolation theorem.

Lemma 1. *For $j = 0, 1$, let X_j and Y_j be Hilbert spaces with continuous inclusions $X_0 \supseteq X_1$ and $Y_0 \supseteq Y_1$. Let $T : X_0 \to Y_0$ be a linear operator, and let the restriction be well-defined as $T : X_1 \to Y_1$. Provided that either operator is continuous with operator norm $c_j = \|T : X_j \to Y_j\|$, the operator $T : X_s \to Y_s$ between the interpolation spaces $X_s = [X_0, X_1]_s$ and $Y_s = [Y_0, Y_1]_s$ is well-defined and continuous with operator norm*

$$\|T : X_s \to Y_s\| \leq c_0^{1-s} c_1^s \tag{14}$$

for all $0 < s < 1$. □

Using interpolation between discrete subspaces of Sobolev spaces, it follows immediately from the finite dimension of the discrete space that the interpolation norm obtained is, in fact, equivalent to the norm obtained by interpolation of the entire Sobolev space. However, we shall need that the norm equivalence constants do not depend on the discrete space, i.e., the number of elements of $\mathcal{T}_\ell$ in our setting. The following abstract lemma from [AL09] provides a suitable criterion whose elementary proof is included for the convenience of the reader.

Lemma 2. *For $j = 0, 1$, let H_j be Hilbert spaces with subspaces $X_j \subseteq H_j$ which satisfy the continuous inclusions $H_0 \supseteq H_1$ and $X_0 \supseteq X_1$. Assume that $P : H_j \to X_j$ is a well-defined* *linear and continuous projection with operator norm $c_j = \|P : H_j \to X_j\|$, for both $j = 0, 1$. Then, there holds equivalence of the interpolation norms*

$$\|v\|_{[H_0,H_1]_s} \le \|v\|_{[X_0,X_1]_s} \le c_0^{1-s} c_1^s \, \|v\|_{[H_0,H_1]_s} \quad \text{for all } v \in X_s = [X_0, X_1]_s \tag{15}$$

and all $0 < s < 1$.

Proof. We consider the identity $I : X_0 \hookrightarrow H_0$ which is also well-defined as $I : X_1 \hookrightarrow H_1$. With the interpolation spaces $X_s = [X_0, X_1]_s$ and $H_s = [H_0, H_1]_s$, the interpolation theorem (14) yields $X_s \subseteq H_s$ and $\|I : X_s \to H_s\| \le 1$, which proves the lower estimate in (15). The upper bound follows from $\|P : H_s \to X_s\| \le c_0^{1-s} c_1^s$ and the projection property $v = Pv$ for all $v \in X_0 \supseteq X_s$. $\square$

3.2. Scott-Zhang projection onto H^s and $\widetilde{H}^s$. In this short section, we generalize the Scott-Zhang operator J_ℓ from [SZ90]. There, J_ℓ is defined on the space $H^s(\Gamma)$ with $s > 1/2$, which is sufficient regularity to define a trace operator and thus to incorporate Dirichlet boundary conditions. In this work, we deal with energy spaces $\widetilde{H}^{1/2}(\Gamma)$ and hence must not use the classical construction from [SZ90] directly. However, it is possible to extend the existing results in two steps. First, one can extend the results from [SZ90] to the space $H^{1/2}(\Gamma)$. For the closed boundary $\Gamma = \partial\Omega$, this is precisely the energy space. In contrast, the energy space in case of an open screen $\Gamma \subsetneqq \partial\Omega$ is $\widetilde{H}^{1/2}(\Gamma)$. Although this space still does not allow for a trace operator, we need to construct a stable projection with range $\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell) = \mathcal{S}^p(\mathcal{T}_\ell) \cap \widetilde{H}^{1/2}(\Gamma)$, which thus incorporates zero boundary conditions.

Lemma 3 (Scott-Zhang projection onto $\mathcal{S}^p(\mathcal{T}_\ell) \subset H^s(\Gamma)$)**.** *There exists a linear projection $J_\ell : L^2(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell)$ such that for all $0 \le s \le 1$*

$$J_\ell V_\ell = V_\ell \quad \text{as well as} \quad \|J_\ell v\|_{H^s(\Gamma)} \le C_{\rm stab}(s)\, \|v\|_{H^s(\Gamma)} \text{ for all } V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell), v \in H^s(\Gamma). \tag{16}$$

For $v \in H^1(\Gamma)$, it holds that

$$\|(1 - J_\ell)v\|_{L^2(T)} \le C_{\rm sz}\, h_\ell(T) \|\nabla v\|_{L^2(\omega_\ell(T))} \text{ and } \|\nabla(1 - J_\ell)v\|_{L^2(T)} \le C_{\rm sz}\, \|\nabla v\|_{L^2(\omega_\ell(T))}. \tag{17}$$

The constant $C_{\rm sz} > 0$ depends only on γ-shape regularity of $\mathcal{T}_\ell$ and p, while $C_{\rm stab}(s) > 0$ additionally depends on Γ and s.

Sketch of proof. In [SZ90], the proof is carried out in the following way: If $\{a_i\}$ is the collection of degrees of freedom for the space $\mathcal{S}^p(\mathcal{T}_\ell)$, one chooses either an element $\sigma_i = T_i$ (in case a_i lies inside T_i), or an edge $\sigma_i = E_i$ (in case $a_i \in E_i$). The nodal value of $(J_\ell v)(a_i)$ is determined by the values $v|_{\sigma_i}$ of v on σ_i. In particular, this definition requires the validity of a trace theorem which fails for the energy case $s = 1/2$. We may extend the result if we always choose an element $\sigma_i = T_i$, regardless of the type of a_i, as long as $a_i \in T_i$. The remaining construction can be carried out as in [SZ90] and, in particular, does not need a trace theorem. Arguing along the lines of [SZ90], one sees that J_ℓ is stable in $L^2(\Gamma)$ as well as in $H^1(\Gamma)$, i.e., (16) holds for $s = 0, 1$, and J_ℓ satisfies (17). By Lemma 1, J_ℓ is also stable in $H^s(\Gamma)$. $\square$

Lemma 4 (Scott-Zhang projection onto $\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell) \subset \widetilde{H}^s(\Gamma)$)**.** *For $\Gamma = \partial\Omega$ resp. $\Gamma \subseteq \mathbb{R}^2$, there exists a linear projection $J_\ell : L^2(\Gamma) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ such that for all $0 \le s \le 1$*

$$J_\ell V_\ell = V_\ell \quad \text{as well as} \quad \|J_\ell v\|_{\widetilde{H}^s(\Gamma)} \le C_{\rm stab}(s)\, \|v\|_{\widetilde{H}^s(\Gamma)} \text{ for all } V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell), v \in \widetilde{H}^s(\Gamma). \tag{18}$$

For $v \in \widetilde{H}^1(\Gamma)$, it holds that

$$\|(1-J_\ell)v\|_{L^2(T)} \le C_{\rm sz}\, h_\ell(T)\|\nabla v\|_{L^2(\omega_\ell(T))} \text{ and } \|\nabla(1-J_\ell)v\|_{L^2(T)} \le C_{\rm sz}\, \|\nabla v\|_{L^2(\omega_\ell(T))}. \tag{19}$$

The constant $C_{\rm sz} > 0$ depends only on γ-shape regularity of $\mathcal{T}_\ell$ and p, while $C_{\rm stab}(s) > 0$ additionally depends on Γ and s.

Proof for $\Gamma = \partial\Omega$. The claim follows immediately from Lemma 3 as $\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell) = \mathcal{S}^p(\mathcal{T}_\ell)$. □

Sketch of proof for $\Gamma \subsetneqq \partial\Omega$. First, we follow the construction in [SZ90], i.e., if $a_i \in \partial\Gamma$, we choose an edge $\sigma_i := E_i \subset \partial\Gamma$ with $a_i \in E_i$ to construct a projection $\widetilde{J}_\ell : H^1(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell)$. The arguments in [SZ90] show that $\widetilde{J}_\ell$ is stable in $H^1(\Gamma)$ and that (19) holds for $\widetilde{J}_\ell$ and all $v \in H^1(\Gamma)$. Second, let J_ℓ be the operator that is constructed analogously to $\widetilde{J}_\ell$, but where we only consider all nodal points $\{a_i\}$ which are *not* on the boundary $\partial\Gamma$. Arguing as in [SZ90], this operator is stable in $L^2(\Gamma)$. The crucial observation is that $J_\ell v = \widetilde{J}_\ell v$ for all $v \in \widetilde{H}^1(\Gamma)$. In particular, J_ℓ satisfies (19) for $v \in \widetilde{H}^1(\Gamma)$ and is stable in $\widetilde{H}^1(\Gamma)$. By Lemma 1, J_ℓ is also stable in $\widetilde{H}^s(\Gamma)$. □

3.3. Inverse estimate in $\boldsymbol{H^s}$. In this section, we prove a local inverse estimate in the H^s-norm. In 2D, such an estimate is also found in [CP07b, Proposition 3.1], and it is generalized to 3D in the following. Moreover, the proof of [CP07b] has a minor gap which is now closed by means of Lemma 2.

Proposition 5. *For all $0 \le s \le 1$, it holds*

$$\|h_\ell^{1-s}\nabla V_\ell\|_{L^2(\Gamma)} \le C_{\rm inv}\|V_\ell\|_{H^s(\Gamma)} \quad \text{for all } V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell). \tag{20}$$

The constant $C_{\rm inv} > 0$ depends only on Γ, p, s, and γ-shape regularity of $\mathcal{T}_\ell$.

Proof. An $\mathcal{T}_\ell$-elementwise scaling argument provides the estimate

$$\|\nabla V_\ell\|_{L^2(h_\ell,\Gamma)} := \|h_\ell \nabla V_\ell\|_{L^2(\Gamma)} \lesssim \|V_\ell\|_{L^2(\Gamma)} \quad \text{for all } V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell),$$

where the hidden constant depends only on γ-shape regularity of $\mathcal{T}_\ell$ and on the polynomial degree p. Together with the trivial estimate $\|\nabla V_\ell\|_{L^2(\Gamma)} \le \|V_\ell\|_{H^1(\Gamma)}$, we see that the gradient is a continuous linear operator $\nabla : X_j \to Y_j$, for $j = 0, 1$, where

$$\begin{aligned} X_0 &= (\mathcal{S}^p(\mathcal{T}_\ell), \|\cdot\|_{L^2(\Gamma)}), & Y_0 &= (L^2(\Gamma), \|\cdot\|_{L^2(h_\ell,\Gamma)}), \\ X_1 &= (\mathcal{S}^p(\mathcal{T}_\ell), \|\cdot\|_{H^1(\Gamma)}), & Y_1 &= (L^2(\Gamma), \|\cdot\|_{L^2(\Gamma)}). \end{aligned}$$

Therefore, the interpolation theorem (14) implies continuity $\nabla : X_s \to Y_s$, where $X_s = [X_0, X_1]_s$ and $Y_s = [Y_0, Y_1]_s$, and the operator norm depends only on γ-shape regularity of $\mathcal{T}_\ell$, p, and s, i.e., $\|\nabla V_\ell\|_{Y_s} \lesssim \|V_\ell\|_{X_s}$. It remains to characterize the interpolation norms involved (which is neglected in the proof of [CP07b, Proposition 3.1]).

First, it is a standard result on interpolation of weighted L^2-spaces that $\|\psi\|_{[L^2(h_\ell,\Gamma),L^2(\Gamma)]_s} \simeq \|h_\ell^{1-s}\psi\|_{L^2(\Gamma)}$ for all $\psi \in L^2(\Gamma)$, see [Tar07, Chapter 23], and norm equivalence constants depend only on s and Γ.

