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Efficient Spectral Methods for the spatially homogeneous Boltzmann equation

Gerhard Kitzler Joachim Schöberl

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Abstract

We present a spectral Petrov-Galerkin method for the spatially homogeneous Boltzmann equation. We approximate the density distribution function by high order multivariate Lagrange polynomials in Gauss Hermite points, multiplied by a Gaussian peak with adjusted mean and width; the test functions are polynomials. Our focus is on an efficient scheme for applying the Boltzmann collision operator. The first improvement is to transform the collision integral to mean and relative velocity which allows to use cheap numerical integration rules for the first one. The second improvement is a fast transformation from Lagrange via Hermite to a hierarchical basis in Polar coordinates. In this basis, the innermost integral operator becomes diagonal. We conclude with a numerical example demonstrating the achieved speed up.

1 Introduction

Dealing with the properties of a gas flow, usually means dealing with a huge amount of particles (e.g. 10^{19} molecules per cm^3 at standard conditions). Therefore, instead of describing each individual particle one introduces a statistical description of the flow: The representation of the gas is given in terms of a positive density distribution function $f = f(t, \mathbf{x}, \mathbf{v})$. The value $f(t_0, \mathbf{x}_0, \mathbf{v}_0)$ holds the average number of particles having a position close to $\mathbf{x}_0$ and a velocity close to $\mathbf{v}_0$ at time t_0. Assuming that the gas of interest is sufficiently dilute, one can neglect collisions involving more than two particles. With this additional assumption, the effect of collisions is given by the Boltzmann collision operator $Q(f)$:

$$Q(f)(t, \mathbf{x}, \mathbf{v}) = \int_{\mathbb{R}^d} \int_{S^{d-1}} B(\mathbf{v}, \mathbf{w}, \mathbf{e}')\big(f(t, \mathbf{x}, \mathbf{v}') f(t, \mathbf{x}, \mathbf{w}') - f(t, \mathbf{x}, \mathbf{v}) f(t, \mathbf{x}, \mathbf{w}) \big)\, d\mathbf{e}'\, d\mathbf{w}, \tag{1}$$

Simplifying, it consists of a gain and a loss term. The gain term contains the availability of post collision velocities, whereas on the loss side the availability of pre collision velocities is found. Both of these availabilities are multiplied with the Boltzmann collision kernel $B(\mathbf{v}, \mathbf{w}, \mathbf{e}') \geq 0$. It holds the probability for a collision that transfers the pre collision velocities $\mathbf{v}$ and $\mathbf{w}$ into the post collision velocities $\mathbf{v}'$ and $\mathbf{w}'$ to happen. The definition of the post collision velocities is done such that mass, momentum and energy are conserved quantities during a binary collision. This conservation properties yield the following representation for these velocities:

$$\begin{aligned} \mathbf{v}' &:= \frac{\mathbf{v} + \mathbf{w}}{2} + \mathbf{e}' \frac{\|\mathbf{v} - \mathbf{w}\|}{2} \\ \mathbf{w}' &:= \frac{\mathbf{v} + \mathbf{w}}{2} - \mathbf{e}' \frac{\|\mathbf{v} - \mathbf{w}\|}{2}, \end{aligned} \tag{2}$$

with the unit scattering vector $\mathbf{e}'$.

It is well known that the kernel of the collision operator $Q(f)$ is formed by the Maxwell distributions:

$$\begin{aligned} Q(f)(t, \mathbf{x}, \mathbf{v}) = 0 \Leftrightarrow \exists\ \rho(t, \mathbf{x}),\ T(t, \mathbf{x}),\ \mathbf{V}(t, \mathbf{x}) : \\ f(t, \mathbf{x}, \mathbf{v}) = \frac{\rho(t, \mathbf{x})}{(\pi T(t, \mathbf{x}))^{\frac{d}{2}}} e^{-\frac{|\mathbf{v} - \mathbf{V}(t, \mathbf{x})|^2}{T(t, \mathbf{x})}}, \end{aligned}$$

where ρ, T, $\mathbf{V}$ are the macroscopic quantities mean density, mean temperature and mean velocity.

The time evolution of f is governed by the Boltzmann equation [1–3]:

$$\frac{\partial f}{\partial t} + \operatorname{div}_{\mathbf{x}}(\mathbf{v}\, f) = Q(f). \tag{3}$$

On the left hand side of (3), there is in general the Boltzmann transport operator $\operatorname{div}_{\mathbf{x}}(\mathbf{v}\, f)$, corresponding to spatial transport of the particles. Although the full Boltzmann equation (3) is of great interest, we focus in this report on efficiency of our method for the spatially homogeneous Boltzmann equation:

$$\frac{\partial f}{\partial t} = Q(f). \tag{4}$$

For a discretization of (3) resp. (4) various methods have been proposed. A majority of them is based on Monte Carlo methods. These methods are of big interest because of their simplicity. The drawback within these methods is the accuracy, i.e. for accurate results many samples are necessary. Whatever the circumstances are, in a hydrodynamic limit, they are almost perfect.

On the other hand there are a lot of deterministic methods. The main problem of such methods is the evaluation of the collision operator arising in (3) resp. (4): Typically the numerical effort to evaluate this operator is $\mathcal{O}(N^4)$ for a N by N grid in the velocity space. For the deterministic methods, fast Fourier transformation is one of the popular methods to evaluate the collision integral. Especially for Maxwellian molecules, where $B = \text{const}$, the collision integral becomes relatively simple in a Fourier representation (see [4]). This representation was used in [5], where a difference scheme was constructed for the solution of (3). The same simplification was also used to derive a deterministic numerical scheme in [6]. To extend this, the authors of [5] have also investigated the Boltzmann collision integral for the hard sphere model by means of the fast fourier transformation [7].
Moreover, within the deterministic methods there are also the discrete velocity methods, which are for instance considered in [8, 9]. They are based on a uniform grid in the velocity domain. Then, a discrete collision mechanics is constructed to preserve the main physical properties. The disadvantage within these methods is their high computational costs, compared to their accuracy.

Our method falls into the category of the deterministic ones and is a Petrov-Galerkin method in the velocity domain. We use a set of global basis functions in the momentum domain, and therefore no assumptions on compact support in the momentum domain are needed. The method conserves macroscopic quantities such as the density, velocity and energy of the gas of interest. In addition to a straightforward discretization of the collision integrals we present a technique for the collision operator with a complexity of only N^3.
In the next section we derive the basic method. Within section 3 we reduce the complexity for the application of the collision integral to N^3. This is done by making use of certain properties of different sets of orthogonal polynomials. For keeping the presentation as simple as possible we let $B \equiv 1$ for the presentation. Especially within section 3, the details have to be adjusted for different models of the collision kernel. Section 4 gives some basic properties of the method. In the last section, we conclude with the presentation of numerical examples.

2 The spatially homogeneous Boltzmann equation

The spatially homogeneous Boltzmann equation is obtained by assuming that $\frac{\partial f}{\partial x_i} = 0$ (i.e. the distribution function is independent of the spatial variable $\mathbf{x}$). Therefore, Boltzmann equation reduces to it's spatially homogeneous case:

$$\frac{\partial f}{\partial t} = Q(f), \tag{5}$$

or more suitable for our purpose

$$\frac{\partial}{\partial t} \int\limits_{\mathbb{R}^d} f\phi \, d\mathbf{v} = \int\limits_{\mathbb{R}^d} Q(f)\phi \, d\mathbf{v} \qquad \forall \phi \text{ suitable.} \tag{6}$$

To keep notation as simple as possible, the $\mathbf{x}$ dependency of f is skipped for the rest of the presentation. For discretization, we perform a Petrov-Galerkin method. The Ansatzspace for the discrete solution f_N is denoted by $V_N^{\mathbf{v}} := e^{-|\mathbf{v}|^2} P^N(\mathbb{R}^d)$, where $P^N(\mathbb{R}^d)$ is the space of polynomials on $\mathbb{R}^d$ with partial degree at most N. $d \in \{2, 3\}$ is the space dimension. Thus the discrete space is a weighted polynomial space and an element $f \in V_N^{\mathbf{v}}$ at a fixed point in time can be expanded into a sum of polynomial basis functions times a Gaussian peak:

$$f(\mathbf{v}) = e^{-|\mathbf{v}|^2} \sum_{m=0}^{n_{\mathbf{v}}} c_m \, L_m(\mathbf{v}), \tag{7}$$

where $L_m,\ \ m = 0 \dots n_{\mathbf{v}}$ denotes a polynomial basis in $V_N^{\mathbf{v}}$. The space for the test functions ϕ, is defined as $W_N^{\mathbf{v}} := e^{|\mathbf{v}|^2} V_N^{\mathbf{v}}$.
Using the above expansion and testing the homogeneous Boltzmann equation with the i-th basis polynomial leads to

$$\frac{\partial}{\partial t} \int\limits_{\mathbb{R}^d} e^{-|\mathbf{v}|^2} \sum_{m=0}^{n_{\mathbf{v}}} c_m(t) L_m(\mathbf{v}) L_i(\mathbf{v})\, d\mathbf{v} = \int\limits_{\mathbb{R}^d} Q(f_N) L_i \, d\mathbf{v} \tag{8}$$

Thus, by reordering the terms on the left hand side of (8), one immediately recognizes a mass matrix $M^{\mathbf{v}} \in \mathbb{R}^{n_{\mathbf{v}}+1 \times n_{\mathbf{v}}+1}$

$$M^{\mathbf{v}} \frac{\partial}{\partial t} \underbrace{\begin{pmatrix} c_0(t) \\ \vdots \\ c_{n_{\mathbf{v}}}(t) \end{pmatrix}}_{:=\mathbf{c}(t)} = \int\limits_{\mathbb{R}^d} Q(f_N) L_i \, d\mathbf{v}, \tag{9}$$

with $M_{i,j}^{\mathbf{v}} = \int\limits_{\mathbb{R}^d} e^{-|\mathbf{v}|^2} L_i(\mathbf{v}) L_j(\mathbf{v})\, d\mathbf{v}$.

Polynomial basis in $V_N^{\mathbf{v}}$:
In order to introduce a suitable polynomial basis for $V_N^{\mathbf{v}}$ some more notation is needed: By the pair $(\omega_k, x_k)_{k=0\dots N}$ a Gauss-Hermite quadrature rule of length $N+1$ is denoted. ω_k represents the integration weights, x_k the integration nodes, thus $\int_{\mathbb{R}} e^{-v^2} p(v) \approx \sum_{i=0}^{N} \omega_k\, p(x_k)$. The Gauss-Hermite quadrature rules evaluate the integrals $\int_{\mathbb{R}} e^{-v^2} p(v)\, dv$ exact for polynomials p up to order $2N+1$.

Now let $h_i(v) \in P^i(\mathbb{R})$ be the Hermite-polynomial of degree i. The Hermite-polynomials are a hierarchical family of polynomials on $\mathbb{R}$ and orthogonal with respect to the weighted L_2 inner product $\int_{\mathbb{R}} e^{-v^2} f(v)\, g(v)\, dv$. In addition by $\mathbb{H}_N^0 := \{x_0, \dots, x_N\}$ the roots of h_{N+1} are denoted and by $\mathbb{L}_N := \{\, l_n(v),\ n = 0, \dots, N\,\}$ the set of Lagrange-Polynomials to the nodes in $\mathbb{H}_N^0$ is defined (i.e $l_m(x_k) = \delta_{m,k}$). The Lagrange polynomials l_m satisfy 2 nice relations:

$$\int\limits_{\mathbb{R}} e^{-v^2} l_i(v)\, l_j(v)\ dv \overset{\text{quad.}}{=} \sum_k \omega_k l_i(x_k)\, l_j(x_k) = \omega_i \delta_{i,j} \tag{10}$$

As stated above, the Gauss-Hermite quadrature rule is exact for polynomials of degree at most $2N+1$ which is satisfied by the product of Lagrange polynomials. Therefore, applying the quadrature rule and using $l_m(x_k) = \delta_{m,k}$, immediately yields the same orthogonality relation as for the Hermite polynomials (up to a constant).
Another useful relation is the next one, which can be seen just as the previous one (Remark that the integral is still exactly evaluated by the quadrature rule).

$$\int\limits_{\mathbb{R}} e^{-v^2} v\, l_i(v)\, l_j(v)\ dv \overset{\text{quad.}}{=} \sum_k \omega_k x_k l_i(x_k)\, l_j(x_k) = \omega_i x_i \delta_{i,j} \tag{11}$$

By forming tensor products of this 1D functions one ends up with the d-dimensional basis functions $L_m(v_x, v_y) := l_i(v_x) l_j(v_y)$, for $d = 2$ where $i, j = 0 \dots N\ \ m = i + n_{\mathbf{v}} j$. In addition we denote the tensored sets of integration resp. collocation nodes as $\mathbb{H}_{N,d}^0$, and the tensored nodes as $x_{j,d}$.