Second, let $P = J_\ell$ be the Scott-Zhang projection onto $\mathcal{S}^p(\mathcal{T}_\ell) \subset H^1(\Gamma)$ from Lemma 3. Then, Lemma 2 yields $\|V_\ell\|_{X_s} \simeq \|V_\ell\|_{[L^2(\Gamma),H^1(\Gamma)]_s} = \|V_\ell\|_{H^s(\Gamma)}$, and the norm equivalence constants depend only on Γ, p, s, and γ-shape regularity of $\mathcal{T}_\ell$.

Altogether, we thus conclude

$$\|h_\ell^{1-s}\nabla V_\ell\|_{L^2(\Gamma)} \simeq \|\nabla V_\ell\|_{Y_s} \lesssim \|V_\ell\|_{X_s} \simeq \|V_\ell\|_{H^s(\Gamma)} \quad \text{for all } V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell),$$

and the hidden constants depend only on Γ, p, s, and γ-shape regularity of $\mathcal{T}_\ell$. $\square$

An inverse estimate in $\widetilde{H}^s(\Gamma)$ follows from Proposition 5 and the continuous inclusion $\widetilde{H}^s(\Gamma) \subseteq H^s(\Gamma)$.

Corollary 6. *For all $0 \le s \le 1$, it holds*

$$\|h_\ell^{1-s}\nabla V_\ell\|_{L^2(\Gamma)} \le C_{\rm inv}\|V_\ell\|_{\widetilde{H}^s(\Gamma)} \quad \text{for all } V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell). \tag{21}$$

The constant $C_{\rm inv} > 0$ depends only on Γ, p, s, and γ-shape regularity of $\mathcal{T}_\ell$. $\square$

3.4. Quasi-interpolation and localization (13) of energy norm. The lower bound in (13a) is an approximation estimate. We first recall that stable projections exhibit an appropriate first-order approximation property with respect to the $\widetilde{H}^s(\Gamma)$-norm. The following lemma slightly improves a result from [KOP13, Theorem 4]. Since the proof in [KOP13] is only stated for the lowest-order case $p = 1$ and since the proof's arguments are also used below (see Lemma 10), we sketch the proof for the convenience of the reader.

Lemma 7. *Each $\widetilde{H}^s(\Gamma)$-stable projection $P_\ell : \widetilde{H}^s(\Gamma) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ onto $\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$, i.e.,*

$$P_\ell V_\ell = V_\ell \quad \text{and} \quad \|P_\ell v\|_{\widetilde{H}^s(\Gamma)} \le C_{\rm stab}\,\|v\|_{\widetilde{H}^s(\Gamma)} \quad \text{for all } V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell) \text{ and } v \in \widetilde{H}^s(\Gamma) \tag{22}$$

satisfies

$$\|(1-P_\ell)v\|_{\widetilde{H}^s(\Gamma)} \le C_{\rm apx} \min_{V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)} \|h_\ell^{1-s}\nabla(v - V_\ell)\|_{L^2(\Gamma)} \quad \text{for } v \in \widetilde{H}^1(\Gamma). \tag{23}$$

The constant $C_{\rm apx} > 0$ depends only on $C_{\rm stab} > 0$, Γ, p, s, and γ-shape regularity of $\mathcal{T}_\ell$. For $\Gamma \subsetneqq \partial\Omega$, the constant $C_{\rm apx}$ additionally depends on the shapes of element patches.

Proof. With the Scott-Zhang projection $J_\ell : \widetilde{H}^s(\Gamma) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ from Lemma 4, it holds that

$$\|v - P_\ell v\|_{\widetilde{H}^s(\Gamma)} = \|(1-P_\ell)(v - J_\ell v)\|_{\widetilde{H}^s(\Gamma)} \lesssim \|v - J_\ell v\|_{\widetilde{H}^s(\Gamma)}.$$

Without loss of generality, we may therefore consider $P_\ell = J_\ell$ only.

Let $\widetilde{H}^1(h_\ell, \Gamma)$ denote the space $\widetilde{H}^1(\Gamma)$ which is now associated with the weighted H^1-norm

$$\|v\|^2_{\widetilde{H}^1(h_\ell,\Gamma)} := \|h_\ell v\|^2_{L^2(\Gamma)} + \|h_\ell \nabla v\|^2_{L^2(\Gamma)} \quad \text{for } v \in \widetilde{H}^1(\Gamma).$$

With this notation, (19) and γ-shape regularity imply for all $v \in \widetilde{H}^1(\Gamma)$

$$\|(1-J_\ell)v\|_{L^2(\Gamma)} \lesssim \|v\|_{\widetilde{H}^1(h_\ell,\Gamma)} \quad \text{and} \quad \|(1-J_\ell)v\|_{\widetilde{H}^1(\Gamma)} \lesssim \|v\|_{\widetilde{H}^1(\Gamma)} \tag{24}$$

These estimates state continuity of the operator $1 - J_\ell : X_j \to Y_j$, for $j = 0, 1$, where

$$X_0 = (\widetilde{H}^1(\Gamma), \|\cdot\|_{\widetilde{H}^1(h_\ell,\Gamma)}), \qquad Y_0 = (L^2(\Gamma), \|\cdot\|_{L^2(\Gamma)}),$$
$$X_1 = (\widetilde{H}^1(\Gamma), \|\cdot\|_{\widetilde{H}^1(\Gamma)}), \qquad Y_1 = (\widetilde{H}^1(\Gamma), \|\cdot\|_{\widetilde{H}^1(\Gamma)}).$$

The operator norms $c_j = \|1 - J_\ell : X_j \to Y_j\|$ depend only on Γ, on p, and γ-shape regularity of $\mathcal{T}_\ell$. The interpolation theorem (14) yields continuity of $1 - J_\ell : X_s \to Y_s$, where $X_s = [\widetilde{H}^1(h_\ell, \Gamma), \widetilde{H}^1(\Gamma)]_s$ and $Y_s = [L^2(\Gamma), \widetilde{H}^1(\Gamma)]_s = \widetilde{H}^s(\Gamma)$. To proceed, we need the following estimate for the interpolation norm on $X_s = [\widetilde{H}^1(h_\ell,\Gamma), \widetilde{H}^1(\Gamma)]_s$ which is proved in [KOP13, Lemmas 1 and 5] in the case of $H^1(\Gamma)$, but the proof transfers verbatim to $\widetilde{H}^1(\Gamma)$:

$$\|v\|_{X_s} = \|v\|_{[\widetilde{H}^1(h_\ell,\Gamma),\widetilde{H}^1(\Gamma)]_s} \lesssim \|h_\ell^{1-s}\nabla v\|_{L^2(\Gamma)} + \|h_\ell^{-s} v\|_{L^2(\Gamma)} \quad \text{for all } v \in \widetilde{H}^1(\Gamma). \tag{25}$$

Here and in the following, the hidden constant depends only on Γ, p, s, and γ-shape regularity of $\mathcal{T}_\ell$. Altogether, we have now proved

$$\|(1-J_\ell)v\|_{\widetilde{H}^s(\Gamma)} \lesssim \|v\|_{X_s} \lesssim \|h_\ell^{1-s}\nabla v\|_{L^2(\Gamma)} + \|h_\ell^{-s} v\|_{L^2(\Gamma)} \quad \text{for all } v \in \widetilde{H}^1(\Gamma).$$

We use the projection property $(1-J_\ell) = (1-J_\ell)^2$ and the local estimates (19) to improve the last estimate to

$$\|(1-J_\ell)v\|_{\widetilde{H}^s(\Gamma)} \lesssim \|h_\ell^{1-s}\nabla(1-J_\ell)v\|_{L^2(\Gamma)} + \|h_\ell^{-s}(1-J_\ell)v\|_{L^2(\Gamma)} \lesssim \|h_\ell^{1-s}\nabla v\|_{L^2(\Gamma)}$$

for all $v \in \widetilde{H}^1(\Gamma)$. Finally, we use the projection property of J_ℓ once more to improve the last estimate to

$$\|(1-J_\ell)v\|_{\widetilde{H}^s(\Gamma)} = \min_{V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)} \|(1-J_\ell)(v-V_\ell)\|_{\widetilde{H}^s(\Gamma)} \lesssim \min_{V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)} \|h_\ell^{1-s}\nabla(v-V_\ell)\|_{L^2(\Gamma)},$$

where the minima are attained due to finite dimension. For $\Gamma = \partial\Omega$, it holds $\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell) = \mathcal{S}^p(\mathcal{T}_\ell)$ which proves (13a). For $\Gamma \subsetneqq \partial\Omega$, the argument is more involved. Let $\widetilde{J}_\ell$ denote the Scott-Zhang projection which was employed in the proof of Lemma 4, and let Π_ℓ denote the L^2-projection onto $\mathcal{P}^{p-1}(\mathcal{T}_\ell)$. In [AFK+13, Proposition 8], it is proved that, for $v \in H^1(\Gamma)$, it holds

$$\|(1-\Pi_\ell)\nabla v\|_{L^2(T)} \le \|\nabla(1-\widetilde{J}_\ell)v\|_{L^2(T)} \lesssim \|(1-\Pi_\ell)\nabla v\|_{L^2(\omega_\ell(T))} \quad \text{for all } T \in \mathcal{T}_\ell. \tag{26}$$

The hidden constants depend only p, γ-shape regularity of $\mathcal{T}_\ell$, and the shapes of the element patches. As $J_\ell v = \widetilde{J}_\ell v$ for $v \in \widetilde{H}^1(\Gamma)$, the estimates (26) remain true for J_ℓ and $v \in \widetilde{H}^1(\Gamma)$. With the techniques used already before, we see, for all $v \in \widetilde{H}^1(\Gamma)$,

$$\begin{aligned}\|(1-J_\ell)v\|_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}\nabla(1-J_\ell)v\|_{L^2(\Gamma)} &\simeq \|(1-\Pi_\ell)\nabla v\|_{L^2(\Gamma)} \\ &\le \min_{V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)} \|h_\ell^{1/2}\nabla(v-V_\ell)\|_{L^2(\Gamma)}.\end{aligned}$$

This concludes the proof. $\square$

Remark 8. ***(i)*** *Lemma 7 holds accordingly for $H^s(\Gamma)$-stable projections onto $\mathcal{S}^p(\mathcal{T}_\ell)$.*
(ii) *For $d = 3$ and newest vertex bisection, only finitely many shapes of triangles and hence element patches occur, see e.g.* [Ste08b]. *Consequently, $C_{\rm apx} > 0$ remains uniformly bounded for $\Gamma \subsetneqq \partial\Omega$. The same holds for $d = 2$ and the bisection algorithm from* [AFF+13].
(iii) *In either case $\Gamma \subsetneqq \partial\Omega$ and $\Gamma = \partial\Omega$, the arguments from the proof of Lemma 7 yield*

$$\|(1-J_\ell)v\|_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \min_{V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)} \|h_\ell^{1/2}\nabla(v-V_\ell)\|_{L^2(\Gamma)} \simeq \|h_\ell^{1/2}(1-\Pi_\ell)\nabla v\|_{L^2(\Gamma)} \tag{27}$$

for all $v \in \widetilde{H}^1(\Gamma)$ and $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^{p-1}(\mathcal{T}_\ell)$ the L^2-orthogonal projection. The hidden constants depend only on Γ, p, s, and γ-shape regularity of $\mathcal{T}_\ell$ as well as on the shapes of element patches.
(iv) *For $\widehat{\mathcal{T}}_\ell$ being the uniform refinement of $\mathcal{T}_\ell$ and $v = \widehat{V}_\ell \in \mathcal{S}^q(\widehat{\mathcal{T}}_\ell)$, one can prove that in either case $\Gamma = \partial\Omega$ and $\Gamma \subsetneqq \partial\Omega$ the constant $C_{\rm apx}$ in* (23) *does* not *depend on the shapes of element patches, see the scaling arguments in the proof of Lemma 10 below.* $\square$

Combining the inverse estimate from Corollary 6 with the approximation estimate of Lemma 7, we are in the position to prove the localization estimate (13).

Proposition 9. *Let $P_\ell \in \{\mathbb{G}_\ell, J_\ell\}$ denote either the Galerkin projection $\mathbb{G}_\ell : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ or the Scott-Zhang projection $J_\ell : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ from Lemma 4. Then, P_ℓ satisfies the localization estimate* (13). *Up to norm equivalence, the constant $C_{\rm apx} > 0$ coincides with that of Lemma 7. The constant $C_{\rm inv} > 0$ depends only on Γ, p, s, and γ-shape regularity of $\mathcal{T}_\ell$.*

Proof of (13a). From the best approximation property of the Galerkin projection $\mathbb{G}_\ell$ and norm equivalence, we infer

$$|||(1-\mathbb{G}_\ell)v||| \le |||(1-J_\ell)v||| \simeq \|(1-J_\ell)v\|_{\widetilde{H}^{1/2}(\Gamma)}$$

Therefore, Lemma 7 for $P_\ell = J_\ell$ and $s = 1/2$ concludes the proof. $\square$

Proof of (13b). Let $\widehat{V}_\ell \in \widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell)$ and $V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$. We employ the projection property of J_ℓ and the local stability (19) to see

$$\|h_\ell^{1/2}\nabla(1-J_\ell)\widehat{V}_\ell\|_{L^2(\Gamma)} = \|h_\ell^{1/2}\nabla(1-J_\ell)(1-\mathbb{G}_\ell)\widehat{V}_\ell\|_{L^2(\Gamma)} \lesssim \|h_\ell^{1/2}\nabla(1-\mathbb{G}_\ell)\widehat{V}_\ell\|_{L^2(\Gamma)}.$$

With $(1-\mathbb{G}_\ell)\widehat{V}_\ell \in \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell)$ and $\widehat{h}_\ell = h_\ell/2$ the inverse estimate from Corollary 6 gives

$$\|h_\ell^{1/2}\nabla(1-\mathbb{G}_\ell)\widehat{V}_\ell\|_{L^2(\Gamma)} \lesssim \|(1-\mathbb{G}_\ell)\widehat{V}_\ell\|_{\widetilde{H}^{1/2}(\Gamma)}.$$

Next, the projection property of $\mathbb{G}_\ell$ yields

$$\|(1-\mathbb{G}_\ell)\widehat{V}_\ell\|_{\widetilde{H}^{1/2}(\Gamma)} = \|(1-\mathbb{G}_\ell)(\widehat{V}_\ell - V_\ell)\|_{\widetilde{H}^{1/2}(\Gamma)}.$$

Norm equivalence on $\widetilde{H}^{1/2}(\Gamma)$ and the best approximation property of $\mathbb{G}_\ell$ yield

$$\|(1-\mathbb{G}_\ell)(\widehat{V}_\ell - V_\ell)\|_{\widetilde{H}^{1/2}(\Gamma)} \simeq |||(1-\mathbb{G}_\ell)(\widehat{V}_\ell - V_\ell)||| \le |||\widehat{V}_\ell - V_\ell|||.$$

Combining the last four estimates and taking the infimum over all V_ℓ, we prove (13b). Due to finite dimension, the infimum is attained. $\square$

3.5. Nodal interpolation and localization (13) of energy norm. By I_ℓ, we denote the Lagrange interpolation operator $I_\ell : C(\overline{\Gamma}) \to \mathcal{S}^p(\mathcal{T}_\ell)$, which is defined by

$$I_\ell v = \sum_{i=1}^N v(a_i)\varphi_i, \tag{28}$$

where $\{a_i\}_{i=1}^N$ are the degrees of freedom of $\mathcal{S}^p(\mathcal{T}_\ell)$ with corresponding Lagrange basis functions $\{\varphi_i\}_{i=1}^N$, i.e., $\varphi_i(a_j) = \delta_{ij}$, for all $i, j = 1, \dots, N$ with Kronecker's delta. In particular, this yields the projection property $I_\ell V_\ell = V_\ell$ for all $V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)$. Moreover, $v|_{\partial\Gamma} = 0$ implies $I_\ell v \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$, i.e., possible zero boundary conditions are preserved.

Since functions in $\widetilde{H}^1(\Gamma)$ are *not* continuous in general for $d = 3$, an approximation result for I_ℓ analogous to Lemma 7 can, in fact, only hold for discrete functions.

Lemma 10. *Let $\widehat{\mathcal{T}}_\ell$ be the uniform refinement of $\mathcal{T}_\ell$. For $q \geq p$ and $0 \leq s \leq 1$, nodal interpolation $I_\ell : \widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ satisfies*

$$\|(1 - I_\ell)\widehat{V}_\ell\|_{\widetilde{H}^s(\Gamma)} \leq C_{\rm apx} \min_{V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)} \|h_\ell^{1-s}\nabla(\widehat{V}_\ell - V_\ell)\|_{L^2(\partial\Omega)} \quad \textit{for all } \widehat{V}_\ell \in \widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell). \tag{29}$$

The constant $C_{\rm apx} > 0$ depends only on Γ, p, q, s, and γ-shape regularity of $\mathcal{T}_\ell$.

Proof. Let $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^{p-1}(\mathcal{T}_\ell)$ be the $L^2(\Gamma)$-orthogonal projection. For $\widehat{V}_\ell \in \widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell)$ and $T \in \mathcal{T}_\ell$, scaling arguments prove

$$\|(1 - I_\ell)\widehat{V}_\ell\|_{L^2(T)} \lesssim h_\ell(T)\|\nabla \widehat{V}_\ell\|_{L^2(T)} \text{ and } \|\nabla(1 - I_\ell)\widehat{V}_\ell\|_{L^2(T)} \lesssim \|(1 - \Pi_\ell)\nabla\widehat{V}_\ell\|_{L^2(T)}, \tag{30}$$

where the hidden constants depend only on q, p and γ-shape regularity of $\mathcal{T}_\ell$. For the second estimate, note that semi-norms on finite dimensional spaces are equivalent if and only if their kernels coincide. With the notation from the proof of Lemma 7, (30) implies

$$\|(1 - I_\ell)\widehat{V}_\ell\|_{L^2(\Gamma)} \lesssim \|\widehat{V}_\ell\|_{\widetilde{H}^1(h_\ell,\Gamma)} \quad \text{and} \quad \|(1 - I_\ell)\widehat{V}_\ell\|_{\widetilde{H}^1(\Gamma)} \lesssim \|\widehat{V}_\ell\|_{\widetilde{H}^1(\Gamma)}.$$

These estimates state continuity of the operator $1 - I_\ell : X_j \to Y_j$, for $j = 0, 1$, where

$$X_0 = (\widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell), \|\cdot\|_{\widetilde{H}^1(h_\ell,\Gamma)}), \qquad Y_0 = (\widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell), \|\cdot\|_{L^2(\Gamma)}),$$
$$X_1 = (\widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell), \|\cdot\|_{\widetilde{H}^1(\Gamma)}), \qquad Y_1 = (\widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell), \|\cdot\|_{\widetilde{H}^1(\Gamma)}).$$

The operator norms $c_j = \|1 - I_\ell : X_j \to Y_j\|$ depend only on q, p, and γ-shape regularity of $\mathcal{T}_\ell$. The interpolation theorem (14) yields continuity of $1 - I_\ell : X_s \to Y_s$, where $X_s = [X_0, X_1]_s$ and $Y_s = [Y_0, Y_1]_s$. Arguing as in the proof of Proposition 5 to identify the discrete interpolation norms, we see, for all $\widehat{V}_\ell \in \widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell)$,

$$\|(1 - I_\ell)\widehat{V}_\ell\|_{\widetilde{H}^s(\Gamma)} \simeq \|(1 - I_\ell)\widehat{V}_\ell\|_{Y_s} \lesssim \|\widehat{V}_\ell\|_{X_s} \simeq \|\widehat{V}_\ell\|_{[\widetilde{H}^1(h_\ell,\Gamma),\widetilde{H}^1(\Gamma)]_s}.$$

From now on, we may follow the lines of the proof of Lemma 7 to conclude the proof. To that end, note that (30) provides the necessary counterpart to (19) and, in particular, improves (26) in the sense that element patches are avoided. □

For $v \in \widetilde{H}^{1/2}(\Gamma) \cap C(\overline{\Gamma})$, nodal interpolation guarantees $I_\ell v \in \widetilde{H}^{1/2}(\Gamma) \cap \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$. With the approximation property of Lemma 10 and the local estimates (30), one may therefore follow the proof of Proposition 9 verbatim to obtain the corresponding result for nodal interpolation. The details are left to the reader.

Proposition 11. *Let $\widehat{\mathcal{T}}_\ell$ be the uniform refinement of $\mathcal{T}_\ell$ and $q \geq p \geq 1$. Let $P_\ell \in \{\mathbb{G}_\ell, I_\ell, J_\ell\}$ denote either the Galerkin projection $\mathbb{G}_\ell$, or the Lagrange interpolation operator I_ℓ, or the Scott-Zhang projection J_ℓ from Lemma 4. Then, P_ℓ satisfies the localization estimate* (13). *The constants $C_{\rm apx}, C_{\rm inv} > 0$ depend only on Γ, p, q, and γ-shape regularity of $\mathcal{T}_\ell$.* □

4. A posteriori error estimation

Throughout this section, let $U_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ denote the Galerkin solution of (9) with respect to $\mathcal{T}_\ell$, and let $\widehat{U}_\ell \in \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell)$ denote the Galerkin solution with respect to the uniform refinement $\widehat{\mathcal{T}}_\ell$ of $\mathcal{T}_\ell$.

4.1. $(h - h/2)$-type error estimators. Suppose that $P_\ell : \widetilde{\mathcal{S}}^q(\widehat{\mathcal{T}}_\ell) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ is a linear projection onto $\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ which satisfies (13). We consider the following four error estimators

$$
\begin{aligned}
\eta_\ell &:= |||\widehat{U}_\ell - U_\ell||| & \mu_\ell &:= \|h_\ell^{1/2}\nabla(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)} \\
\widetilde{\eta}_\ell &:= |||(1 - P_\ell)\widehat{U}_\ell||| & \widetilde{\mu}_\ell &:= \|h_\ell^{1/2}\nabla(1 - P_\ell)\widehat{U}_\ell\|_{L^2(\Gamma)}
\end{aligned}
\tag{31}
$$

The estimators η_ℓ and $\widetilde{\eta}_\ell$ serve for error estimation only, whereas the local contributions of μ_ℓ and $\widetilde{\mu}_\ell$ can also serve as refinement indicators in the adaptive mesh-refining algorithm of Section 5. From a computational point of view, the estimators $\widetilde{\eta}_\ell$ and $\widetilde{\mu}_\ell$ are more attractive, since they do not require the computation of the coarse mesh-solution U_ℓ.

Although $(h - h/2)$-type estimators are conceptually simple, they have firstly been proposed by [FLP08] for weakly-singular integral equations. Their analysis has been transferred to hypersingular integral equations with $d = 2$ and $p = 1$ in [EFGP13]. The following theorem generalizes the latter result to $d = 2, 3$ and $p \geq 1$.

Theorem 12. *With the constants $C_{\mathrm{apx}}, C_{\mathrm{inv}} > 0$ from* (13)*, it holds*

$$
\eta_\ell \leq \widetilde{\eta}_\ell \leq C_{\mathrm{apx}}\,\widetilde{\mu}_\ell \quad \text{and} \quad \max\{\mu_\ell, \widetilde{\mu}_\ell\} \leq C_{\mathrm{inv}}\,\eta_\ell. \tag{32}
$$

While efficiency

$$
\eta_\ell \leq |||u - U_\ell||| \tag{33}
$$

is always satisfied with constant 1*, reliability*

$$
|||u - U_\ell||| \leq C_{\mathrm{rel}}\,\eta_\ell \tag{34}
$$

with some constant $C_{\mathrm{rel}} > 0$ is equivalent to the saturation assumption

$$
|||u - \widehat{U}_\ell||| \leq q_{\mathrm{sat}}\,|||u - U_\ell||| \tag{35}
$$

with some constant $0 < q_{\mathrm{sat}} < 1$.