Now, using these basis functions for discretization, the mass matrix on the left hand side decouples into a diagonal matrix ($d = 2$):

$$
\begin{aligned}
M^{\mathbf{v}}_{u,v} &= \int\limits_{\mathbb{R}^d} L_u(\mathbf{v}_x,\mathbf{v}_y)L_v(\mathbf{v}_x,\mathbf{v}_y)\,d(\mathbf{v}_x,\mathbf{v}_y) = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} l_{i_u}(\mathbf{v}_x)l_{j_u}(\mathbf{v}_y)l_{i_v}(\mathbf{v}_x)l_{j_v}(\mathbf{v}_y)\,d\mathbf{v}_x\,d\mathbf{v}_y \\
&= \underbrace{\int\limits_{\mathbb{R}} l_{i_u}(\mathbf{v}_x)l_{i_v}(\mathbf{v}_x)d\mathbf{v}_x}_{=\delta_{i_u,i_v}\omega_{i_u}} \underbrace{\int\limits_{\mathbb{R}} l_{j_u}(\mathbf{v}_y)l_{j_v}(\mathbf{v}_y)\,d\mathbf{v}_y}_{=\delta_{j_u,j_v}\omega_{j_u}} = \delta_{u,v}\tilde{\omega}_u
\end{aligned} \tag{12}
$$

The collision operator on the right hand side is investigated via a different but equal representation of $\int\limits_{\mathbb{R}^d} Q(f_N)\phi$:

Theorem 1. *For all suitable ϕ there holds*

$$
\begin{aligned}
\int\limits_{\mathbb{R}^d} Q(f_N)(\mathbf{v})\phi(\mathbf{v})\,d\mathbf{v} &= \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{S^{d-1}} B(\mathbf{v},\mathbf{w},\mathbf{e}')f_N(\mathbf{v})f_N(\mathbf{w})[\phi(\mathbf{v}')-\phi(\mathbf{v})]\,d\mathbf{e}'\,d\mathbf{w}\,d\mathbf{v} \\
&= \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{S^{d-1}} B(\mathbf{v},\mathbf{w},\mathbf{e}')[f_N(\mathbf{v})f_N(\mathbf{w})-f_N(\mathbf{v}')f_N(\mathbf{w}')]\times \\
&\qquad [\phi(\mathbf{v}')+\phi(\mathbf{w}')-\phi(\mathbf{v})-\phi(\mathbf{w})]\,d\mathbf{e}'\,d\mathbf{w}\,d\mathbf{v}.
\end{aligned}, \tag{13}
$$

with the post collision velocites defined in (2)

Proof. A more general version of Theorem 1 is proven in [1]. □

Representation (13) has no dependency on the post collision velocity $\mathbf{w}'$, moreover the post collision velocities have been transferred to the test function. Using this representation one finds that the discretization of $\int\limits_{\mathbb{R}^d} Q(f_N)\phi$ becomes a 3$^{\text{rd}}$-order tensor:

$$
\begin{aligned}
\int\limits_{\mathbb{R}^d} Q(f_N)\phi(\mathbf{v})\,d\mathbf{v} &= \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{S^{d-1}} f_N(\mathbf{v})f_N(\mathbf{w})[\phi(\mathbf{v}')-\phi(\mathbf{v})]\,d\mathbf{e}'\,d\mathbf{w}\,d\mathbf{v} \\
&= \sum_{m,n=0}^{n_{\mathbf{v}}} c_m c_n \underbrace{\int\limits_{\mathbb{R}^d} e^{-|\mathbf{v}|^2}\int\limits_{\mathbb{R}^d} e^{-|\mathbf{w}|^2}\int\limits_{S^{d-1}} L_m(\mathbf{v})L_n(\mathbf{w})[L_j(\mathbf{v}')-L_j(\mathbf{v})]\,d\mathbf{e}'\,d\mathbf{w}\,d\mathbf{v}}_{=:q_{m,n,j}} \\
&= \sum_{m,n=0}^{n_{\mathbf{v}}} c_m c_n q_{m,n,j} =: Q^{n_{\mathbf{v}}}(\mathbf{c})_j
\end{aligned} \tag{14}
$$

(14) can also be written as a matrix-vector multiplication: $Q^{n_{\mathbf{v}}}(\mathbf{c}) = Q\tilde{\mathbf{c}}$, where $Q \in \mathbb{R}^{n_{\mathbf{v}}+1\times(n_{\mathbf{v}}+1)^2}$ and $\tilde{\mathbf{c}} \in \mathbb{R}^{(n_{\mathbf{v}}+1)^2}$:

$$\tilde{\mathbf{c}} = \begin{pmatrix} \mathbf{c}_0\mathbf{c}_0 \\ \vdots \\ \mathbf{c}_{n_\mathbf{v}}\mathbf{c}_0 \\ \vdots \\ \mathbf{c}_0\mathbf{c}_{n_\mathbf{v}} \\ \vdots \\ \mathbf{c}_{n_\mathbf{v}}\mathbf{c}_{n_\mathbf{v}} \end{pmatrix} \qquad \text{and } Q = \begin{pmatrix} q_{0,0,0}, & \cdots & q_{n_\mathbf{v},0,0} & \cdots & q_{0,n_\mathbf{v},0} & \cdots & q_{n_\mathbf{v},n_\mathbf{v},0} \\ q_{0,0,1}, & \cdots & q_{n_\mathbf{v},0,1} & \cdots & q_{0,n_\mathbf{v},1} & \cdots & q_{n_\mathbf{v},n_\mathbf{v},1} \\ \vdots & & & & & \vdots & \\ \vdots & & & & & \vdots & \\ q_{0,0,n_\mathbf{v}}, & \cdots & q_{n_\mathbf{v},0,n_\mathbf{v}} & \cdots & q_{0,n_\mathbf{v},n_\mathbf{v}} & \cdots & q_{n_\mathbf{v},n_\mathbf{v},n_\mathbf{v}} \end{pmatrix} \tag{15}$$

Thus, $\approx n_\mathbf{v}^3 \approx N^6$ (N corresponds to the polynomial order in one space direction, $n_\mathbf{v} = (N+1)^2 - 1$) floating point operations are performed for the application of the discrete collision operator.

Collecting everything together, a system of Ode's for the coefficients $\mathbf{c}_j$ is found:

$$M^\mathbf{v}\dot{\mathbf{c}} = Q\tilde{\mathbf{c}}$$

For time integration a simple forward euler scheme is used. Denote by $\mathbf{c}^j \approx \mathbf{c}(t_j), \quad j = 0 \dots j_t$ the approximation to the coefficient vector at time t_j, $\mathbf{c}^0$ holds the initial distribution. Now the euler scheme yields the following representation for $\mathbf{c}^{j+1}$:

$$\mathbf{c}^{j+1} = \mathbf{c}^j + \Delta_t\, M_\mathbf{v}^{-1} Q\tilde{\mathbf{c}}^j, \qquad j \geq 1$$

where $\Delta_{t_j} = t_j - t_{j-1}$. As has already been seen, the inverse $M_\mathbf{v}^{-1}$ is easy to compute since $M_\mathbf{v}$ is a diagonal matrix.

In the following the amount for the application of the collision operator will be reduced to $\approx n_\mathbf{v}^3$ operations:

2.1 Collision integral in terms of relative velocities

By transforming to mean and relative velocities,

$$\bar{\mathbf{v}} := \tfrac{\mathbf{v}+\mathbf{w}}{2}, \qquad \hat{\mathbf{v}} := \tfrac{|\mathbf{v}-\mathbf{w}|}{2} \tag{16}$$

the collision integral takes the form

$$\begin{aligned} \int\limits_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\, d\mathbf{v} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2}\int\limits_{S^1} f(\mathbf{v})f(\mathbf{w})[\phi(\mathbf{v}') - \phi(\mathbf{v})]\, d\mathbf{e}'\, d\mathbf{v} d\mathbf{w} \\ &= 4\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2}\int\limits_{S^1} f(\bar{\mathbf{v}}+\hat{\mathbf{v}})f(\bar{\mathbf{v}}-\hat{\mathbf{v}})[\phi(\bar{\mathbf{v}}+\mathbf{e}'|\hat{\mathbf{v}}|) - \phi(\bar{\mathbf{v}}+\hat{\mathbf{v}})]\, d\mathbf{e}' d\hat{\mathbf{v}}\, d\bar{\mathbf{v}}, \end{aligned} \tag{17}$$

Now by defining the shifted functions $f^{\bar{\mathbf{v}}}(\hat{\mathbf{v}}) = f(\bar{\mathbf{v}} + \hat{\mathbf{v}})$, the integral becomes

$$\int\limits_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\, d\mathbf{v} = 4\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2}\int\limits_{S^1} f^{\bar{\mathbf{v}}}(\hat{\mathbf{v}})f^{\bar{\mathbf{v}}}(-\hat{\mathbf{v}})[\phi^{\bar{\mathbf{v}}}(\mathbf{e}'|\hat{\mathbf{v}}|) - \phi^{\bar{\mathbf{v}}}(\hat{\mathbf{v}})]\, d\mathbf{e}' d\hat{\mathbf{v}}\, d\bar{\mathbf{v}}$$

In terms of the basis polynomials, the product of f evaluations is a polynomial of the double degree than the polynomials in the Ansatzspace. Thus, this polynomial can be represented exactly by Lagrange polynomials of the double degree. Representing $f^{(2)}(\mathbf{v}) := f^{\bar{\mathbf{v}}}(\hat{\mathbf{v}})f^{\bar{\mathbf{v}}}(-\hat{\mathbf{v}})$, one arrives with

$$\int_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v} = 4\int_{\mathbb{R}^2}\underbrace{\int_{\mathbb{R}^2}\int_{S^1} f^{(2)}(\hat{\mathbf{v}})[\phi^{\bar{\mathbf{v}}}(\mathbf{e}'|\hat{\mathbf{v}}|) - \phi^{\bar{\mathbf{v}}}(\hat{\mathbf{v}})]\,d\mathbf{e}'d\hat{\mathbf{v}}}_{:=Q^I(f)(\bar{\mathbf{v}})}\,d\bar{\mathbf{v}}$$

Now let

$$f(\bar{\mathbf{v}}+\hat{\mathbf{v}}) = e^{-|\bar{\mathbf{v}}+\hat{\mathbf{v}}|^2}\sum_{i=0}^{n_{\mathbf{v}}} c_i L_i(\bar{\mathbf{v}}+\hat{\mathbf{v}}) = e^{-|\bar{\mathbf{v}}+\hat{\mathbf{v}}|^2}\sum_{i=0}^{n_{\mathbf{v}}} \tilde{c}_i L_i(\hat{\mathbf{v}}) = f^{\bar{\mathbf{v}}}(\hat{\mathbf{v}})$$

With $\hat{\mathbf{v}} = x_{j,d}$ and $x_{j,d}$ a node out of the already mentioned Gauss-Hermite quadrature rule of appropriate length, one has

$$\tilde{c}_j = \sum_{i=0}^{n_{\mathbf{v}}} c_i L_i(\bar{\mathbf{v}} + x_{j,d})$$

or in more compact form

$$\tilde{\mathbf{c}} = S^{\bar{\mathbf{v}}}\mathbf{c}$$

with $S^{\bar{\mathbf{v}}} \in \mathbb{R}^{n_{\mathbf{v}}+1\times n_{\mathbf{v}}+1}$, $S^{\bar{\mathbf{v}}}_{i,j} = L_j(\bar{\mathbf{v}} + x_{i,d})$, and $\mathbf{c}$ resp. $\tilde{\mathbf{c}}$ being the vector quantities formed of c_i resp. $\tilde{c}_i$.

Remark 2. *A direct calculation of this matrix-vector multiplication needs $n_{\mathbf{v}}^2 \approx N^4$ floating point operations. Using the tensor product structure of $V_N^{\mathbf{v}}$ this calculation can be performed in $\approx n_{\mathbf{v}}^{3/2}$ floating point operations: Let $\mathbf{c}^{mat} \in \mathbb{R}^{N\times N}$, with ${\mathbf{c}^{mat}}_{i,j} = \mathbf{c}_{i(N+1)+j}$ be a matrix wise representation of the coefficient vector $\mathbf{c}$. In addition define the 1D shift matrices $S^{\bar{\mathbf{v}}_x} \in \mathbb{R}^{N\times N}$ resp. $S^{\bar{\mathbf{v}}_y} \in \mathbb{R}^{N\times N}$, with $S^{\bar{\mathbf{v}}_{x/y}}_{i,j} = l_j(\bar{\mathbf{v}}_{x/y} + x_i^{x/y})$. Then the resulting matrix $\tilde{\mathbf{c}}^{mat}$ of the product*

$$S^{\bar{\mathbf{v}}_y}\mathbf{c}^{mat}(S^{\bar{\mathbf{v}}_x})^T = \tilde{\mathbf{c}}^{mat},$$

is a matrix wise representation of the coefficient vector $\tilde{\mathbf{c}}$, such that $\tilde{\mathbf{c}}_{i(N+1)+j} = \tilde{\mathbf{c}}^{mat}_{i,j}$. Both matrix multiplications can be performed in $N^3 \approx n_{\mathbf{v}}^{3/2}$ operations.