Proof. Note that the definition of the Galerkin projection (10) ensures $\mathbb{G}_\ell\widehat{U}_\ell = U_\ell$. Therefore, the best approximation property of $\mathbb{G}_\ell$ and (13a) yield

$$
\eta_\ell = |||(1 - \mathbb{G}_\ell)\widehat{U}_\ell||| \leq \widetilde{\eta}_\ell \leq C_{\mathrm{apx}}\,\widetilde{\mu}_\ell
$$

On the other hand, the inverse estimate (13b) yields

$$
\widetilde{\mu}_\ell = \|h_\ell^{1/2}\nabla(1 - P_\ell)\widehat{U}_\ell\|_{L^2(\Gamma)} \leq C_{\mathrm{inv}} \min_{V_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)} |||\widehat{U}_\ell - V_\ell||| = C_{\mathrm{inv}}\,\eta_\ell,
$$

where the last equality follows from the best approximation property of $\mathbb{G}_\ell$. Since $\mathbb{G}_\ell$ also satisfies (13), the same argument also proves $\mu_\ell \leq C_{\mathrm{inv}}\,\eta_\ell$.

The remaining claims (33)–(35) follow from the Pythagoras theorem

$$
|||u - U_\ell|||^2 = |||u - \widehat{U}_\ell|||^2 + |||\widehat{U}_\ell - U_\ell|||^2 = |||u - \widehat{U}_\ell|||^2 + \eta_\ell^2
$$

and reveal the identity $q_{\mathrm{sat}} = (1 - C_{\mathrm{rel}}^2)^{1/2}$ resp. $C_{\mathrm{rel}} = (1 - q_{\mathrm{sat}})^{-1/2}$. □

Remark 13. *The saturation assumption* (35) *is essentially equivalent to the fact that the numerical scheme has reached an asymptotic regime* [FLP08, Section 5.2]*. It can be proved for FEM model problems* [DN02, FLOP10] *if the given data are sufficiently resolved. For smooth data and $\widehat{\mathcal{T}}_\ell$ being the k-times uniform refinement of $\mathcal{T}_\ell$, it is proved in* [AFF+13, Appendix] *for weakly-singular integral equations in 2D, where k depends only on Γ.* □

4.2. Two-level error estimators. In this subsection, we first recall the two-level error estimator from [MS00, HMS01, Heu02]. While the analysis of [MS00] is restricted to uniform meshes and lowest-order elements, the analysis of [HMS01, Heu02] already works for hp-refinement on γ-shape regular meshes. We consider only h-refinement and provide, in this frame, an alternate proof to those of [MS00, HMS01, Heu02]. In contrast to and generalization of [HMS01, Heu02], our proof avoids any strong assumptions on the two-level basis functions.

Let $\widehat{\mathcal{T}}_\ell$ be the uniform refinement of $\mathcal{T}_\ell$. Suppose that $\{\phi_1, \dots, \phi_n\}$ is a basis of $\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ and $\{\phi_1, \dots, \phi_n, \varphi_1, \dots, \varphi_N\}$ is a basis of $\widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell)$. We suppose that $\varphi_j(z) = 0$ for all vertices $z \in \mathcal{N}_\ell$ and $j = 1, \dots, N$ and that the number of overlapping supports $\operatorname{supp}(\varphi_j)$ is uniformly bounded in terms of the γ-shape regularity of $\mathcal{T}_\ell$. To employ scaling arguments in the proof of Lemma 15 below, we assume that there are only finitely many reference functions $\varphi_k^{\rm ref}$ such that $\varphi_j \circ F_T \in \{0, \varphi_1^{\rm ref}, \dots, \varphi_M^{\rm ref}\}$ for all $j = 1, \dots, n$. We remark that these assumptions are met for Lagrange bases, but also satisfied for, e.g., the usual basis of $\mathcal{S}^2(\widehat{\mathcal{T}}_\ell)$ which consists of hat functions plus edge bubble functions.

Under these assumptions, the two-level error estimator reads

$$\tau_\ell = \Big(\sum_{j=1}^N \frac{|\langle f - WU_\ell\,,\, \varphi_j \rangle_{L^2(\Gamma)}|^2}{|||\varphi_j|||^2} \Big)^{1/2}. \tag{36}$$

The local contributions $\tau_\ell(T)$, where the sum in (36) is taken only over the indices j with $|T \cap \operatorname{supp}(\varphi_j)| > 0$, can serve as refinement indicators for the adaptive algorithm of Section 5.

Theorem 14. *The two-level error estimator τ_ℓ satisfies*

$$C_1^{-1} \eta_\ell \le \tau_\ell \le C_2 \eta_\ell. \tag{37}$$

The constants $C_1, C_2 > 0$ depend only on Γ, p, and γ-shape regularity of $\mathcal{T}_\ell$. In particular, τ_ℓ is always efficient with constant C_2, while reliability is equivalent to the saturation assumption (35).

To prove the preceding theorem, we need two auxiliary results.

Lemma 15. *For $\widehat{V}_{\ell,j} \in \mathcal{S}^p_{\ell,j} := \operatorname{span}\{\varphi_j\} \subset \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell)$ holds*

$$C_3^{-1} \Big(\sum_{j=1}^N |||\widehat{V}_{\ell,j}|||^2 \Big)^{1/2} \le \Big|\Big|\Big| \sum_{j=1}^N \widehat{V}_{\ell,j} \Big|\Big|\Big| \le C_4 \Big(\sum_{j=1}^N |||\widehat{V}_{\ell,j}|||^2 \Big)^{1/2} \tag{38}$$

The constants $C_3, C_4 > 0$ depend only on Γ, p, and γ-shape regularity of $\mathcal{T}_\ell$.

Proof. Let $I_\ell : C(\overline{\Gamma}) \to \mathcal{S}^1(\mathcal{T}_\ell)$ denote the nodal interpolation operator onto the lowest-order BEM space $\mathcal{S}^1(\mathcal{T}_\ell)$. Note that $\widehat{V}_{\ell,j} = (1 - I_\ell)\widehat{V}_{\ell,j}$ by assumption on φ_j. Therefore, the localization estimate (13) yields

$$|||\widehat{V}_{\ell,j}||| \simeq \|h_\ell^{1/2} \nabla \widehat{V}_{\ell,j}\|_{L^2(\Gamma)} \quad \text{as well as} \quad \Big|\Big|\Big| \sum_{j=1}^N \widehat{V}_{\ell,j} \Big|\Big|\Big| \simeq \Big\| h_\ell^{1/2} \nabla \sum_{j=1}^N \widehat{V}_{\ell,j} \Big\|_{L^2(\Gamma)}.$$

Moreover, for each $T \in \mathcal{T}_\ell$, it holds that

$$\Big\|\nabla \sum_{j=1}^N \widehat{V}_{\ell,j}\Big\|_{L^2(T)}^2 \lesssim \sum_{j=1}^N \|\nabla \widehat{V}_{\ell,j}\|_{L^2(T)}^2 \lesssim \Big\|\nabla \sum_{j=1}^N \widehat{V}_{\ell,j}\Big\|_{L^2(T)}^2.$$

The constant in the lower estimate depends only on γ-shape regularity of $\mathcal{T}_\ell$, since the number of overlapping supports $\operatorname{supp}(\widehat{V}_{\ell,j}) = \operatorname{supp}(\varphi_j)$ is uniformly bounded. The upper estimate follows from a scaling argument and equivalence of norms on finite dimensional spaces. To see this, note that $\operatorname{supp}(\widehat{V}_{\ell,j}) \cap T \neq \emptyset$ holds only for finitely many j and that the $\widehat{V}_{\ell,j}$ are linearly independent with $\sum_{j=1}^N \widehat{V}_{\ell,j}(z) = 0$ for all nodes $z \in \mathcal{N}_\ell$. The hidden constants thus depend only on γ-shape regularity of $\mathcal{T}_\ell$, but *not* on N. Summing the last estimate over all $T \in \mathcal{T}_\ell$, we see

$$\Big|\!\Big|\!\Big| \sum_{j=1}^N \widehat{V}_{\ell,j}\Big|\!\Big|\!\Big|^2 \simeq \Big\|h_\ell^{1/2}\nabla \sum_{j=1}^N \widehat{V}_{\ell,j}\Big\|_{L^2(\Gamma)}^2 \simeq \sum_{j=1}^N \|h_\ell^{1/2}\nabla \widehat{V}_{\ell,j}\|_{L^2(\Gamma)}^2 \simeq \sum_{j=1}^N |\!|\!| \widehat{V}_{\ell,j}|\!|\!|^2$$

and conclude the proof. $\square$

Lemma 16. *Let* $\mathbb{G}_{\ell,j} : \widetilde{H}^{1/2}(\Gamma) \to \mathcal{S}^p_{\ell,j} := \operatorname{span}\{\varphi_j\}$ *denote the Galerkin projection onto* $\mathcal{S}^p_{\ell,j}$ *for* $j = 1, \dots, N$ *and* $\mathbb{G}_{\ell,0} := \mathbb{G}_\ell$. *Then, it holds*

$$C_5^{-1}\, |\!|\!|\widehat{V}_\ell|\!|\!| \le \Big(\sum_{j=0}^N |\!|\!|\mathbb{G}_{\ell,j}\widehat{V}_\ell|\!|\!|^2\Big)^{1/2} \le C_6\, |\!|\!|\widehat{V}_\ell|\!|\!| \quad \text{for all } \widehat{V}_\ell \in \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell). \tag{39}$$

The constants $C_5, C_6 > 0$ *depend only on* Γ, p, *and* γ*-shape regularity of* $\mathcal{T}_\ell$.

Proof. Let $I_\ell : C(\Gamma) \to \mathcal{S}^1(\mathcal{T}_\ell)$ denote the nodal interpolation operator onto the lowest-order BEM space $\mathcal{S}^1(\mathcal{T}_\ell)$. With $\lambda_1 := 1$, $\varphi_0 := I_\ell \widehat{V}_\ell$, and $\sum_{j=1}^N \lambda_j \varphi_j = (1 - I_\ell)\widehat{V}_\ell$, we have $\widehat{V}_\ell = \sum_{j=0}^N \lambda_j \varphi_j$. Therefore, the Cauchy-Schwarz inequality yields

$$\begin{aligned}|\!|\!|\widehat{V}_\ell|\!|\!|^2 = \langle\!\langle \widehat{V}_\ell, \widehat{V}_\ell\rangle\!\rangle = \sum_{j=0}^N \langle\!\langle \widehat{V}_\ell, \lambda_j\varphi_j\rangle\!\rangle &= \sum_{j=0}^N \langle\!\langle \mathbb{G}_{\ell,j}\widehat{V}_\ell, \lambda_j\varphi_j\rangle\!\rangle \\ &\le \Big(\sum_{j=0}^N |\!|\!|\mathbb{G}_{\ell,j}\widehat{V}_\ell|\!|\!|^2\Big)^{1/2}\Big(\sum_{j=0}^N |\!|\!|\lambda_j\varphi_j|\!|\!|^2\Big)^{1/2}.\end{aligned}$$

The lower estimate in Lemma 15 and the localization estimate (13) yield

$$\sum_{j=0}^N |\!|\!|\lambda_j\varphi_j|\!|\!|^2 \lesssim |\!|\!|I_\ell\widehat{V}_\ell|\!|\!|^2 + |\!|\!|\sum_{j=1}^N \lambda_j\varphi_j|\!|\!|^2 = |\!|\!|I_\ell\widehat{V}_\ell|\!|\!|^2 + |\!|\!|(1 - I_\ell)\widehat{V}_\ell|\!|\!|^2 \lesssim |\!|\!|\widehat{V}_\ell|\!|\!|^2.$$

Combining the last two estimates, we prove the first estimate in (39). To see the second estimate, we note that

$$\sum_{j=0}^N |\!|\!|\mathbb{G}_{\ell,j}\widehat{V}_\ell|\!|\!|^2 = \sum_{j=0}^N \langle\!\langle \mathbb{G}_{\ell,j}\widehat{V}_\ell, \mathbb{G}_{\ell,j}\widehat{V}_\ell\rangle\!\rangle = \sum_{j=0}^N \langle\!\langle \mathbb{G}_{\ell,j}\widehat{V}_\ell, \widehat{V}_\ell\rangle\!\rangle \le |\!|\!|\sum_{j=0}^N \mathbb{G}_{\ell,j}\widehat{V}_\ell|\!|\!|\,|\!|\!|\widehat{V}_\ell|\!|\!|.$$

The upper estimate in Lemma 15 therefore yields

$$|||\sum_{j=0}^{N} \mathbb{G}_{\ell,j}\widehat{V}_\ell||| \le |||\mathbb{G}_{\ell,0}\widehat{V}_\ell||| + |||\sum_{j=1}^{N} \mathbb{G}_{\ell,j}\widehat{V}_\ell||| \le \sqrt{2}\Big(|||\mathbb{G}_{\ell,0}\widehat{V}_\ell|||^2 + |||\sum_{j=1}^{N} \mathbb{G}_{\ell,j}\widehat{V}_\ell|||^2\Big)^{1/2}$$
$$\lesssim \Big(\sum_{j=0}^{N} |||\mathbb{G}_{\ell,j}\widehat{V}_\ell|||^2\Big)^{1/2}$$

and concludes the proof. □

Proof of Theorem 14. Let $\widehat{V}_\ell = \widehat{U}_\ell - U_\ell \in \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell)$ and note that $\mathbb{G}_{\ell,0}\widehat{V}_\ell = 0$, since $\mathbb{G}_{\ell,0}\widehat{U}_\ell = U_\ell$. Lemma 16 thus yields

$$\eta_\ell = |||\widehat{V}_\ell||| \simeq \Big(\sum_{j=1}^{N} |||\mathbb{G}_{\ell,j}\widehat{V}_j|||^2\Big)^{1/2}.$$

Since $\mathbb{G}_{\ell,j}$ is the orthogonal projection onto the *one-dimensional* space spanned by φ_j, it holds

$$\mathbb{G}_{\ell,j}\widehat{V}_\ell = \frac{\langle\!\langle \widehat{U}_\ell - U_\ell\,,\varphi_j\rangle\!\rangle}{|||\varphi_j|||^2}\,\varphi_j.$$

Finally, $\varphi_j \in \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell)$ and the Galerkin formulation (9) for $\widehat{U}_\ell \in \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell)$ reveal

$$|||\mathbb{G}_{\ell,j}\widehat{V}_j||| = \frac{|\langle f - WU_\ell\,,\,\varphi_j\rangle_\Gamma|}{|||\varphi_j|||}.$$

For $\Gamma = \partial\Omega$, the stabilization disappears as $\langle 1\,,\,U_\ell\rangle_\Gamma = 0$. This concludes the proof. □

4.3. Averaging on large patches. The $(h - h/2)$-type estimators from Section 4.1 and the two-level error estimator from Section 4.2 are unsatisfactory in the sense that one computes the Galerkin matrix and the right-hand side vector with respect to the uniform refinement $\widehat{\mathcal{T}}_\ell$, while only the error $|||u - U_\ell|||$ with respect to $\mathcal{T}_\ell$ is controlled. Moreover, for either estimator its reliability is equivalent to the saturation assumption (35) which is mathematically open and may fail to hold in general because of some possible preasymptotic convergence behavior.

In this section, we consider the strategy proposed in [CP07b] and extend the analysis of [CP07b, EFGP13] from 2D and $p = 1$ to $d = 2, 3$ and $p \ge 1$. Unlike the previous sections, the estimators considered now, control the error $|||u-\widehat{U}_\ell|||$ on the uniform refinement. Throughout, $U_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ and $\widehat{U}_\ell \in \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell)$ denote the Galerkin solutions with respect to $\mathcal{T}_\ell$ and $\widehat{\mathcal{T}}_\ell$. With the Galerkin projection

(40) $\quad \overline{\mathbb{G}}_\ell : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{\mathcal{S}}^{p+1}(\mathcal{T}_\ell)$

onto the higher-order space $\widetilde{\mathcal{S}}^{p+1}(\mathcal{T}_\ell)$ with respect to the coarse mesh $\mathcal{T}_\ell$, we define the error estimator

(41) $\quad \alpha_\ell := |||(1 - \overline{\mathbb{G}}_\ell)\widehat{U}_\ell|||.$

The following proposition is proved in [CP07a, CP07b] by abstract Hilbert space arguments.

Proposition 17. *With the quantities*

$$q_\ell := \frac{|||(1-\overline{\mathbb{G}}_\ell)u|||}{|||u-\widehat{U}_\ell|||} \quad \textit{and} \quad \lambda_\ell := \max_{\overline{V}_\ell \in \widetilde{\mathcal{S}}^{p+1}(\mathcal{T}_\ell)\setminus\{0\}} \min_{\widehat{V}_\ell \in \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell)} \frac{|||\overline{V}_\ell - \widehat{V}_\ell|||}{|||\overline{V}_\ell|||}, \tag{42}$$

it holds efficiency

$$\alpha_\ell \le (1+q_\ell)\,|||u-\widehat{U}_\ell|||. \tag{43}$$

Provided that $q_\ell^2 + \lambda_\ell^2 < 1$, there also holds reliability

$$|||u-\widehat{U}_\ell||| \le \frac{1}{\sqrt{1-\lambda_\ell^2} - q_\ell}\,\alpha_\ell. \tag{44}$$

□

As the assumption $q_\ell^2 + \lambda_\ell^2 < 1$ can hardly be checked in practice, the interpretation of this result is as follows: If the exact solution u is sufficiently smooth or if the mesh $\mathcal{T}_\ell$ is appropriately graded towards the singularities of u, it holds $q_\ell \to 0$. Moreover, instead of a *single* uniform refinement (12), one can obtain $\widehat{\mathcal{T}}_\ell$ by k successive uniform refinements, i.e. $\widehat{h}_\ell = 2^{-k}h_\ell$. Then, the approximation result of Lemma 7 and the inverse estimate of Corollary 6 show for $s = 1/2$ and $\overline{V}_\ell \in \widetilde{\mathcal{S}}^{p+1}(\mathcal{T}_\ell)$

$$\frac{\|(1-\widehat{\mathbb{G}}_\ell)\overline{V}_\ell\|_{\widetilde{H}^{1/2}(\Gamma)}}{\|\overline{V}_\ell\|_{\widetilde{H}^{1/2}(\Gamma)}} \lesssim \frac{\|\widehat{h}_\ell^{1/2}\nabla\overline{V}_\ell\|_{L^2(\Gamma)}}{\|\overline{V}_\ell\|_{\widetilde{H}^{1/2}(\Gamma)}} = 2^{-k/2}\,\frac{\|h_\ell^{1/2}\nabla\overline{V}_\ell\|_{L^2(\Gamma)}}{\|\overline{V}_\ell\|_{\widetilde{H}^{1/2}(\Gamma)}} \lesssim 2^{-k/2}$$

and hence $\lambda_\ell \lesssim 2^{-k/2}$. In particular, the assumptions $q_\ell^2 + \lambda_\ell^2 < 1$ is satisfied asymptotically and for sufficiently large k which depends on p, Γ, and γ-shape regularity of $\mathcal{T}_\ell$.

The numerical experiments in [CP07b] give evidence that —at least for $d = 2$ and lowest-order polynomials $p = 1$— the choice $k = 1$ and hence $\widehat{h}_\ell = h_\ell/2$ is sufficient. In the following, we restrict to the case $k = 1$, but stress that the crucial norm localization estimate (13) remains valid for any fixed $k \ge 1$, where $C_{\rm inv} > 0$ then additionally depends on k.

In addition to the computationally expensive estimator α_ℓ from (41), we define the following localized variants,

$$\beta_\ell := \|h_\ell^{1/2}\nabla(1-\overline{P}_\ell)\widehat{U}_\ell\|_{L^2(\Gamma)} \quad \text{and} \quad \widetilde{\beta}_\ell := \|h_\ell^{1/2}(1-\overline{\Pi}_\ell)\nabla\widehat{U}_\ell\|_{L^2(\Gamma)}, \tag{45}$$

where $\overline{P}_\ell : \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell) \to \widetilde{\mathcal{S}}^{p+1}(\mathcal{T}_\ell)$ satisfies (13) and where $\overline{\Pi}_\ell : L^2(\Gamma) \to \mathcal{P}^p(\mathcal{T}_\ell)$ denotes the $L^2(\Gamma)$-orthogonal projection.

The following theorem has been proved for $d = 2$ and $p = 1$ in [EFGP13], while the equivalence $\alpha_\ell \simeq \beta_\ell$ under the same restrictions is already found in [CP07b]. It is now transferred to the general case $d = 2, 3$ and $p \ge 1$.

Theorem 18. *It holds*

$$C_7^{-1}\,\alpha_\ell \le \widetilde{\beta}_\ell \le \beta_\ell \le C_{\rm inv}\,\alpha_\ell. \tag{46}$$

In particular, $\beta_\ell, \widetilde{\beta}_\ell$ are reliable and efficient estimators for the fine-mesh error $|||u-\widehat{U}_\ell|||$ in the sense of Proposition 17. Moreover, it holds

$$C_{\rm inv}^{-1}\widetilde{\beta}_\ell \le \eta_\ell \le C_8\,\widetilde{\beta}_\ell. \tag{47}$$

In particular, $\alpha_\ell, \beta_\ell, \widetilde{\beta}_\ell$ are also reliable and efficient estimators for the coarse-mesh error $|||u - U_\ell|||$ in the sense of Theorem 12. The constant $C_{\rm inv} > 0$ is that from (13)*, while $C_7, C_8 > 0$ depend only on $C_{\rm apx} > 0$ and on p and γ-shape regularity of $\mathcal{T}_\ell$.*

Proof. According to Proposition 9, the localization estimate (13) holds for both $\overline{P}_\ell$ and $\overline{\mathbb{G}}_\ell$. This proves $C_{\rm apx}^{-1}\alpha_\ell \le \beta_\ell \le C_{\rm inv}\,\alpha_\ell$. Moreover, it holds $\widetilde{\beta}_\ell \le \beta_\ell$, since $\nabla\overline{P}_\ell\widehat{U}_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$ and $\overline{\Pi}_\ell$ is the $\mathcal{T}_\ell$-elementwise L^2-bestapproximation operator. It thus only remains to prove that $\beta_\ell \lesssim \widetilde{\beta}_\ell$. Since all operators $\overline{P}_\ell$ with (13) lead to equivalent estimators, we may assume that $\overline{P}_\ell$ is the Lagrange nodal interpolation operator onto $\mathcal{S}^{p+1}(\mathcal{T}_\ell)$. For $T \in \mathcal{T}$, a scaling argument and equivalence of semi-norms on finite dimensional spaces prove

$$\|\nabla(1-\overline{P}_\ell)\widehat{U}_\ell\|_{L^2(T)} \lesssim \|(1-\Pi_\ell)\nabla\widehat{U}_\ell\|_{L^2(T)},$$

since both semi-norms have the same kernel. The hidden constant depends only on the polynomial degree p (resp. $p+1$) and γ-shape regularity of $\mathcal{T}_\ell$. This concludes the proof of (46). To prove (47), we use the Lagrange nodal interpolation operator $P_\ell : \widetilde{\mathcal{S}}^p(\widehat{\mathcal{T}}_\ell) \to \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ onto $\mathcal{S}^p(\mathcal{T}_\ell)$ for the definition of μ_ℓ. Arguing as before, we see

$$\widetilde{\beta}_\ell \le \mu_\ell \le C_{\rm inv}\,\eta_\ell \quad \text{as well as} \quad C_{\rm apx}^{-1}\,\eta_\ell \le \mu_\ell \lesssim \widetilde{\beta}_\ell$$

where the second estimate is proved $\mathcal{T}_\ell$-elementwise as for $\beta_\ell \simeq \widetilde{\beta}_\ell$. This concludes the proof. $\square$