For the calculation of the function $f^{(2)}(\hat{\mathbf{v}})$ we denote the Lagrange polynomials corresponding to the double degree and the corresponding quadrature nodes with $\hat{L}_m$ resp. $\hat{x}_{k,d}$. The resulting Gaussian peak of the function $f^{(2)}$ is given by the product of the shifted Gauss peaks:

$$\begin{aligned} f^{(2)}(\hat{\mathbf{v}}) &= e^{-|\bar{\mathbf{v}}+\hat{\mathbf{v}}|^2}\sum_{i=0}^{n_{\mathbf{v}}}\tilde{c}_i L_i(\hat{\mathbf{v}}) e^{-|\bar{\mathbf{v}}-\hat{\mathbf{v}}|^2}\sum_{i=0}^{n_{\mathbf{v}}}\tilde{c}_i L_i(-\hat{\mathbf{v}}) \\ &= e^{-2|\bar{\mathbf{v}}|^2-2|\hat{\mathbf{v}}|^2}\sum_{i=0}^{n_{\mathbf{v}}}\tilde{c}_i L_i(\hat{\mathbf{v}})\sum_{i=0}^{n_{\mathbf{v}}}\tilde{c}_i L_i(-\hat{\mathbf{v}}) \end{aligned}$$

Thus, the appropriate Gauss peak for $f^{(2)}$ is given by $e^{-2|\bar{\mathbf{v}}|^2-2|\hat{\mathbf{v}}|^2}$ and therefore, the Ansatz for $f^{(2)}$ is:

$$f^{(2)}(\hat{\mathbf{v}}) = e^{-2|\bar{\mathbf{v}}|^2-2|\hat{\mathbf{v}}|^2}\sum_i \hat{c}_i \hat{L}_i(\hat{\mathbf{v}}). \tag{18}$$

For the tensor product of double degree polynomials, the corresponding number of basis functions is given by $4(n_{\mathbf{v}} - N) := n_{\mathbf{v}}^{(2)}$, yielding

$$\sum_{i=0}^{n_{\mathbf{v}}} \tilde{c}_i L_i(\hat{\mathbf{v}}) \sum_{i=0}^{n_{\mathbf{v}}} \tilde{c}_i L_i(-\hat{\mathbf{v}}) = \sum_{i=0}^{4n_{\mathbf{v}}^{(2)}} \hat{c}_i \hat{L}_i(\hat{\mathbf{v}})$$

for the polynomial part of $f^{(2)}$. Again, as for the shifting matrices we use $\hat{\mathbf{v}} = \hat{x}_{k,d}$, to end up with

$$\hat{c}_k = \sum_{i=0}^{n_{\mathbf{v}}} \tilde{c}_i L_i(\hat{x}_{k,d}) \sum_{i=0}^{n_{\mathbf{v}}} \tilde{c}_i L_i(-\hat{x}_{k,d}) \qquad k = 0, \ldots, n_{\mathbf{v}}^{(2)}.$$

This calculation now needs $n_{\mathbf{v}} n_{\mathbf{v}}^{(2)} \approx N^4$ floating point operations.

Remark 3. *The presented approach uses basis functions of the same polynomial degree for both, the shifted and non shifted functions. An alternative is given by the usage of Lagrange polynomials $\hat{L}_m$ of the double degree for the representation of the shifted function. This makes the shifting of course more expensive in the sense of floating point operations, but still bounded by N^3. The benefit of the alternative lies in the calculation of the coefficients of $f^{(2)}$: The resulting representation of $f^{\bar{\mathbf{v}}}(\hat{\mathbf{v}}) = e^{-|\bar{\mathbf{v}}+\hat{\mathbf{v}}|^2} \sum_{i=0}^{n_{\mathbf{v}}} \tilde{c}_i \hat{L}_i(\hat{\mathbf{v}})$ yields (using the symmetry of the collocation nodes, i.e. $\hat{x}_{k,d} = -\hat{x}_{n_{\mathbf{v}}^{(2)}-k,d}$):*

$$\begin{aligned} \hat{c}_k &= \underbrace{\sum_{i=0}^{n_{\mathbf{v}}} \tilde{c}_i \hat{L}_i(\hat{x}_{k,d})}_{=\tilde{c}_k} \underbrace{\sum_{i=0}^{n_{\mathbf{v}}} \tilde{c}_i \hat{L}_i(-\hat{x}_{k,d})}_{=\tilde{c}_{n_{\mathbf{v}}^{(2)}-k}} \qquad k = 0, \ldots, n_{\mathbf{v}}^{(2)} \\ &= \tilde{c}_k \tilde{c}_{n_{\mathbf{v}^{(2)}}-k} \end{aligned} \tag{19}$$

Within this alternative the calculation of $f^{(2)}$ is performed within N^3 floating point operations.

Plugging (18) into the collision integral one finds

$$\begin{aligned} \int_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v} &= 4 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{S^1} f^{(2)}(\hat{\mathbf{v}})[\phi^{\bar{\mathbf{v}}}(\mathbf{e}'|\hat{\mathbf{v}}|) - \phi^{\bar{\mathbf{v}}}(\hat{\mathbf{v}})]\,d\mathbf{e}' d\hat{\mathbf{v}}\,d\bar{\mathbf{v}} \\ &= 4 \sum_{i=0}^{n_{\mathbf{v}}^{(2)}} \int_{\mathbb{R}^2} \hat{c}_i e^{-2|\bar{\mathbf{v}}|^2} \int_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|^2} \int_{S^1} \hat{L}_i(\hat{\mathbf{v}})\,[\phi^{\bar{\mathbf{v}}}(\mathbf{e}'|\hat{\mathbf{v}}|) - \phi^{\bar{\mathbf{v}}}(\hat{\mathbf{v}})]\,d\mathbf{e}' d\hat{\mathbf{v}}\,d\bar{\mathbf{v}} \end{aligned} \tag{20}$$

For the shifted test functions, the procedure is very similar: With the Ansatz

$$L_j(\bar{\mathbf{v}} + \mathbf{v}) = \sum_{k=0}^{n_{\mathbf{v}}} \phi_k^j L_k(\mathbf{v})$$

and $\mathbf{v} = x_{k,d}$, one finds $\phi_k^j = L_j(\bar{\mathbf{v}} + x_{k,d})$ and ends up with:

$$L_j(\bar{\mathbf{v}} + \mathbf{v}) = \sum_{k=0}^{n_{\mathbf{v}}} L_j(\bar{\mathbf{v}} + x_{k,d}) L_k(\mathbf{v})$$

Again, in compact form

$$\mathbf{L}(\bar{\mathbf{v}}+\mathbf{v}) = S^{\bar{\mathbf{v}},\phi}\mathbf{L}(\bar{\mathbf{v}}),$$

with $S^{\bar{\mathbf{v}},\phi}_{i,j} = L_i(\bar{\mathbf{v}}+x_{j,d}),\ i,j=0\dots n_{\mathbf{v}}$, thus there holds $S^{\bar{\mathbf{v}},\phi} = S^{\bar{\mathbf{v}}T}$.
Testing with all test functions simultaneously, the collision integral takes the form

$$\begin{aligned}\int\limits_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v} &= 4\sum_{i=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_i \int\limits_{\mathbb{R}^2} e^{-2|\bar{\mathbf{v}}|^2} \int\limits_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|^2} \int\limits_{S^1} \hat{L}_i(\hat{\mathbf{v}}) \left[S^{\bar{\mathbf{v}}T}\mathbf{L}(\mathbf{e}'|\hat{\mathbf{v}}|) - S^{\bar{\mathbf{v}}T}\mathbf{L}(\hat{\mathbf{v}})\right] d\mathbf{e}' d\hat{\mathbf{v}}\, d\bar{\mathbf{v}} \\ &= 4\sum_{i=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_i \int\limits_{\mathbb{R}^2} e^{-2|\bar{\mathbf{v}}|^2} S^{\bar{\mathbf{v}}T} \int\limits_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|^2} \int\limits_{S^1} \hat{L}_i(\hat{\mathbf{v}}) \left[\mathbf{L}(\mathbf{e}'|\hat{\mathbf{v}}|) - \mathbf{L}(\hat{\mathbf{v}})\right] d\mathbf{e}' d\hat{\mathbf{v}}\, d\bar{\mathbf{v}}\end{aligned},$$

with $\mathbf{L} = (L_0 \dots L_{n_{\mathbf{v}}})^T$.
For numerical purposes (stability of the oversampling process) it is reasonable to calculate the integrals after scaling the integral by a factor $\frac{1}{\sqrt{2}}$, thus let $\hat{\mathbf{w}} = \sqrt{2}\hat{\mathbf{v}}$ resp. $\bar{\mathbf{w}} = \sqrt{2}\bar{\mathbf{v}}$:

$$\int\limits_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v} = \sum_{i=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_i \int\limits_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} S^{\frac{\bar{\mathbf{v}}}{\sqrt{2}}T} \int\limits_{\mathbb{R}^2} e^{-|\hat{\mathbf{v}}|^2} \int\limits_{S^1} \hat{L}_i\left(\frac{\hat{\mathbf{v}}}{\sqrt{2}}\right) \left[\mathbf{L}\left(\frac{\mathbf{e}'|\hat{\mathbf{v}}|}{\sqrt{2}}\right) - \mathbf{L}\left(\frac{\hat{\mathbf{v}}}{\sqrt{2}}\right)\right] d\mathbf{e}' d\hat{\mathbf{v}}\, d\bar{\mathbf{v}}$$

Now by pre assembling the integrals with respect to the variables $\mathbf{e}'$ and $\hat{\mathbf{v}}$, the following representation is found for the application of the collision integral:

$$\int\limits_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v} = 4\int\limits_{\mathbb{R}^2} e^{-|\mathbf{v}|^2} S^{\frac{\bar{\mathbf{v}}}{\sqrt{2}}T} Q^I \hat{\mathbf{c}}\, d\bar{\mathbf{v}},$$

with $Q^I \in \mathbb{R}^{n_{\mathbf{v}} \times n_{\mathbf{v}}{}^{(2)}}$ and $Q^I_{i,j} = \int\limits_{\mathbb{R}^2} e^{-|\hat{\mathbf{v}}|^2} \int\limits_{S^{d-1}} \hat{L}_j\left(\frac{\hat{\mathbf{v}}}{\sqrt{2}}\right) \left[L_i\left(\frac{\mathbf{e}'|\hat{\mathbf{v}}|}{\sqrt{2}}\right) - L_i\left(\frac{\hat{\mathbf{v}}}{\sqrt{2}}\right)\right] d\mathbf{e}' d\hat{\mathbf{v}}$.
The matrix-vector multiplication $Q^I\hat{\mathbf{c}}$ needs $\approx N^4$ floating point operations. Note that we used the same symbol for denoting the inner collision integrals with respect to $\mathbf{e}'$ and $\hat{\mathbf{v}}$ and for the matrix resulting from these inner collision integrals. The back shift, corresponding to the multiplication with $S^{\frac{\bar{\mathbf{v}}}{\sqrt{2}}T}$ can again be factorised in $\mathbf{v}_x$ and $\mathbf{v}_y$ direction to save operations and result in cN^3 operations for the back shift.
Now within the time loop, the integral with respect to $\bar{\mathbf{v}}$ is evaluated via a quadrature rule (a Gauss-Hermite rule). Therefore we end up in a total application cost for the application of the collision operator of $\approx N^4 n_{\text{ip}}$, where n_{ip} is the number of integration points used in the above mentioned quadrature rule.

2.2 Collision integral in polar coordinates - polar basis for $V_N^{\mathbf{v}}$

One more optimization can be achieved by making use of polar coordinates, in the collision integral and in the space $V_N^{\mathbf{v}}$. With this transformation one additional power of N can be saved in applying the inner collision operator. Transforming the collision integral into polar coordinates yields:

$$\int\limits_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v} = \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^+} r\Big[\int\limits_{S^1} f^{(2)}(r\mathbf{e})\,d\mathbf{e} \int\limits_{S^1} \phi^{\bar{\mathbf{v}}}(r\mathbf{e})\,d\mathbf{e} - 2\pi \int\limits_{S^1} f^{(2)}(r\mathbf{e})\phi^{\bar{\mathbf{v}}}(r\mathbf{e})\,d\mathbf{e}\Big]\,dr\,d\bar{\mathbf{v}} \quad (21)$$

In (r, φ) coordinates, $f^{(2)}$ is expressed via

$$f^{(2)}(r, \varphi) = e^{-2|\bar{\mathbf{v}}|-2r^2} \sum_{m=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_m \hat{L}_m(r\mathbf{e}), \tag{22}$$

where $\mathbf{e} = (\cos(\varphi), \sin(\varphi))^T$.