Remark 19. *Let P_ℓ and $\overline{P}_\ell$ denote the nodal interpolation operators onto $\mathcal{S}^p(\mathcal{T}_\ell)$ resp. $\mathcal{S}^{p+1}(\mathcal{T}_\ell)$. The proof of Theorem 18 shows that β_ℓ, $\widetilde{\beta}_\ell$, and $\widetilde{\mu}_\ell$ are* locally *equivalent, i.e.*

$$\widetilde{\beta}_\ell(T) \le \min\{\beta_\ell(T), \widetilde{\mu}_\ell(T)\} \le \max\{\beta_\ell(T), \widetilde{\mu}_\ell(T)\} \lesssim \widetilde{\beta}_\ell(T) \quad \textit{for all } T \in \mathcal{T}_\ell,$$

where, e.g., $\widetilde{\beta}_\ell(T) := \mathrm{diam}(T)^{1/2}\,\|(1-\overline{\Pi}_\ell)\nabla\widehat{U}_\ell\|_{L^2(T)}$ and β_ℓ, μ_ℓ are defined accordingly. The hidden constant depends only on p and γ-shape regularity of $\mathcal{T}_\ell$. $\square$

4.4. ZZ-type error estimator. Since the seminal work of Zienkiewicz and Zhu [ZZ87], averaging techniques for FEM a posteriori error control became quite popular among the engineering community. For BEM, ZZ-type error estimators have only recently been proposed in [FFKP13], where the analysis is, however, restricted to $d = 2$ and lowest-order ansatz functions $p = 1$.

We denote the set of interior edges ($d = 3$) resp. interior nodes ($d = 2$) of $\mathcal{T}_\ell$ by $\mathcal{E}_\ell := \big\{E = T \cap T' \,:\, T, T' \in \mathcal{T}_\ell \text{ with } T \neq T'\big\}$. For $E = T \cap T'$, let $\omega_\ell(E) := T \cup T'$ denote the corresponding patch. The error estimator is then defined elementwise via

$$\zeta_\ell^2 := \sum_{E\in\mathcal{E}_\ell} \zeta_\ell(E)^2 \quad \text{with} \quad \zeta_\ell(E)^2 := \begin{cases} \sum_{j=1}^p \mathrm{diam}(E)^{2j}\|[\partial_n^{(j)}U_\ell]\|_{L^2(E)}^2 & \text{for } d=3,\\ \sum_{j=1}^p \mathrm{diam}(\omega_\ell(E))^{2j}\big|[U_\ell^{(j)}](E)\big|^2 & \text{for } d=2.\end{cases}$$

For $d = 2$, $[U_\ell^{(j)}]$ denotes the jump of the j-th arclength derivative at the node $E \in \mathcal{E}_\ell = \mathcal{N}_\ell$. For $d = 3$ the jump of the j-th normal derivative $[\partial_n^{(j)}(\cdot)]$ is defined as follows: Given $x \in E$, let n, n' denote tangential unit vectors with $x + tn \in T$ and $x + tn' \in T'$ for sufficiently small $t > 0$ such that $n \perp E$ and $n' \perp E$. Then,

$$[\partial_n^{(j)}U_\ell](x) := \frac{\partial^j}{\partial t^j}\big(t \mapsto \frac{1}{2t}(U_\ell(x+tn) - U_\ell(x+tn'))\big)(0).$$

Theorem 20. *The ZZ-type estimator ζ_ℓ satisfies efficiency up to higher-order terms*

$$C_{\rm eff}^{-1}\zeta_\ell \le |||u-U_\ell||| + \Big(\sum_{E\in\mathcal{E}_\ell}\min_{V\in\mathcal{P}^{p+1}(\omega_\ell(E))}\|u-V\|^2_{H^{1/2}(\omega_\ell(E))}\Big)^{1/2}, \tag{48}$$

where $C_{\rm eff}>0$ depends only on the polynomial degree p and on the γ-shape regularity of $\mathcal{T}_\ell$. Moreover, suppose that there exists a coarsening $\overline{\mathcal{T}}_\ell$ of $\mathcal{T}_\ell$, such that

$$|||u-U_\ell||| \le q_{\rm sat}|||u-\overline{U}_\ell||| \quad \textit{and} \quad \#\big\{T\in\mathcal{T}_\ell\,:\,T\subseteq T'\big\}\le C_{\rm coarse} \textit{ for all } T'\in\overline{\mathcal{T}}_\ell. \tag{49}$$

Then, there holds reliability

$$|||u-U_\ell||| \le |||u-\overline{U}_\ell||| \le (1-q_{\rm sat}^2)^{-1/2}C_{\rm rel}\zeta_\ell, \tag{50}$$

where $C_{\rm rel}>0$ depends only on the polynomial degree p, on the γ-shape regularity of $\mathcal{T}_\ell$, on the shapes of the element patches, and on $C_{\rm coarse}$.

Before we come to the proof, we need a technical result.

Lemma 21. *Let $0\le s\le 1$. There exists a constant $C_{\rm loc}>0$ such that*

$$\sum_{E\in\mathcal{E}_\ell}\|v\|^2_{H^s(\omega_\ell(E))} \le C_{\rm loc}\|v\|^2_{H^s(\Gamma)} \quad \textit{for all } v\in H^s(\Gamma), \tag{51}$$

where the H^s-norms for $0<s<1$ are defined either via interpolation or by use of the Sobolev-Slobodeckij seminorm. The constant $C_{\rm loc}$ depends only on the γ-shape regularity of $\mathcal{T}_\ell$.

Proof. The cases $s=0$ and $s=1$ are obvious. A colouring argument from [CMS01] provides a finite number of sets $\mathcal{E}_\ell=\mathcal{F}_1\cup\ldots\cup\mathcal{F}_n$, where $n\in\mathbb{N}$ depends only on the γ-shape regularity of $\mathcal{T}_\ell$, such that $\omega_\ell(E)\cap\omega_\ell(E')=\emptyset$ for all $E,E'\in\mathcal{F}_j$ with $E\ne E'$ and all $1\le j\le n$. By definition of the Sobolev-Slobodeckij seminorm, there holds

$$\sum_{E\in\mathcal{E}_\ell}\|v\|^2_{H^s(\omega_\ell(E))} = \sum_{j=1}^n\sum_{E\in\mathcal{F}_j}\|v\|^2_{H^s(\omega_\ell(E))} \le \sum_{j=1}^n\|v\|^2_{H^s(\Gamma)} = n\|v\|^2_{H^s(\Gamma)}.$$

To prove the result for interpolation norms, the essential observation is that the product of interpolated spaces is the interpolation of the products, even with equal norms [Tar07]. The sum on the left-hand side of (51) can be written as a product norm

$$\sum_{E\in\mathcal{E}_\ell}\|v\|^2_{H^s(\omega_\ell(E))} = \|v\|^2_{\Pi_{E\in\mathcal{E}_\ell}H^s(\omega_\ell(E))}.$$

For $s\in\{0,1\}$ it holds

$$\|v\|^2_{\Pi_{E\in\mathcal{F}_j}H^s(\omega_\ell(E))} \le \|v\|^2_{H^s(\Gamma)}.$$

Consequently, interpolation applies and concludes the proof. □

Proof of efficiency (48). Let $V\in\mathcal{P}^{p+1}(\omega_\ell(E))$, $E=T\cap T'$, and $h_E:=\operatorname{diam}(E)$ for $d=3$ and $h_E:=\operatorname{diam}(\omega_\ell(E))$ for $d=2$. Given $1\le j\le p$, scaling arguments and norm equivalence on finite dimensional spaces prove for $d=3$

$$h_E^j\|[\partial_n^{(j)}(U_\ell-V)]\|_{L^2(E)} \lesssim \|U_\ell-V\|_{H^{1/2}(\omega_\ell(E))}.$$

The hidden constant depends only on p, the shape of $\omega_\ell(E)$, and γ-shape regularity of $\mathcal{T}_\ell$. For $d = 2$, a similar argument shows

$$h_E^j \big|[\partial_n^{(j)}(U_\ell - V)](E)\big| \lesssim \|U_\ell - V\|_{H^{1/2}(\omega_\ell(E))}.$$

With $V_E \in \mathcal{P}^{p+1}(\omega_\ell(E))$ and 51 for $v = u - U_\ell$, this leads to

$$\begin{aligned}\zeta_\ell^2 &= \sum_{E\in\mathcal{E}_\ell}\sum_{j=1}^{p}\zeta_\ell(E)^2\\ &\lesssim \sum_{E\in\mathcal{E}_\ell}\|U_\ell - V_E\|^2_{H^{1/2}(\omega_\ell(E))} \lesssim \|u - U_\ell\|^2_{H^{1/2}(\Gamma)} + \sum_{E\in\mathcal{E}_\ell}\|u - V_E\|^2_{H^{1/2}(\omega_\ell(E))}.\end{aligned}$$

Taking the infimum over all V_E, which is attained due to finite dimension, we conclude efficiency. □

Proof of reliability (50). Given $T \in \overline{\mathcal{T}}$, define $\mathcal{E}_T := \big\{E \in \mathcal{E}_\ell \,:\, E \subseteq \omega_\ell(T)\big\}$ and note $\#\mathcal{E}_T \lesssim C_{\rm coarse}$. On $\widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$, consider the seminorms $|\cdot|_1, |\cdot|_2$,

$$|\cdot|_1 := \sum_{j=1}^{p}\sum_{E\in\mathcal{E}_T} h_E^{2j}\begin{cases}\|[\partial_n^{(j)}(\cdot)]\|^2_{L^2(E)} & \text{for } d=3,\\ \big|[(\cdot)^{(j)}](E)\big|^2 & \text{for } d=2,\end{cases}$$

$$|\cdot|_2 := h_T\|\nabla(1-\overline{J}_\ell)(\cdot)\|^2_{L^2(T)},$$

where $\overline{J}_\ell : L^2(\Gamma) \to \widetilde{\mathcal{S}}^p(\overline{\mathcal{T}}_\ell)$ is the Scott-Zhang projection from Lemma 4. For $V \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$, assume $|V|_1 = 0$. This implies that $V \in \mathcal{P}^p(\omega_\ell(T))$ and therefore $\overline{J}_\ell V|_T = V|_T$. This yields $|V|_2 = 0$ and hence

$$|\cdot|_2 \lesssim |\cdot|_1.$$

A scaling argument shows that the hidden constant depends only on the polynomial degree $p \in \mathbb{N}$ and the shape of the patch $\omega_\ell(T)$ as well as the shape of $\mathcal{E}_T$. Since $\#\mathcal{E}_T \lesssim C_{\rm coarse}$, this constant depends only on the shape of the patches of finitely many elements $T' \cap \bigcup\mathcal{E}_T \neq \emptyset$. Altogether and with (13), this leads to

$$|||U_\ell - \overline{U}_\ell|||^2 \le |||(1-\overline{J}_\ell)U_\ell|||^2 \lesssim \sum_{T\in\overline{\mathcal{T}}_\ell}|U_\ell|_2^2 \lesssim \sum_{T\in\overline{\mathcal{T}}_\ell}|U_\ell|_1^2 \simeq \zeta_\ell^2.$$

By assumption, the reliability follows from

$$|||u - \overline{U}_\ell|||^2 = |||u - U_\ell|||^2 + |||U_\ell - \overline{U}_\ell|||^2 \le q_{\rm sat}^2|||u - \overline{U}_\ell|||^2 + C_{\rm rel}^2\zeta_\ell^2.$$