Definition 4. *By* $\Psi_{j,k}^{\cos/\sin}(\mathbf{v})$ *the Polar-Laguerre functions are denoted:*

$$\Psi_{j,k}^{\cos}(\mathbf{v}) := \begin{cases} \cos(2j\varphi) r^{2j} \mathcal{L}_{\frac{k}{2}-j}^{(2j)}(r^2), & k \in 2\mathbb{N} \\ \cos((2j+1)\varphi) r^{2j+1} \mathcal{L}_{\frac{k-1}{2}-j}^{(2j+1)}(r^2), & k \in 2\mathbb{N}+1 \end{cases} \quad \textit{and}$$

$$\Psi_{j,k}^{\sin}(\mathbf{v}) := \begin{cases} \sin(2j\varphi) r^{2j} \mathcal{L}_{\frac{k}{2}-j}^{(2j)}(r^2), & k \in 2\mathbb{N} \\ \sin((2j+1)\varphi) r^{2j+1} \mathcal{L}_{\frac{k-1}{2}-j}^{(2j+1)}(r^2), & k \in 2\mathbb{N}+1 \end{cases}$$

The corresponding sets in addition are denoted by:

$$\mathbb{L}_k^{\cos} := \{\Psi_{j,k}^{\cos} : j = 0 \ldots \left[\tfrac{k}{2}\right]\}$$
$$\mathbb{L}_k^{\sin} := \{\Psi_{j,k}^{\sin} : j = 1 - (k \mod 2) \ldots \left[\tfrac{k}{2}\right]\}.$$

$\mathcal{L}_k^{(j)}(x)$ *is the associated Laguerre polynomial of order* k*. They satisfy [10]:*

$$\int_{\mathbb{R}^+} e^{-x} x^{\alpha} \mathcal{L}_k^{(\alpha)}(x) \mathcal{L}_{k'}^{(\alpha)}(x)\, dx = \delta_{k,k'} \frac{\Gamma(k+\alpha+1)}{k!} \overset{\alpha \in \mathbb{N}}{=} \delta_{k,k'} \frac{(k+\alpha)!}{k!}$$

The next few Lemmata state some basic properties of the functions in $\mathbb{L}_k^{\sin/\cos}$:

Lemma 5. *The Polar-Laguerre functions* $\Psi_{j,k}^{\cos/\sin}$ *are polynomials in Cartesian coordinates of total degree* k*.*

Proof. Use, that

$$\cos(n\varphi) = \sum_{j=0}^{[\frac{n}{2}]} \binom{n}{2j} \sin(\varphi)^{2j} \cos(\varphi)^{n-2j} \qquad \sin(n\varphi) = \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n}{2j+1} \sin(\varphi)^{2j+1} \cos(\varphi)^{n-2j-1}$$

With this representation of the trigonometric part one finds for the elements of $\mathbb{L}_k^{\cos}$ (for even k):

$$\begin{aligned} f \in \mathbb{L}_k^{\cos} \Leftrightarrow f &= \sum_{i=0}^{j} \binom{2j}{2i} \sin(\varphi)^{2i} \cos(\varphi)^{2j-2i} \mathcal{L}_{\frac{k}{2}-j}^{2j}(r^2) r^{2j} \\ &= \sum_{i=0}^{j} \binom{2j}{2i} \underbrace{\sin(\varphi)^{2i} r^{2i}}_{y^{2i}} \underbrace{\cos(\varphi)^{2j-2i} r^{2j-2i}}_{x^{2j-2i}} \mathcal{L}_{\frac{k}{2}-j}^{2j}(r^2) \\ &= \sum_{i=0}^{j} \binom{2j}{2i} y^{2i} x^{2j-2i} \mathcal{L}_{\frac{k}{2}-j}^{2j}(x^2+y^2) \end{aligned}$$

The monomial $y^{2i}x^{2j-2i}$ is a polynomial of total degree $2j$. Multiplying with the Laguerre polynomial $\mathcal{L}^{(2j)}_{\frac{k}{2}-j}(x^2+y^2)$, which is a polynomial in cartesian coordinates of degree $2(\frac{k}{2}-j)=k-2j$, results in a polynomial of total degree k. For the functions in $\mathbb{L}^{\sin}_k$, and also for odd k the strategy is the same. $\square$

Lemma 6. *The set* $\mathbb{L}_K := \{\mathbb{L}^{\cos}_k \cup \mathbb{L}^{\sin}_k : k = 0 \dots K\}$ *forms a basis of the polynomial space on* $\mathbb{R}^2$ *of total degree at most* K.

Proof. Observe that the polynomials in $\mathbb{L}^{\cos/\sin}_k$ are hierarchical, thus functions in $\mathbb{L}_k$ and $\mathbb{L}_{k'}$ are linearly independent. Thus for linearly independence it remains to show, that the functions in $\mathbb{W}_k := \mathbb{L}^{\cos}_k \cup \mathbb{L}^{\sin}_k$ are linearly independent. For this purpose we use the fact, that the roots of the Laguerre polynomial $\mathcal{L}^{(j)}_n$ are positive and real for natural j, and that they are bounded by $n+j+(n-1)\sqrt{n+j}$. Now checking for linear independence results in

$$\sum c_j \cos(2j\varphi) r^{2j} \mathcal{L}^{(2j)}_{\frac{k}{2}-j}(r^2) + s_j \sin(2j\varphi) r^{2j} \mathcal{L}^{(2j)}_{\frac{k}{2}-j}(r^2) \stackrel{!}{=} 0$$

Choosing r_0 sufficiently large yields

$$\sum \tilde{c}_j \cos(2j\varphi) + \tilde{s}_j \sin(2j\varphi) \stackrel{!}{=} 0,$$

with $\tilde{c}_j = c_j \mathcal{L}^{(2j)}_{\frac{k}{2}-j}(r_0^2) r_0^{2j}$ and $\tilde{s}_j = s_j \mathcal{L}^{(2j)}_{\frac{k}{2}-j}(r_0^2) r_0^{2j}$. The linear independence of the trigonometric functions immediately yields $\tilde{c}_j$ resp. $\tilde{s}_j = 0,\ j = 0 \dots \frac{k}{2}$ and since $\mathcal{L}^{(2j)}_{\frac{k}{2}-j}(r^2) \neq 0$, one concludes c_j resp. $s_j = 0,\ j = 0 \dots \frac{k}{2}$. Doing the same for odd k and counting the dimension of the space $\mathbb{L}_k$ finally yields the basis property. $\square$

Lemma 7. *The Polar-Laguerre polynomials are orthogonal with respect to the inner product* $< f, g > := \int_{\mathbb{R}^2} e^{-|\mathbf{v}|^2} f(\mathbf{v}) g(\mathbf{v})\, d\mathbf{v}$.

Proof. We calculate the inner product $< \Psi^a_{j,k}, \Psi^b_{j',k'} >,\ \ a, b \in \{\cos, \sin\}$ for even k and k':

$$\begin{aligned}
< \Psi^a_{j,k}, \Psi^b_{j',k'} > &= \int_{\mathbb{R}^2} e^{-|\mathbf{v}|^2} \Psi^a_{j,k}(\mathbf{v}) \Psi^b_{j',k'}(\mathbf{v})\, d\mathbf{v} \\
&\stackrel{\mathbf{v}=r\binom{\cos(\varphi)}{\sin(\varphi)}}{=} \int_0^\infty r e^{-r^2} \int_0^{2\pi} a(2j\varphi) \mathcal{L}^{(2j)}_{\frac{k}{2}-j}(r^2) b(2j'\varphi) \mathcal{L}^{(2j')}_{\frac{k'}{2}-j'}(r^2)\, d\varphi\, dr \\
&= (1+\delta_{0,j})\delta_{j,j'}\delta_{a,b}\pi \int_0^\infty r e^{-r^2} \mathcal{L}^{(2j)}_{\frac{k}{2}-j}(r^2) \mathcal{L}^{(2j)}_{\frac{k'}{2}-j}(r^2)\, dr \\
&\stackrel{r^2=\tilde{r}}{=} \tfrac{1}{2}(1+\delta_{0,j})\delta_{j,j'}\delta_{a,b}\pi \int_0^\infty e^{-r} \mathcal{L}^{(2j)}_{\frac{k}{2}-j}(r) \mathcal{L}^{(2j)}_{\frac{k'}{2}-j}(r)\, dr \\
&= \tfrac{1}{2}(1+\delta_{0,j})\delta_{j,j'}\delta_{a,b}\delta_{k,k'}\pi \frac{(\frac{k}{2}+j)!}{(\frac{k}{2}-j)!}
\end{aligned}$$

As with the proof of Lemma 6, the situation is quite similar for odd k and also for a combination of even and odd k resp. k'. $\square$

Now, remember the fact that $f^{(2)}$ is a polynomial of total degree $4N$. Thus, using the Polar-Laguerre basis $\mathbb{L}_{4N}$, the function $f^{(2)}$ can be represented exactly. The same holds for the test polynomials, which are polynomials of total degree $2N$, just as the function f itself. This enables us to also represent the test functions exactly in the Polar-Laguerre basis.
Let

$$f^{(2)}(\hat{\mathbf{v}}) = e^{-2|\bar{\mathbf{v}}|^2-2|\hat{\mathbf{v}}|^2} \sum_{n=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_m \hat{L}_m(\hat{\mathbf{v}}) = e^{-2|\bar{\mathbf{v}}|^2-2r^2} \sum_{\substack{k=0 \\ j\leq[\frac{k}{2}]}}^{4N} f_{j,k}^{\cos}\Psi_{j,k}^{\cos}(\sqrt{2}r\mathbf{e}) + f_{j,k}^{\sin}\Psi_{j,k}^{\sin}(\sqrt{2}r\mathbf{e}) \tag{23}$$

The scaling in the arguments of the basis functions is to obtain a sparse inner collision operator. Similar, for the test functions $\phi^{\bar{\mathbf{v}}}$ one has the representation

$$\phi^{\bar{\mathbf{v}}} = \sum_{j=0}^{2N} \phi_{j,k}^{\cos}\Psi_{j,k}^{\cos}(\sqrt{2}r\mathbf{e}) + \phi_{j,k}^{\sin}\Psi_{j,k}^{\sin}(\sqrt{2}r\mathbf{e}). \tag{24}$$

Now, the orthogonality relations of the polar basis polynomials are used to proof the following

Lemma 8. *With the representations (23) and (24), the contribution to* $\int_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v}$ *vanishes for polynomial degree larger than* $2N$.

Proof. Plugging representations (23) and (24) into (21) yields:

$$\int_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v}$$

$$= \sum_{\substack{k=0 \\ j\leq[\frac{k}{2}]}}^{4N} \sum_{\substack{k'=0 \\ j'\leq[\frac{k'}{2}]}}^{2N} \int_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} \int_{\mathbb{R}^+} re^{-r^2} \Big[\int_{S^1} f_{j,k}^{\cos}\Psi_{j,k}^{\cos}(r\mathbf{e}) + f_{j,k}^{\sin}\Psi_{j,k}^{\sin}(r\mathbf{e})\,d\mathbf{e} \int_{S^1} \phi_{j',k'}^{\cos}\Psi_{j',k'}^{\cos}(r\mathbf{e}) + \phi_{j',k'}^{\sin}\Psi_{j',k'}^{\sin}(r\mathbf{e})\,d\mathbf{e}$$

$$- 2\pi \int_{S^1} \big(f_{j,k}^{\cos}\Psi_{j,k}^{\cos}(r\mathbf{e}) + f_{j,k}^{\sin}\Psi_{j,k}^{\sin}(r\mathbf{e})\big)\big(\phi_{j',k'}^{\cos}\Psi_{j',k'}^{\cos}(r\mathbf{e}) + \phi_{j',k'}^{\sin}\Psi_{j',k'}^{\sin}(r\mathbf{e})\big)\ d\mathbf{e}\Big]\,dr\,d\bar{\mathbf{v}}$$

Due to the line integrals along S^{d-1} the above equation simplifies to

$$\int_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v}$$

$$= \sum_{k\in 2\mathbb{N}}^{4N} \sum_{k'\in 2\mathbb{N}}^{2N} 4\pi^2 \int_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} \int_{\mathbb{R}^+} re^{-r^2} f_{0,k}^{\cos}\mathcal{L}_{\frac{k}{2}}^{(0)}(r^2)\phi_{0,k'}^{\cos}\mathcal{L}_{\frac{k'}{2}}^{(0)}(r^2)\,dr\,d\bar{\mathbf{v}}$$

$$- 2\pi^2(1+\delta_{0,j'}) \sum_{k=0}^{4N} \sum_{\substack{k'=0 \\ j'\leq[\frac{k'}{2}]}}^{2N} \int_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} \int_{\mathbb{R}^+} re^{-r^2}\big(f_{j',k}^{\cos}\phi_{j',k'}^{\cos} + f_{j',k}^{\sin}\phi_{j',k'}^{\sin}\big)\mathcal{L}_{[\frac{k}{2}]-j'}^{(2j')}(r^2)\mathcal{L}_{[\frac{k'}{2}]-j'}^{(2j')}(r^2)r^{4j'}\,dr\,d\bar{\mathbf{v}}$$

By the substitution $\tilde{r} = r^2$, yielding $2r\,dr = d\tilde{r}$ one obtains finally (tilde sign was removed):

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v} \\
&= 2\pi^2 \sum_{k\in 2\mathbb{N}}^{4N} \sum_{k'\in 2\mathbb{N}}^{2N} \int\limits_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} \int\limits_{\mathbb{R}^+} e^{-r} f^{\cos}_{0,k} \mathcal{L}^{(0)}_{\frac{k}{2}}(r) \phi^{\cos}_{0,k'} \mathcal{L}^{(0)}_{\frac{k'}{2}}(r)\,dr\,d\bar{\mathbf{v}} \\
&- \pi^2(1+\delta_{0,j'}) \sum_{k=0}^{4N} \sum_{\substack{k'=0 \\ j'\le[\frac{k'}{2}]}}^{2N} \int\limits_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} \int\limits_{\mathbb{R}^+} e^{-r} \left(f^{\cos}_{j',k}\phi^{\cos}_{j',k'} + f^{\sin}_{j',k}\phi^{\sin}_{j',k'} \right) \mathcal{L}^{(2j')}_{[\frac{k}{2}]-j'}(r) \mathcal{L}^{(2j')}_{[\frac{k'}{2}]-j'}(r) r^{2j'}\,dr\,d\bar{\mathbf{v}} \\
&= 2\pi^2 \sum_{k'\in 2\mathbb{N}}^{2N} \int\limits_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} f^{\cos}_{0,k'}\phi^{\cos}_{0,k'}\,d\bar{\mathbf{v}} - \pi^2(1+\delta_{0,j'}) \sum_{\substack{k'=0 \\ j'\le[\frac{k'}{2}]}}^{2N} \frac{(\frac{k'}{2}+j')!}{(\frac{k'}{2}-j')!} \int\limits_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} (f^{\cos}_{j',k}\phi^{\cos}_{j',k'} + f^{\sin}_{j',k'}\phi^{\sin}_{j',k'})\,d\bar{\mathbf{v}}
\end{aligned} \tag{25}$$

□

Lemma 8 plays an important role, when transforming $f^{(2)}$ to the Polar-Laguerre basis: Using a basis of order $4N$, clearly $f^{(2)}$ and the test functions are represented exactly. The contribution of a basis polynomial with total degree smaller than $4N$ to $f^{(2)}$ is the same, regardless whether a polar basis of order $4N$ or smaller is used. Thus, it is natural to use a Polar-Laguerre basis of order $2N$, since even if $f^{(2)}$ is not represented exactly by these polar polynomials (in contrast to the test functions, which can be expressed exactly), the effect of the collision is still exact.

3 Fast transformations of polynomial bases

As has been seen in Lemma 8, there is no collision effect on basis functions corresponding to a polynomial degree $\geq 2N$. The overall effect of the collision is the same if a transformation of $f^{(2)}$ to a Polar-Laguerre basis of order $4N$ or $2N$ is performed. Of course, the latter needs less operations in practice.

For speeding up the transformations, a second basis, $\mathbb{H}_K$ of the space span$\{\mathbb{L}_K\}$ is introduced. Let h_j denote the (1D) scaled Hermite polynomial of order j, such that $\|h_j\|_{L_2(\mathbb{R},e^{-v^2})} = 1$. By forming the $s+1$ multivariate polynomials of total order s

$$H_{s,j}(\mathbf{v}) := h_j(\mathbf{v}_x)h_{s-j}(\mathbf{v}_y), \qquad j = 0\dots s,$$

another basis of span$\{\mathbb{L}_K\}$ is given by

$$\mathbb{H}_K := \{H_{s,j}(\mathbf{v}) : \quad j = 0\dots s, s = 0\dots K\}.$$

These Hermite polynomials are now used as intermediate basis while transforming from the Lagrange to the Polar-Laguerre basis. The benefit of these Hermite polynomials is their hierarchical behaviour and the separated variables. The separated variables result in a cheap transformation from Lagrange basis to the Hermite basis by making use of the tensor product structures. The hierarchical behaviour yields sparse transformation matrices when transforming from Hermite to polar basis.

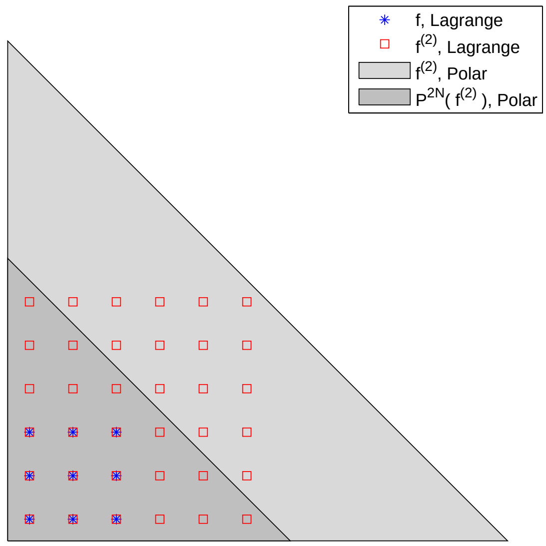

Figure 1: Polynomial orders for f, $f^{(2)}$ described in Lagrange basis and in a hierarchical basis. The red resp. blue markers represent the basis functions of partial order N resp. $2N$. The gray shaded domains correspond to the polynomial orders, when describing $f^{(2)}$ in polar resp. Hermite basis. The lighter gray domain corresponds to an exact representation, while the darker domain corresponds to an inexact representation of $f^{(2)}$ but is still exact for $\int_{\mathbb{R}^2} Q(f)\phi$, with $\phi \in P^{2N}$.

3.1 Nodal to Hermite Transformation

We start with $f^{(2)} = e^{-2|\hat{\mathbf{v}}|} \sum_{i=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_i \hat{L}_i(\hat{\mathbf{v}})$ and are looking for coefficients $h_{s,j}$, such that

$$f^{(2)}(\hat{\mathbf{v}}) = e^{-2|\hat{\mathbf{v}}|} \sum_{i=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_i \hat{L}_i(\hat{\mathbf{v}}) = e^{-2|\hat{\mathbf{v}}|} \sum_{\substack{k=0 \\ j \leq k}}^{2N} h_{k,j} H_{k,j}(\sqrt{2}\hat{\mathbf{v}})$$

Note that the Lagrange polynomials cannot be represented exactly by the Hermite polynomials of total degree smaller than $2N$ ($4N$ would fit). But as has been worked out before, the higher polynomial degrees would not cause any contribution to the collision integrals.
Now we use the L_2-orthogonal projection onto the space of polynomials of total degree smaller than $2N$ (with the Hermite polynomials forming the basis):

$$\sum_{i=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_i \int_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|} \hat{L}_i(\hat{\mathbf{v}}) H_{k_0,j_0}(\sqrt{2}\hat{\mathbf{v}})\, d\hat{\mathbf{v}} = \sum_{\substack{k=0 \\ j \leq k}}^{2N} h_{k,j} \int_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|} H_{k,j}(\sqrt{2}\hat{\mathbf{v}}) H_{k_0,j_0}(\sqrt{2}\hat{\mathbf{v}})\, d\hat{\mathbf{v}}$$

The right hand side turns into

$$\sum_{\substack{k=0\\j\leq k}}^{2N} h_{k,j} \int\limits_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|^2} H_{k,j}(\sqrt{2}\hat{\mathbf{v}}) H_{k_0,j_0}(\sqrt{2}\hat{\mathbf{v}})\, d\hat{\mathbf{v}} =$$

$$\sum_{\substack{k=0\\j\leq k}}^{2N} h_{k,j} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} e^{-2\hat{\mathbf{v}}_x^2-2\hat{\mathbf{v}}_y^2} h_{k-j}(\sqrt{2}\hat{\mathbf{v}}_y) h_j(\sqrt{2}\hat{\mathbf{v}}_x) h_{k_0-j_0}(\sqrt{2}\hat{\mathbf{v}}_x) h_{j_0}(\sqrt{2}\hat{\mathbf{v}}_x)\, d\hat{\mathbf{v}}_x\, d\hat{\mathbf{v}}_y =$$

$$\sum_{\substack{k=0\\j\leq k}}^{2N} h_{k,j} \underbrace{\int\limits_{\mathbb{R}} e^{-2\hat{\mathbf{v}}_x^2} h_j(\sqrt{2}\hat{\mathbf{v}}_x) h_{j_0}(\sqrt{2}\hat{\mathbf{v}}_x)\, d\hat{\mathbf{v}}_x}_{=\delta_{j,j_0}\frac{1}{\sqrt{2}}} \underbrace{\int\limits_{\mathbb{R}} e^{-2\hat{\mathbf{v}}_y^2} h_{k-j}(\sqrt{2}\hat{\mathbf{v}}_y) h_{k_0-j_0}(\sqrt{2}\hat{\mathbf{v}}_y)\, d\hat{\mathbf{v}}_y}_{=\delta_{k-j,k_0-j_0}\frac{1}{\sqrt{2}}} = \frac{1}{2} h_{k_0,j_0}$$

On the left hand side it is useful to factorise the sum via

$$\sum_{i=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_i \int\limits_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|} \hat{L}_i(\hat{\mathbf{v}}) H_{k_0,j_0}(\sqrt{2}\hat{\mathbf{v}})\, d\hat{\mathbf{v}} =$$

$$\sum_{m=0}^{2N} \sum_{\tilde{m}=0}^{2N} \hat{c}_{m(2N+1)+\tilde{m}} \underbrace{\int\limits_{\mathbb{R}} \hat{l}_m(\hat{\mathbf{v}}_x) h_{k_0-j_0}(\sqrt{2}\hat{\mathbf{v}}_x) e^{-2\hat{\mathbf{v}}_x^2}\, d\mathbf{v}_x}_{:=\mathrm{N2H}_{m,k_0-j_0}} \underbrace{\int\limits_{\mathbb{R}} \hat{l}_{\tilde{m}}(\hat{\mathbf{v}}_y) h_{j_0}(\sqrt{2}\hat{\mathbf{v}}_y) e^{-2\hat{\mathbf{v}}_y^2}\, d\mathbf{v}_y}_{:=\mathrm{N2H}_{\tilde{m},j_0}}$$

Now, by replacing the sum with an ”row · column”-wise product, one ends up with

$$\frac{1}{2} h_{k_0,j_0} = \sum_{m=0}^{2N} \mathrm{N2H}_{m,k_0-j_0} \begin{pmatrix} \mathrm{N2H}_{0,j_0}, \mathrm{N2H}_{1,j_0} \dots \mathrm{N2H}_{2N,j_0} \end{pmatrix} \begin{pmatrix} \hat{c}_{m(2N+1)} \\ \hat{c}_{m(2N+1)+1} \\ \vdots \\ \hat{c}_{m(2N+1)+2N} \end{pmatrix}$$

Going one step further, ordering the coefficients $\hat{c}_{m(2N+1)+\tilde{m}}$ into a matrix representation, the left hand side can be computed as a matrix · matrix · matrix-product:

$$\frac{1}{2} h_{k_0,j_0} = \left[\mathrm{N2H} \cdot \hat{c}^{\mathrm{mat}} \cdot \mathrm{N2H}^T\right]_{k_0-j_0,j_0},$$

where

$$\mathrm{N2H}_{i,j} = \int\limits_{\mathbb{R}} \hat{l}_j(v) h_i(\sqrt{2}v) e^{-2v^2}\, dv \qquad i,j = 0 \dots 2N$$

and

$$\hat{c}^{\mathrm{mat}}_{i,j} = \hat{c}_{i(2N+1)+j} \qquad i,j = 0 \dots 2N$$

Thus, by first calculating the matrix-matrix product $m := c_{\mathrm{mat}} \cdot \mathrm{N2H}^T$ and afterwards the product $\mathrm{N2H} \cdot m$, the floating point operations needed for the transformations are bounded by $\approx (2N)^3$.