This concludes the proof of reliability. □

Remark 22. *With the same techniques as in the proof of Theorem 20, one may prove*

$$\overline{\eta}_\ell \lesssim \zeta_\ell$$

with $\overline{\eta}_\ell$ denoting the $(h{-}h/2)$-error estimator from Theorem 12, which corresponds to $\overline{\mathcal{T}}_\ell$. □

Remark 23. *For the lowest-order case $p = 1$, the estimator ζ_ℓ is equivalent to the error estimator from* [FFKP13] *which reads*

$$\widetilde{\zeta}_\ell := \|h_\ell^{1/2}(1-\mathcal{A}_\ell)\nabla U_\ell\|_{L^2(\Gamma)} \qquad (52)$$

with some arbitrary averaging operator $\mathcal{A}_\ell : \mathcal{P}^0(\mathcal{T}_\ell) \to \mathcal{S}^1(\mathcal{T}_\ell)$ which satisfies $(\mathcal{A}_\ell\Psi)|_T = \Psi|_T$ for all $\Psi \in \mathcal{P}^0(\mathcal{T}_\ell)$ with $\Psi|_{\omega_\ell(T)} \in C(\omega_\ell(T))$ and all $T \in \mathcal{T}_\ell$. This can easily be seen by scaling arguments and norm equivalence on finite dimensional spaces. For higher-order elements $p > 1$, the analogous analysis needs an averaging operator $\mathcal{A}_\ell : \mathcal{P}^{p-1}(\mathcal{T}_\ell) \to \mathcal{S}^p(\mathcal{T}_\ell) \cap H^p(\Gamma)$ with more regularity to establish the equivalence. □

Remark 24. *The assumption* (49) *is, for instance, guaranteed if $\mathcal{T}_\ell := \widehat{\mathcal{T}}_k$ is the uniform refinement of some mesh $\mathcal{T}_k$ and the classical saturation assumption*

$$|||u - \widehat{U}_k||| \le q_{\rm sat}|||u - U_k|||$$

holds for some $0 < q_{\rm sat} < 1$. Then, (49) *holds with $\overline{\mathcal{T}}_\ell := \mathcal{T}_k$. Moreover, if the adaptive algorithm converges linearly, i.e.,*

$$|||u - U_\ell||| \le q_{\rm sat}|||u - U_{\ell-1}|||,$$

the assumption (49) *is satisfied with $\overline{\mathcal{T}}_\ell := \mathcal{T}_{\ell-1}$.* □

5. Numerical experiments

We consider different numerical examples ($d = 2, 3$) and compare uniform versus adaptive strategies as well as some selected a posteriori error estimators presented in this work.

We use some local error estimator ρ_ℓ with $\rho_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \rho_\ell(T)^2$ to steer the following adaptive algorithm. Here, $\rho_\ell \in \{\mu_\ell, \widetilde{\mu}_\ell, \tau_\ell, \beta_\ell, \widetilde{\beta}_\ell, \zeta_\ell, \widetilde{\zeta}_\ell\}$ denotes either an $(h - h/2)$-type error estimator from Section 4.1, the two-level error estimator τ_ℓ from Section 4.2, the averaging error estimator β_ℓ resp. $\widetilde{\beta}_\ell$ from Section 4.3, or the ZZ-type error estimators ζ_ℓ and $\widetilde{\zeta}_\ell$ from Section 4.4. The quantities $\rho_\ell(T)$ are the contributions of ρ_ℓ associated with the elements $T \in \mathcal{T}_\ell$.

Algorithm 25. ***Input:*** *Initial triangulation $\mathcal{T}_0$, counter $\ell := 0$, parameter $0 < \theta < 1$.*

(i) *Compute Galerkin solution $U_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$.*
(ii) *Compute local refinement indicators $\rho_\ell(T)$ for all $T \in \mathcal{T}_\ell$.*
(iii) *Choose some set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ (of minimal cardinality) such that*

$$\theta\rho_\ell^2 \le \sum_{T\in\mathcal{M}_\ell} \rho_\ell(T)^2. \tag{53}$$

(iv) *Refine at least all marked elements $T \in \mathcal{M}_\ell$ to obtain the refined mesh $\mathcal{T}_{\ell+1}$, increase the counter $\ell \mapsto \ell + 1$, and goto* (i).

Output: *Sequence of nested triangulations $\mathcal{T}_\ell$ with corresponding Galerkin solutions U_ℓ and error estimators ρ_ℓ, for $\ell = 0, 1, 2, \dots$.*

For $d = 2$, we use the bisection algorithm from [AFF$^+$13, Section 3]. For $d = 3$, we employ 2D newest vertex bisection, see e.g. [KPP] and the references therein. Either refinement strategy guarantees that the meshes $\mathcal{T}_\ell$ obtained, are uniformly γ-shape regular, where γ depends only on the initial mesh $\mathcal{T}_0$. Moreover, only finitely many shapes of patches occur. In particular, all constants in the a posteriori analysis of Section 4 remain uniformly bounded.

5.1. Experiment on 2D open slit. For the open boundary $\Gamma = (-1, 1)\times\{0\}$, we consider the hypersingular integral equation (1) with $f = 1$ and exact solution $u(x, 0) = 2\sqrt{1 - x^2}$.

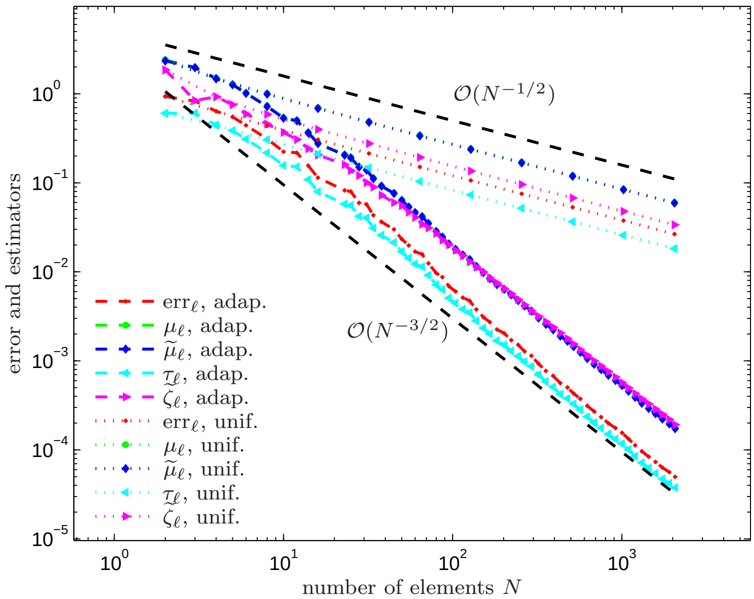

FIGURE 1. Hypersingular integral equation from Section 5.1 on the slit $\Gamma = (-1,1)\times\{0\}$ with right-hand side $f = 1$, exact solution $u(x,0) = 2\sqrt{1-x^2}$, and $p = 1$. The adaptive refinement is steered by the ZZ-type error estimator $\widetilde{\zeta}_\ell$.

Since the energy norm $|||u|||^2 = \pi$ can be computed analytically, we compute the energy error between u and the Galerkin approximation $U_\ell \in \widetilde{\mathcal{S}}^p(\mathcal{T}_\ell)$ by use of the Galerkin orthogonality

$$\mathrm{err}_\ell^2 := |||u - U_\ell|||^2 = |||u|||^2 - |||U_\ell|||^2 = \pi - |||U_\ell|||^2. \tag{54}$$

We stress that $u \in \widetilde{H}^{1/2}(\Gamma) \cap H^{1-\varepsilon}(\Gamma)$ for all $\varepsilon > 0$, but $u \notin H^1(\Gamma)$. Thus, theory predicts the convergence order $\alpha = 1/2$ for uniform mesh refinement, i.e. $\mathrm{err}_h = \mathcal{O}(h^{1/2}) = \mathcal{O}(N^{-1/2})$, where N denotes the number of elements in the *uniform* triangulation $\mathcal{T}_\ell$. In contrast to that, adaptive strategies are likely to regain the optimal convergence order $\alpha = 1/2 + p$. In Figure 1 resp. Figure 2, we compare uniform vs. adaptive mesh-refinement for $p = 1$ resp. $p = 2$, where we use the local refinement indicators

$$\widetilde{\zeta}_\ell(T)^2 = \mathrm{diam}(T)\|(1-\mathcal{A}_\ell)u_\ell'\|_{L^2(T)}^2 \quad \text{for } T \in \mathcal{T}_\ell \tag{55}$$

of the ZZ-type error estimator from (52) in Section 4.4 and $\theta = 0.25$ to steer the adaptive algorithm. Here, $\mathcal{A}_\ell : L^2(\Gamma) \to \mathcal{S}^1(\mathcal{T}_\ell)$ denotes the standard Clément operator defined nodewise for $z \in \mathcal{N}_\ell$ via

$$(\mathcal{A}v)(z) := |\omega_\ell(z)|^{-1} \int_{\omega_\ell(z)} v\,dx \quad \text{with } \omega_\ell(z) := \bigcup\{T \in \mathcal{T}_\ell \,:\, z \in T\} \tag{56}$$

for $p = 1$ resp. the Scott-Zhang projection $\mathcal{A}_\ell = J_\ell$ from Lemma 3 onto the space $\mathcal{S}^p(\mathcal{T}_\ell)$ for $p = 2$. Additionally, we also compute the $(h - h/2)$-type error estimators $\mu_\ell, \widetilde{\mu}_\ell$ from Section 4.1 as well as the two-level estimator τ_ℓ from Section 4.2. We use the nodal interpolation operator $P_\ell = I_\ell$ for the computation of $\widetilde{\mu}_\ell$. As can be seen from Figures 1–2, the

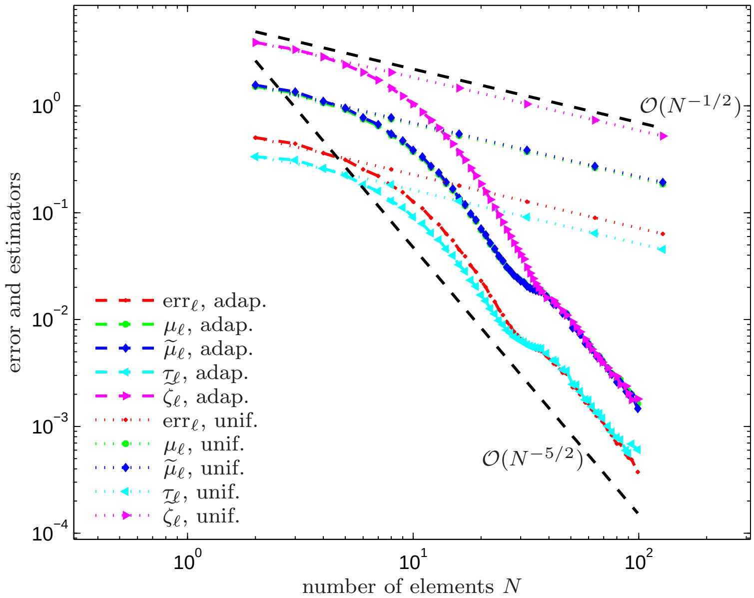

FIGURE 2. Hypersingular integral equation from Section 5.1 on the slit $\Gamma = (-1, 1) \times \{0\}$ with right-hand side $f = 1$, exact solution $u(x, 0) = 2\sqrt{1 - x^2}$, and $p = 2$. The adaptive refinement is steered by the ZZ-type error estimator $\widetilde{\zeta}_\ell$.