Transformation of test functions:
Again, for the test functions $L_k, k = 0, \dots, n_{\mathbf{v}}$ the procedure is quite similar: We take an arbitrary Lagrange basis polynomial $l_i(\mathbf{v}_x)l_j(\mathbf{v}_y), 0 \le i, j \le N$ and write

$$l_i(\mathbf{v}_x)\, l_j(\mathbf{v}_y) = \sum_{\substack{k=0 \\ j \le k}}^{2N} h_{k,j} h_{k-j}(\sqrt{2}\mathbf{v}_y) h_j(\sqrt{2}\mathbf{v}_x)$$

Note, that the 2-dimensional index corresponds to coefficients, while the 1d to the Hermite polynomials. Now, testing both sides with $e^{-2|\mathbf{v}|^2} h_{k_0-j_0}(\sqrt{2}\mathbf{v}_y) h_{j_0}(\sqrt{2}\mathbf{v}_x)$ yields:

$$\int_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} h_{k_0-j_0}(\sqrt{2}\mathbf{v}_y) h_{j_0}(\sqrt{2}\mathbf{v}_x) l_i(\mathbf{v}_x)\, l_j(\mathbf{v}_y)\, d(\mathbf{v}_x,\, \mathbf{v}_y) =$$
$$\sum_{\substack{k=0 \\ j \le k}}^{2N} h_{k,j} \int_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} h_{k_0-j_0}(\sqrt{2}\mathbf{v}_y) h_{j_0}(\sqrt{2}\mathbf{v}_x) h_{k-j}(\sqrt{2}\mathbf{v}_y) h_j(\sqrt{2}\mathbf{v}_x)\, d(\mathbf{v}_x,\, \mathbf{v}_y)$$

As for $f^{(2)}$, the right hand side turns into

$$\sum_{\substack{k=0 \\ j \le k}}^{2N} h_{k,j} \int_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} h_{k_0-j_0}(\sqrt{2}\mathbf{v}_y) h_{j_0}(\sqrt{2}\mathbf{v}_x) h_{k-j}(\sqrt{2}\mathbf{v}_y) h_j(\sqrt{2}\mathbf{v}_x)\, d(\mathbf{v}_x,\, \mathbf{v}_y) = \frac{1}{2} h_{k_0,j_0},$$

resulting in

$$\frac{1}{2} h_{k_0,j_0} = \mathrm{N2H}_{j_0,i} \cdot \mathrm{N2H}_{k_0-j_0,j}$$

and therefore $l_m(\mathbf{v}_x)\, l_n(\mathbf{v}_y)$ has the representation

$$l_m(\mathbf{v}_x)\, l_n(\mathbf{v}_y) = \sum_{\substack{k=0 \\ j \le k}}^{2N} \mathrm{N2H}_{j,m} \cdot \mathrm{N2H}_{k-j,n} 2 h_{k-j}(\sqrt{2}\mathbf{v}_y) h_j(\sqrt{2}\mathbf{v}_x)$$

Summarizing, this can again be computed simultaneously:

$$l_m(\mathbf{v}_x)\, l_n(\mathbf{v}_y) = \left[\mathrm{N2H}^T \cdot H \cdot \mathrm{N2H}\right]_{m,n}$$

with

$$\mathrm{N2H}_{i,j} = \int_{\mathbb{R}} e^{-x^2} h_i(\sqrt{2}x) l_j(x)\, dx, \qquad i = 0, \dots, 2N,\ j = 0, \dots, N$$

and

$$H \in \mathbb{R}^{2N+1 \times 2N+1} \qquad H_{i,j} = \begin{cases} 2h_i(\sqrt{2}\mathbf{v}_x)\, h_j(\sqrt{2}\mathbf{v}_y) & i + j \le 2N \\ 0 & \text{else} \end{cases}$$

The transformation of the test functions is useful when having calculated the integrals in terms of the Hermite polynomials resp. Laguerre polynomials as test functions. The result within these test functions are then transformed to back, to the Lagrange polynomials as test functions.

The next step is now the transformation to the polar basis:

3.2 Hermite to Polar-Laguerre Transformation

With $f^{(2)}(\hat{\mathbf{v}}) = e^{-2|\hat{\mathbf{v}}|} \sum_{\substack{k=0 \\ j \le k}}^{2N} h_{k,j} H_{k,j}(\sqrt{2}\hat{\mathbf{v}})$, we're looking for coefficients $f_{j,k}^{\cos}$ resp. $f_{j,k}^{\sin}$ such that

$$f^{(2)}(\hat{\mathbf{v}}) = e^{-2|\hat{\mathbf{v}}|} \sum_{\substack{k=0 \\ j \le k}}^{2N} h_{k,j} H_{k,j}(\sqrt{2}\hat{\mathbf{v}}) = e^{-2r^2} \sum_{\substack{k=0 \\ j \le [\frac{k}{2}]}}^{2N} f_{j,k}^{\cos} \Psi_{j,k}^{\cos}(\sqrt{2}r\mathbf{e}) + f_{j,k}^{\sin} \Psi_{j,k}^{\sin}(\sqrt{2}r\mathbf{e}). \tag{26}$$

For this purpose – as already has been done for the Hermite polynomials – we scale the Laguerre polynomials $\mathcal{L}_n^j$ by a factor, such that $\|\mathcal{L}_n^j\|_{L_2(\mathbb{R}, e^{-v^2})} = 1,\ j = 1 \cdots [\frac{k}{2}]$. For $j = 0$ and even n, the scaling factor is chosen such that $\|\mathcal{L}_n^0\|_{L_2, e^{-v^2}} = \frac{1}{\sqrt{2}}$.
Now, formulating (26) in terms of a L_2 orthogonal projection one ends up with (tested with a $\Psi_{j_0,k_0}^{\cos/\sin},\ k_0 \le 2N$ basis function):

$$\begin{aligned}
&\sum_{\substack{k=0 \\ j \le k}}^{2N} h_{k,j} \int_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|} H_{k,j}(\sqrt{2}\hat{\mathbf{v}}) \Psi_{j_0,k_0}^{\cos/\sin}(\sqrt{2}r\mathbf{e})\, d\hat{\mathbf{v}} \\
&= \sum_{\substack{k=0 \\ j \le \left[\frac{k}{2}\right]}}^{2N} \int_{\mathbb{R}^2} e^{-2r^2} f_{j,k}^{\cos} \Psi_{j,k}^{\cos}(\sqrt{2}r\mathbf{e}) \Psi_{j_0,k_0}^{\cos/\sin}(\sqrt{2}r\mathbf{e}) + f_{j,k}^{\sin} \Psi_{j,k}^{\sin}(\sqrt{2}r\mathbf{e}) \Psi_{j_0,k_0}^{\cos/\sin}(\sqrt{2}r\mathbf{e})\, d\hat{\mathbf{v}}
\end{aligned} \tag{27}$$

Due to the orthogonality properties of the Polar Laguerre basis polynomials, (27) turns into

$$\sum_{\substack{k=0 \\ j \le k}}^{2N} h_{k,j} \int_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|} H_{k,j}(\sqrt{2}\hat{\mathbf{v}}) \Psi_{j_0,k_0}^{\cos/\sin}\, d\hat{\mathbf{v}} = \frac{\pi}{4} f_{j_0,k_0}^{\cos/\sin} \tag{28}$$

The next step consists of splitting the sum on the left hand side of (28) into

$$\begin{aligned}
&\underbrace{\sum_{\substack{k=0 \\ j \le k}}^{k_0-1} h_{k,j} \int_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|} H_{k,j}(\sqrt{2}\hat{\mathbf{v}}) \Psi_{j_0,k_0}^{\cos/\sin}(\sqrt{2}r\mathbf{e})\, d\hat{\mathbf{v}}}_{:=A} \\
&+ \underbrace{\sum_{\substack{k=k_0 \\ j \le k}} h_{k,j} \int_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|} H_{k,j}(\sqrt{2}\hat{\mathbf{v}}) \Psi_{j_0,k_0}^{\cos/\sin}(\sqrt{2}r\mathbf{e})\, d\hat{\mathbf{v}}}_{:=B} \\
&+ \underbrace{\sum_{\substack{k=k_0+1 \\ j \le k}}^{2N} h_{k,j} \int_{\mathbb{R}^2} e^{-2|\hat{\mathbf{v}}|} H_{k,j}(\sqrt{2}\hat{\mathbf{v}}) \Psi_{j_0,k_0}^{\cos/\sin}(\sqrt{2}r\mathbf{e})\, d\hat{\mathbf{v}}}_{:=C} = \frac{\pi}{4} f_{j_0,k_0}^{\cos/\sin}
\end{aligned} \tag{29}$$

For an analysis of A, use

$$H_{k,j}(\sqrt{2}\mathbf{v}) = \sum_{\substack{u=0 \\ v \le \left[\frac{u}{2}\right]}}^{k_0-1} c_{v,u}^{\cos} \Psi_{v,u}^{\cos}(\sqrt{2}\mathbf{v}) + c_{v,u}^{\sin} \Psi_{v,u}^{\sin}(\sqrt{2}\mathbf{v}).$$

Plugging this into A, using the orthogonality properties of the Polar Laguerre basis, one finds $A = 0$. For C we use

$$\Psi^{\cos/\sin}_{j_0,k_0}(\sqrt{2}\mathbf{v}) = \sum_{\substack{u=0\\ v\leq u}}^{k_0} c^{\cos/\sin}_{u,v} H_{u,v}(\sqrt{2}\mathbf{v})$$

Again as for A, the orthogonality properties of the Hermite polynomials yields $C = 0$. Thus, the coefficients $f^{\cos/\sin}_{k_0,j}$ in the polar basis, corresponding to polynomial degree k_0 depend only on the coefficients in the hermite basis corresponding to the same order k_0. Writing

$$\mathbf{f} = \mathrm{H2P}\cdot\mathbf{h} \qquad \text{with } \mathbf{f} = \begin{pmatrix} f^{\cos}_{0,0} \\ f^{\cos}_{0,1} \\ f^{\sin}_{0,1} \\ \vdots \\ f^{\cos}_{0,2N} \\ \vdots \\ f^{\cos}_{N,2N} \\ f^{\sin}_{N,2N} \end{pmatrix} \qquad \text{and } \mathbf{h} = \begin{pmatrix} h_{0,0} \\ h_{0,1} \\ h_{1,1} \\ \vdots \\ h_{0,2N} \\ h_{1,2N} \\ \vdots \\ h_{2N,2N} \end{pmatrix},$$

one finds the following structure in the matrix H2P:

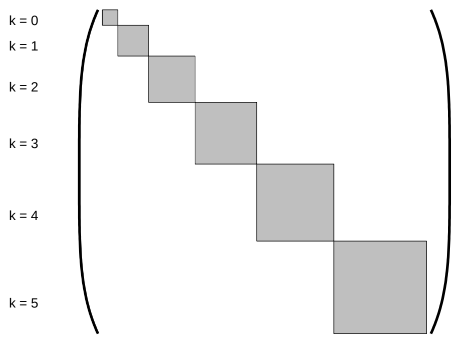

Figure 2: The transformation matrix H2P. The gray shaded blocks correspond to the none zero entries of the matrix. Due to the above result, one finds this special structure, highlighting again, the fact that coefficients corresponding to a specific order are only dependent on coefficients of the same order. The size of the block for order k is $(k+1)\times(k+1)$.

For a fixed order k_0, the corresponding block in the matrix H2P is given by (exemplary for even k_0):

$$
\mathrm{H2P}_k = \frac{4}{\pi}\begin{pmatrix}
\int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,0}(\sqrt{2}\mathbf{v})\Psi^{\cos}_{0,k_0} & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,1}(\sqrt{2}\mathbf{v})\Psi^{\cos}_{0,k_0} & \dots & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,k_0}(\sqrt{2}\mathbf{v})\Psi^{\cos}_{0,k_0} \\
\int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,0}(\sqrt{2}\mathbf{v})\Psi^{\cos}_{1,k_0} & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,1}(\sqrt{2}\mathbf{v})\Psi^{\cos}_{1,k_0} & \dots & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,k_0}(\sqrt{2}\mathbf{v})\Psi^{\cos}_{1,k_0} \\
\int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,0}(\sqrt{2}\mathbf{v})\Psi^{\sin}_{1,k_0} & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,1}(\sqrt{2}\mathbf{v})\Psi^{\sin}_{1,k_0} & \dots & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,k_0}(\sqrt{2}\mathbf{v})\Psi^{\sin}_{1,k_0} \\
\vdots & & & \\
\int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,0}(\sqrt{2}\mathbf{v})\Psi^{\cos}_{\frac{k_0}{2},k_0} & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,1}(\sqrt{2}\mathbf{v})\Psi^{\cos}_{\frac{k_0}{2},k_0} & \dots & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,k_0}(\sqrt{2}\mathbf{v})\Psi^{\cos}_{\frac{k_0}{2},k_0} \\
\int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,0}(\sqrt{2}\mathbf{v})\Psi^{\sin}_{\frac{k_0}{2},k_0} & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,1}(\sqrt{2}\mathbf{v})\Psi^{\sin}_{\frac{k_0}{2},k_0} & \dots & \int\limits_{\mathbb{R}^2} e^{-2|\mathbf{v}|^2} H_{k_0,k_0}(\sqrt{2}\mathbf{v})\Psi^{\sin}_{\frac{k_0}{2},k_0}
\end{pmatrix}
$$

With the matrix H2P, it is also possible to count the floating point operations performed for the Hermite-to-Polar transformation: For a fixed order k, the transformation can be written as a matrix $\cdot$ vector multiplication with dimension $k+1$. Thus, $(k+1)^2$ operations are performed for the transformation of the k-th order, resulting in $\sum_{k=0}^{2N}(k+1)^2 = \frac{2N(2N+1)(4N+1)}{6} \approx cN^3$ floating point operations for the transformation.