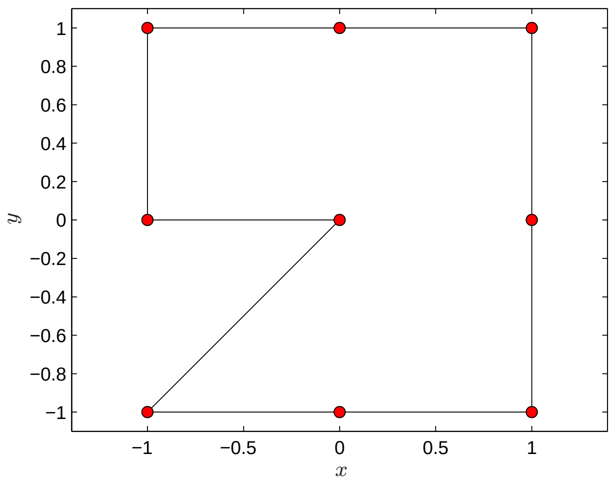

FIGURE 3. Z-shaped domain Ω with boundary $\Gamma = \partial\Omega$ and initial triangulation of Γ into 9 boundary elements for the numerical experiment from Section 5.2.

uniform mesh-refinement strategy leads — as expected — to the suboptimal convergence order $\alpha = 1/2$, whereas the adaptive strategy regains the optimal order of convergence of $\alpha = 3/2$ resp. $\alpha = 5/2$.

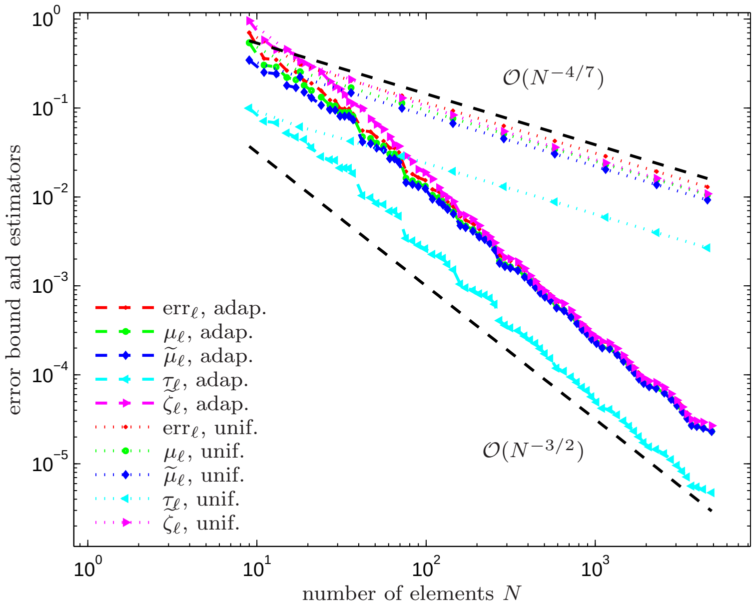

FIGURE 4. Hypersingular integral equation from Section 5.2 on the Z-shaped domain, sketched in Figure 3, with $p = 1$. The adaptive refinement is steered by the ZZ-type error estimator $\widetilde{\zeta}_\ell$.

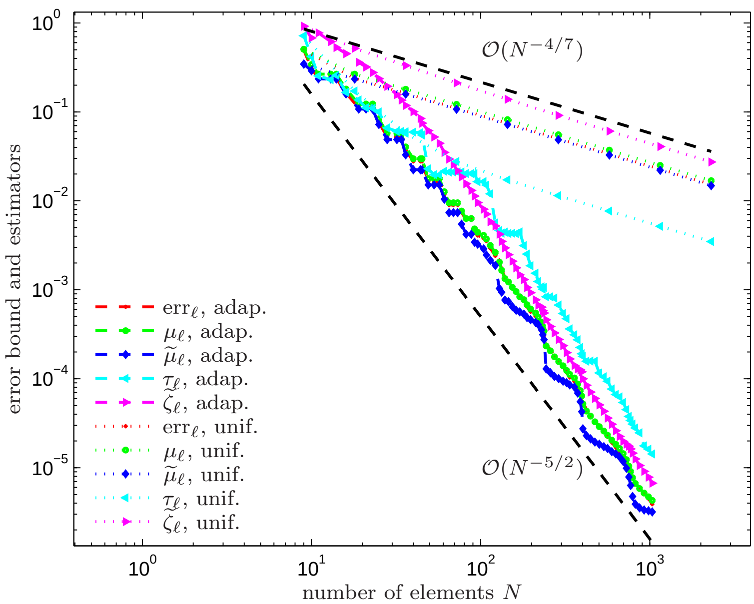

FIGURE 5. Hypersingular integral equation from Section 5.2 on the Z-shaped domain, sketched in Figure 3, with $p = 2$. The adaptive refinement is steered by the ZZ-type error estimator $\widetilde{\zeta}_\ell$.

5.2. Experiment on closed boundary of Z-shaped domain in 2D. Let Ω be the Z-shaped domain from Figure 3 with reentrant corner at the origin $(0,0)$. Let $\Gamma = \partial\Omega$ denote its boundary. We consider the hypersingular integral equation (1) with $f = (1/2 - K')\phi$. Here, K' denotes the adjoint double-layer potential and $\phi = \partial_{\boldsymbol{n}} w$ is the normal derivative of the function

$$w(x) = r^{4/7}\cos(4/7\varphi), \tag{57}$$

where (r,φ) are the polar coordinates of $x \in \Gamma$ with respect to the origin $(0,0)$. By choice of f, the exact solution u of (1) is, up to some additive constant, the trace of w. Note that u admits a generic singularity in the origin $(0,0)$. Since $u \in \widetilde{H}^{1/2}(\Gamma) \cap H^{4/7+1/2-\varepsilon}(\Gamma)$ for all $\varepsilon > 0$, theory predicts a convergence order $\alpha = 4/7$ for uniform mesh-refinement. Because the exact value of the energy norm $|||u|||$ is unknown, we employ Lemma 7 for $s = 1/2$ and estimate the energy error between the exact solution u and the Galerkin approximation $U_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)$ by

$$|||u - U_\ell||| \lesssim \|h_\ell^{1/2}(u - U_\ell)'\|_{L^2(\Gamma)} =: \mathrm{err}_\ell. \tag{58}$$

In Figure 4 resp. Figure 5, we compare uniform vs. adaptive mesh-refinement for $p = 1$ resp. $p = 2$. The adaptive algorithm is steered by the local contributions (55) of the ZZ-type error estimator $\widetilde{\zeta}_\ell$ from (52) and $\theta = 0.25$. We observe that the uniform strategy leads to the suboptimal convergence order $\alpha = 4/7$ for both $p = 1$ and $p = 2$, whereas the adaptive strategy regains the optimal order $\alpha = 1/2 + p$. For comparison, we also plot the $(h - h/2)$-type error estimators $\mu_\ell, \widetilde{\mu}_\ell$ (with P_ℓ the nodal interpolation operator) from Section 4.1 as well as the two-level error estimator τ_ℓ from Section 4.2. Here, we use the nodal interpolation operator I_ℓ for the computation of $\widetilde{\mu}_\ell$.

5.3. Experiment on 3D screen problem. We choose Γ to be the square screen $(0,1)^2 \times \{0\}$ with an initial mesh shown in Figure 6. We consider the hypersingular integral equation (1) with $f = 1$. The energy norm is known to be approximately $|||u|||^2 \approx 0.45486722$, such that the energy error between u and the Galerkin approximation $U_\ell \in \widetilde{\mathcal{S}}^1(\mathcal{T}_\ell)$ can be computed by

$$\mathrm{err}_\ell^2 := |||u - U_\ell|||^2 = |||u|||^2 - |||U_\ell|||^2.$$

For the exact solution u, it is well known that $u \in \widetilde{H}^{1/2}(\Gamma) \cap H^{1-\varepsilon}(\Gamma)$ for all $\varepsilon > 0$, but $u \notin H^1(\Gamma)$. Thus, theory predicts the convergence order $\alpha = 1/2 - \varepsilon$ for uniform mesh refinement, i.e., $\mathrm{err}_h = \mathcal{O}(h^{1/2-\varepsilon}) = \mathcal{O}(N^{-1/4-\varepsilon/2})$, where N denotes the number of elements in the *uniform* triangulation $\mathcal{T}_\ell$. In contrast to that, adaptive strategies based on isotropic mesh refinement are known to regain the convergence order $\alpha = 1$. In Figure 7, we compare uniform vs. adaptive mesh-refinement for $p = 1$, where we use the local contributions (55) of the ZZ-type error estimator $\widetilde{\zeta}_\ell$ from (52). Here, $\mathcal{A}_\ell$ denotes the standard Clément operator. As before, we also compute the $(h - h/2)$-type error estimators $\mu_\ell, \widetilde{\mu}_\ell$ (with P_ℓ the nodal interpolation operator) from Section 4.1 as well as the two-level estimator τ_ℓ from Section 4.2 As can be seen from Figure 7, the uniform mesh-refinement strategy leads — as expected — to the suboptimal convergence order $\alpha = 1/4$, whereas the adaptive strategy regains the optimal order of convergence of $\alpha = 1/2$.

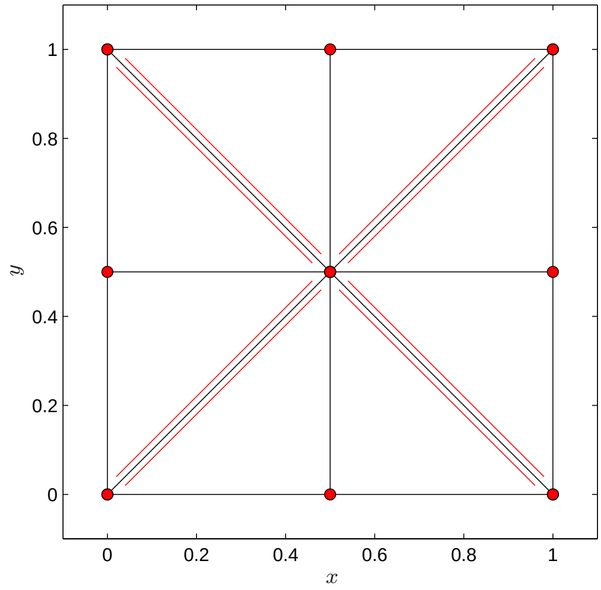

FIGURE 6. Initial mesh for the 3D screen problem from Section 5.3. The reference edges of each triangle used for newest vertex bisection, are indicated by double lines.

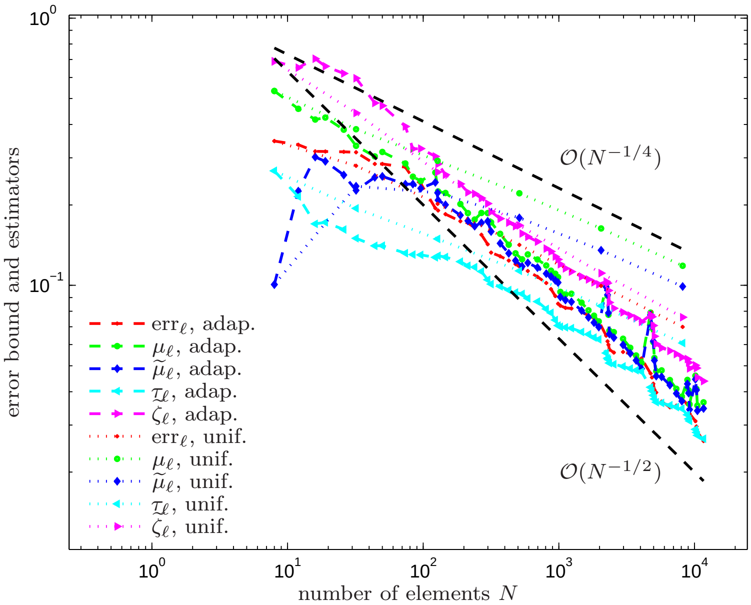

FIGURE 7. Hypersingular integral equation from Section 5.3 on the screen $\Gamma = (0,1)^2 \times \{0\}$ with right-hand side $f = 1$ and $p = 1$. The adaptive refinement is steered by the ZZ-type error estimator ζ_ℓ.

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