Transformation of the test functions:
In the next step we consider the transformation of the test functions in the Hermite basis to the Polar-Laguerre basis. There holds (exemplary for even k):

$$
\begin{pmatrix} H_{k,0}(\sqrt{2}\mathbf{v}) \\ H_{k,1}(\sqrt{2}\mathbf{v}) \\ \vdots \\ H_{k,k}(\sqrt{2}\mathbf{v}) \end{pmatrix} = \mathrm{H2P}_k^T \begin{pmatrix} \Psi^{\cos}_{0,k}(\sqrt{2}\mathbf{v}) \\ \Psi^{\cos}_{1,k}(\sqrt{2}\mathbf{v}) \\ \Psi^{\sin}_{1,k}(\sqrt{2}\mathbf{v}) \\ \vdots \\ \Psi^{\cos}_{\frac{k}{2},k}(\sqrt{2}\mathbf{v}) \\ \Psi^{\sin}_{\frac{k}{2},k}(\sqrt{2}\mathbf{v}) \end{pmatrix}
$$

Thus, testing with the Polar-Laguerre basis functions can be transferred back to testing with the Hermite polynomials.

3.3 Collision algorithm within N^3 operations

Now, we denote by Ψ_j $j = 0 \dots n_{\text{polar}}$ the Polar-Laguerre basis functions. $\mathbf{L}(\cdot)$ denotes the Lagrange test polynomials in a matrix representation as it is natural for the transformations, thus $\mathbf{L}(\cdot)_{m,n} = l_m(\cdot_x)l_n(\cdot_y)$.
By taking into account the already mentioned transformations, the collision operator can now be written as:

$$\int_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v} = \sum_{i=0}^{n_{\mathbf{v}}^{(2)}} \hat{c}_i \int_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} S^{\frac{\bar{\mathbf{v}}}{\sqrt{2}}^T} \int_{\mathbb{R}^2} e^{-|\hat{\mathbf{v}}|^2} \int_{S^1} \hat{L}_i(\tfrac{\hat{\mathbf{v}}}{\sqrt{2}}) \left[\mathbf{L}(\tfrac{\mathbf{e}'|\hat{\mathbf{v}}|}{\sqrt{2}}) - \mathbf{L}(\tfrac{\hat{\mathbf{v}}}{\sqrt{2}})\right]\, d\mathbf{e}' d\hat{\mathbf{v}} d\bar{\mathbf{v}}$$

$$= 2\sum_{j=0}^{n_{\text{polar}}} f_j \int_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} S^{\frac{\bar{\mathbf{v}}}{\sqrt{2}}^T} \int_{\mathbb{R}^2} e^{-|\hat{\mathbf{v}}|^2} \int_{S^1} \Psi_j(\hat{\mathbf{v}}) \cdot \text{N2H}^T \cdot \text{H2P}^T \cdot \left[\mathbf{\Psi}(\mathbf{e}'|\hat{\mathbf{v}}|) - \mathbf{\Psi}(\hat{\mathbf{v}})\right] \cdot \text{N2H}\, d\mathbf{e}' d\hat{\mathbf{v}} d\bar{\mathbf{v}}$$

$$= 2\sum_{j=0}^{n_{\text{polar}}} f_j \int_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} S^{\frac{\bar{\mathbf{v}}}{\sqrt{2}}^T} \cdot \text{N2H}^T \cdot \text{H2P}^T \cdot \underbrace{\int_{\mathbb{R}^2} e^{-|\hat{\mathbf{v}}|^2} \int_{S^1} \Psi_j(\hat{\mathbf{v}}) \left[\mathbf{\Psi}(\mathbf{e}'|\hat{\mathbf{v}}|) - \mathbf{\Psi}(\hat{\mathbf{v}})\right]\, d\mathbf{e}' d\hat{\mathbf{v}}}_{:=Q_{\text{polar}}(f)(\bar{\mathbf{v}}) \in \mathbb{R}^{n_{\text{polar}} \times n_{\text{polar}}}} \cdot \text{N2H}\, d\bar{\mathbf{v}}$$

$$= 2\int_{\mathbb{R}^2} e^{-|\bar{\mathbf{v}}|^2} S^{\frac{\bar{\mathbf{v}}}{\sqrt{2}}^T} \cdot \text{N2H}^T \cdot \text{H2P}^T \cdot Q_{\text{polar}}(f)(\bar{\mathbf{v}})\, \hat{\mathbf{f}} \cdot\ \text{N2H}\, d\bar{\mathbf{v}}$$

Now, the integration with respect to $\bar{\mathbf{v}}$ is evaluated by a Gauss-Hermite quadrature rule. Thus, by using the notation from the beginning, this results in:

$$\int_{\mathbb{R}^2} Q(f)\phi(\mathbf{v})\,d\mathbf{v} = 2\sum_{\text{ip}=0}^{n_{\text{ip}}} \omega_{ip,2} S^{\frac{x_{\text{ip},2}}{\sqrt{2}}^T} \cdot \text{N2H}^T \cdot \text{H2P}^T \cdot Q_{\text{polar}}(f)(x_{\text{ip},2})\, \hat{\mathbf{f}} \cdot\ \text{N2H}$$

Each summand can be evaluated within $\mathcal{O}(N^3)$ operations, thus the total effort is $\mathcal{O}(n_{\text{ip}}N^3)$.

Remark 9. *To have better approximation properties, one can make use of the macroscopic unknown quantities $\mathbf{V}$ and T, by expanding f via:*

$$f_{T,\mathbf{V}}(\mathbf{v}) = e^{-\frac{|\mathbf{v}-\mathbf{V}|^2}{T}} \sum_m c_m L_m(\tfrac{\mathbf{v}-\mathbf{V}}{\sqrt{T}})$$

In addition let

$$f_0(\mathbf{v}) := f_{T,\mathbf{V}}(\sqrt{T}\mathbf{v} + \mathbf{V}) = e^{-|\mathbf{v}|^2} \sum_m c_m L_m(\mathbf{v}).$$

and also for the test functions let $\phi_{T,\mathbf{V}}(\mathbf{v}) = L_j(\frac{\mathbf{v}-\mathbf{V}}{\sqrt{T}})$ and $\phi_0 = L_j(\mathbf{v})$.
Then for the corresponding mass integrals one arrives with:

$$\int_{\mathbb{R}^2} f_{T,\mathbf{V}}(\mathbf{v})\phi_{T,\mathbf{V}}(\mathbf{v})\,d\mathbf{v} = T\int_{\mathbb{R}^2} f_0(\mathbf{v})\phi_0(\mathbf{v})\,d\mathbf{v}$$

Similar, by making use of the transformations $\mathbf{v} = \sqrt{T}\tilde{\mathbf{v}} + \mathbf{V}$ and $\mathbf{w} = \sqrt{T}\tilde{\mathbf{w}} + \mathbf{V}$, one finds

$$\mathbf{v}' = \sqrt{T}\underbrace{\left(\frac{\tilde{\mathbf{v}} + \tilde{\mathbf{w}}}{2} + \mathbf{e}'\frac{|\tilde{\mathbf{v}} - \tilde{\mathbf{w}}|}{2}\right)}_{=\tilde{\mathbf{v}}'} + \mathbf{V}.$$

Therefore, for the entries of the collision operator there holds:

$$\int_{\mathbb{R}^2} Q(f_{T,\mathbf{V}})\phi_{T,\mathbf{V}}(\mathbf{v})\,d\mathbf{v} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{S^1} f_{T,\mathbf{V}}(\mathbf{v}) f_{T,\mathbf{V}}(\mathbf{w})[\phi_{T,\mathbf{V}}(\mathbf{v}') - \phi_{T,\mathbf{V}}(\mathbf{v})]\,d\mathbf{e}'\,d\mathbf{v}\,d\mathbf{w}$$

$$= T^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{S^1} f_0(\mathbf{v}) f_0(\mathbf{w})[\phi_0(\mathbf{v}') - \phi_0(\mathbf{v})]\,d\mathbf{e}'\,d\mathbf{v}\,d\mathbf{w},$$

So both, mass and collision simply are scaled by a factor T resp. T^2.

4 Basic properties

Moment conservation:
By making use of the basic collision invariants $\phi_0(\mathbf{v}) \equiv 1$, $\phi_1(\mathbf{v}) = \mathbf{v}_x$, $\phi_2(\mathbf{v}) = \mathbf{v}_y$ and $\phi_3(\mathbf{v}) = |\mathbf{v}|^2$, one arrives with the moment equations for the Boltzmann equation: There holds

$$\frac{\partial}{\partial t} \underbrace{\int_{\mathbb{R}^2} f}_{:=\rho(t)} = 0$$

$$\frac{\partial}{\partial t} \underbrace{\frac{1}{\rho(t)} \int_{\mathbb{R}^2} \mathbf{v} f}_{:=\mathbf{V}(t)} = 0$$

$$\frac{\partial}{\partial t} \underbrace{\frac{1}{\rho(t)} \int_{\mathbb{R}^2} |\mathbf{v} - \mathbf{V}|^2 f}_{:=T(t)} = 0,$$

The quantities are the mean particle density, the mean bulk velocity and last but not least the mean temperature, proportional to the mean energy of the gas. Within our discretization we also conserve these quantities, what is simply a conclusion of having the basic collision invariant functions within the test functions:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} \int_{\mathbb{R}^2} f\, 1 = \int_{\mathbb{R}^2} Q(f)\, 1\, d\mathbf{v} = \sum_{\text{ip}} \omega_{\text{ip}} \int_{\mathbb{R}+} r \Big[\int_{S^1} f^{(2)}(r\mathbf{e})\, d\mathbf{e} \int_{S^1} 1 - 2\pi \int_{S^1} f^{(2)}(r\mathbf{e}) \Big] dr = 0,$$

and thus the total mass is conserved over time. More over by using $\phi = \mathbf{v}_{x/y}$ one gets:

$$\frac{\partial \left(\rho \mathbf{V}_i\right)}{\partial t} = \frac{\partial}{\partial t} \int_{\mathbb{R}^2} f\, \mathbf{v}_i = \int_{\mathbb{R}^2} Q(f)\, \mathbf{v}_i\, d\mathbf{v} = \sum_{\text{ip}} \omega_{\text{ip}} \int_{\mathbb{R}+} r \Big[\int_{S^1} f^{(2)}(r\mathbf{e})\, d\mathbf{e} \int_{S^1} r\mathbf{e}_i - 2\pi \int_{S^1} f^{(2)}(r\mathbf{e}) r\mathbf{e}_i \Big] dr$$

Now, notice that the first part consisting of the factor $\int_{S^1} r\mathbf{e}_i = 0$ vanishes. For the second part notice that $g(r, \mathbf{e}) = f^{(2)}(r\mathbf{e}) r\mathbf{e}_i)$ satisfies $g(r, \mathbf{e}) = -g(r, -\mathbf{e})$, and thus also $\int_{S^1} g(r, \mathbf{e})\, d\mathbf{e} = 0$ and finally

$$\frac{\partial \left(\rho \mathbf{V}_i\right)}{\partial t} = 0$$

is obtained. Since $\rho(t) \equiv \rho_0$ this immediately yields $\rho_0 \frac{\partial \mathbf{V}_i}{\partial t} = 0$ resp.

$$\frac{\partial \mathbf{V}}{\partial t} = 0.$$

Now, we use $|\mathbf{v} - \mathbf{V}|^2$ as a test function

$$\frac{\partial \rho T}{\partial t} = \frac{\partial}{\partial t} \int_{\mathbb{R}^2} |\mathbf{v} - \mathbf{V}|^2 f = \frac{\partial}{\partial t} \Big(\int_{\mathbb{R}^2} |\mathbf{v}|^2 f - \int_{\mathbb{R}^2} |\mathbf{V}|^2 f \Big) = \int_{\mathbb{R}^2} Q(f)\, |\mathbf{v}|^2 - \underbrace{\int_{\mathbb{R}^2} Q(f) |\mathbf{V}|^2\, d\mathbf{v}}_{=0}$$

$$= \sum_{\text{ip}} \omega_{\text{ip}} \int_{\mathbb{R}+} r \Big[\int_{S^1} f^{(2)}(r\mathbf{e})\, d\mathbf{e} \int_{S^1} r^2 - 2\pi \int_{S^1} f^{(2)}(r\mathbf{e}) r^2 \Big] dr = 0$$

to finally end up with conservation of energy resp. temperature.

Remark 10. *The conservation properties correspond to the fact, that the Boltzmann collision invariants are also collision invariants within the discrete level.*

H-theorem: For the Boltzmann collision operator exists the famous H-theorem, stating that

$$\int_{\mathbb{R}^2} Q(f)\log(f) \leq 0$$

holds for all suitable functions $f > 0$. This is usually done by using (13) (see [1] for instance). Basically the same techniques are applicable to our reduced model of the collision integral and yield the following result:

$$\int_{\mathbb{R}^2} Q_N(f)\log(f) \leq 0$$

for all suitable $f > 0$.

5 Implementation remarks

An implementation of the proposed method requires some additional, careful treatment of the transformations. It turns out, that in the presented form, the described transformation matrices are very ill conditioned. To overcome this problem we reformulated the transformations in terms of the coefficients $\mathbf{d}$: Let

$$f(\mathbf{v}) = e^{-|\mathbf{v}|^2} \sum_{m=0}^{n_{\mathbf{v}}} c_m\, L_m(\mathbf{v}),$$

then, the coefficients d_j are given by $d_j := c_j e^{-|\mathbf{v}|^2/2}$. Moreover, the Hermite polynomials h_j have to be replaced by the Hermite functions [10]:

$$h_j^f(v) = e^{-v^2/2} h_j(v),$$

where h_j^f denotes the Hermite functions.
In addition, the shift matrices have a better condition if the Lagrange polynomials of degree $2N$ are defined via collocation nodes of the form $\hat{y}_{j,d} := \frac{\hat{x}_{j,d}}{\sqrt{2}}$. Otherwise a calculation of the shifting matrices would end up in the evaluation of the Lagrange polynomials of order N outside of the initial collocation points:

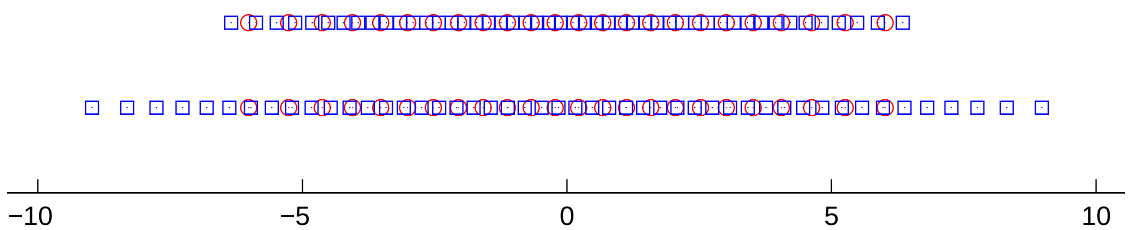

Figure 3: **Upper line:** Collocation nodes for order N (red) and scaled nodes of order $2N$ (blue).
Lower line: Collocation nodes for order N (red) and order $2N$ (blue).

6 Numerical results

For this example, an initial distribution of the form

$$f_0(\mathbf{v}) = \frac{1}{2\pi T_0} e^{-\frac{\left(\mathbf{v}_x - \frac{d}{2}\right)^2 + \mathbf{v}_y^2}{T_0}} + \frac{1}{2\pi T_0} e^{-\frac{\left(\mathbf{v}_x + \frac{d}{2}\right)^2 + \mathbf{v}_y^2}{T_0}}$$

was used. By moment conservation within the collision integrals, the temperature T_∞ of the final gauss distribution

$$f_\infty(\mathbf{v}) = \frac{1}{\pi T_\infty} e^{-\frac{|v|^2}{T_\infty}}$$

is given by $T_\infty = \frac{d^2}{4} + T_0$. The distance of the initial peaks is given by $d = 2$, the initial temperature of the peaks is $T_0 = \frac{1}{5}$.
Referring to remark 9, f is expanded as

$$f = e^{-\frac{|\mathbf{v}|^2}{T_\infty}} \sum_m c_m L_m\left(\frac{\mathbf{v}}{\sqrt{T_\infty}}\right)$$

The following results were obtained by using 2500 degrees of freedom, corresponding to a polynomial degree of 49 for both directions, and 3 integration points w.r.t. $\bar{\mathbf{v}}$ for each Cartesian direction. Time integration was performed by a simple forward euler scheme:

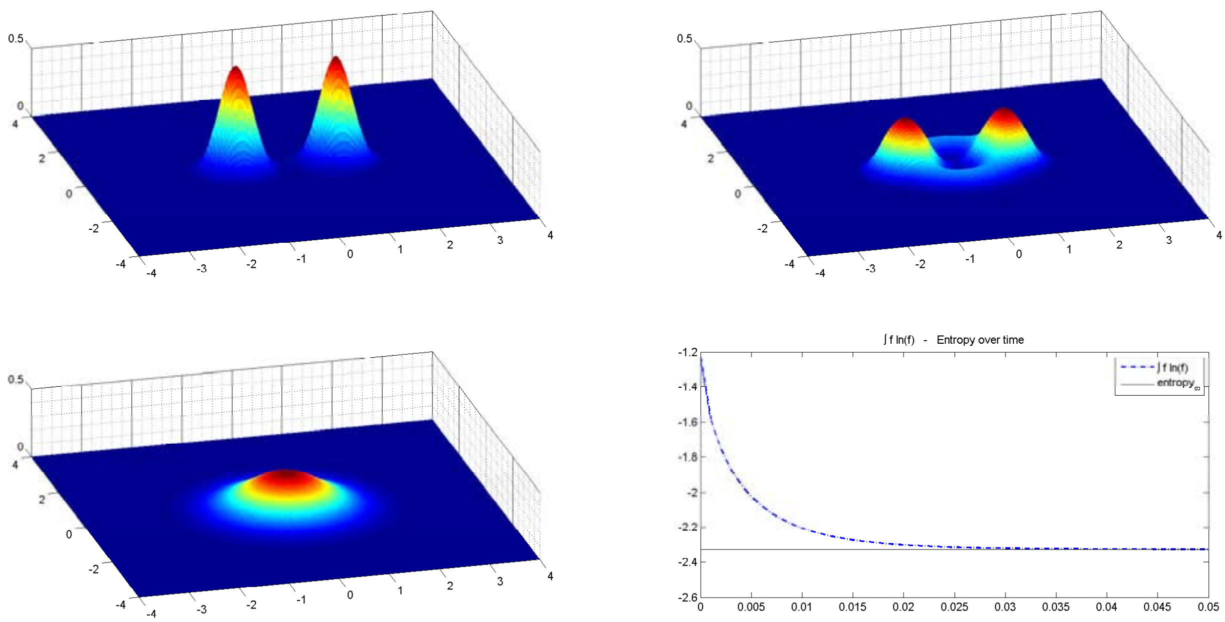

Figure 4: **Upper left:** Initial distribution as described above. **Upper right:**The distribution immediately after the simulation begin. The gain along the circle is due to mass, momentum and energy conservation during the binary collisions and the specific shape of the initial values. **Lower left:** Gauss function as a stationery solution. **Lower right:** Entropy over time.

The above results were also obtained by using higher order integration rule w.r.t $\bar{\mathbf{v}}$. The figure shows the entropy for different orders of these integration formulars:

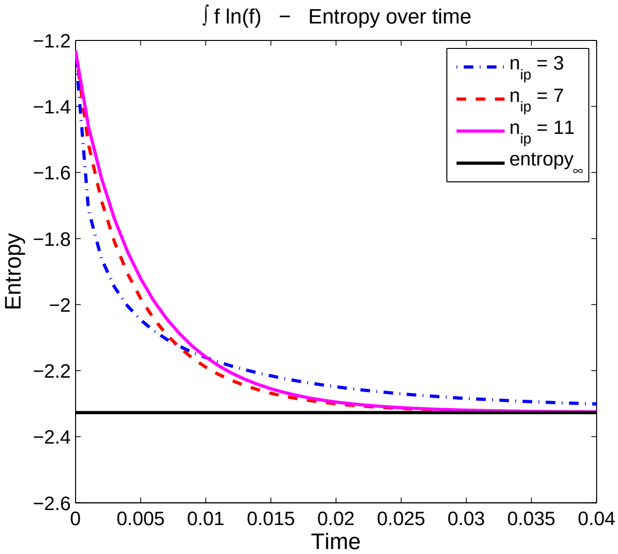

Figure 5: shows the entropies for different orders of integration w.r.t $\bar{\mathbf{v}}$. The chosen quadrature rules are of Gauss-Hermite type. The number n_{ip} corresponds to the number of integration points for each Cartesian direction, thus the total integration points are given by n_{ip}^2. As can be seen, all entropies converge to the same stationary value, the curves for 11 resp. 15 points are even hard to distinguish. Moreover, all of them are bounded from below by the entropy of the Gauss function with the same mean density, velocity and energy.

The behavior of the L_2-error $\|f_{\text{nip}} - f_\infty\|_{L_2}$ can be seen in the following figure. Here f_∞ denotes a reference solution and f_{nip} is a solution obtained by using an integration formular of order nip for the $\bar{\mathbf{v}}$ integral. Moreover also the error in the corresponding entropy functionals is displayed. The calculation of the "exact" solution f_∞ was also performed with the same formulation of the collision integral, but with a much higher integration order of $\text{nip} = 25$.

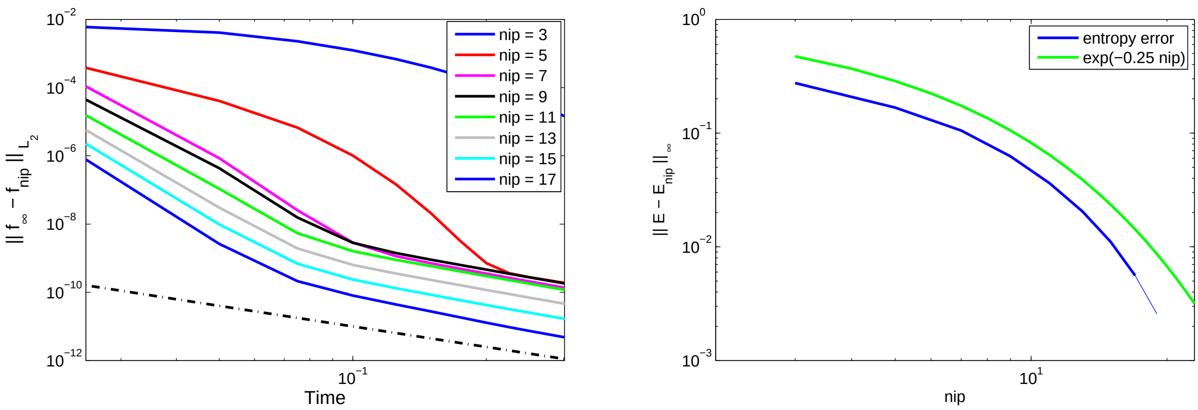

Figure 6: **Left:** shows the L_2 norm of the error $e := f_{\text{nip}} - f_\infty$. The black dotted line corresponds to the function $\frac{1}{t^2}$. Thus, we expect quadratic (w.r.t. to time) convergence for the solution function itself. **Right:** shows the ∞ norm of the entropy error for different numbers of nip. For the entropy we expect due to this result exponential convergence.

A comparison of the performance of the reduced algorithm for the collision integral with the straight forward discretization is presented in the next figure:

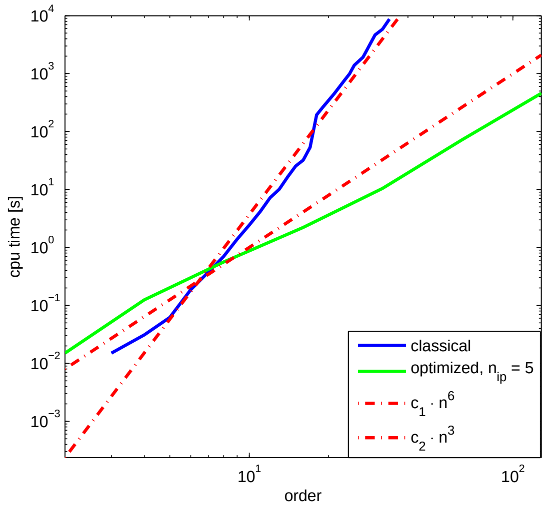

Figure 7: shows a comparison of the CPU-times for the different versions of the collision operator: The blue curve corresponds to the time consumed by the straight forward discretization for 1000 applications. The green curve on the other hand corresponds to the improved algorithm (order 3 quadrature w.r.t. $\bar{\mathbf{v}}$). The blue curve displays an $\mathcal{O}(N^6)$ asymptotic, the green one instead displays only an $O(N^3)$ asymptotic. Within polynomial order 8, the optimized algorithm is already faster than the straight forward discretization.

References

[1] Sergej Rjasanow and Wolfgang Wagner. *Stochastic numerics for the Boltzmann equation*, volume 37 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[2] Carlo Cercignani. *Mathematical methods in kinetic theory*. Plenum Press, New York, second edition, 1990.

[3] Carlo Cercignani, Reinhard Illner, and Mario Pulvirenti. *The mathematical theory of dilute gases*, volume 106 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[4] A. V. Bobylëv. The method of the Fourier transform in the theory of the Boltzmann equation for Maxwell molecules. *Dokl. Akad. Nauk SSSR*, 225(6):1041–1044, 1975.

[5] A. Bobylev and S. Rjasanow. Difference scheme for the Boltzmann equation based on the fast Fourier transform. *European J. Mech. B Fluids*, 16(2):293–306, 1997.

[6] Lorenzo Pareschi and Benoit Perthame. A fourier spectral method for homogeneous boltzmann equations. *Transport Theory and Statistical Physics*, 25(3-5):369–382, 1996.

[7] A. V. Bobylev and S. Rjasanow. Fast deterministic method of solving the Boltzmann equation for hard spheres. *Eur. J. Mech. B Fluids*, 18(5):869–887, 1999.

[8] C. Buet. A discrete-velocity scheme for the Boltzmann operator of rarefied gas dynamics. *Transport Theory Statist. Phys.*, 25(1):33–60, 1996.

[9] Vladislav A. Panferov and Alexei G. Heintz. A new consistent discrete-velocity model for the Boltzmann equation. *Math. Methods Appl. Sci.*, 25(7):571–593, 2002.

[10] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.