Existence and Uniqueness of Solutions to Weakly Singular Integral-Algebraic and Integro-Differential Equations

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Existence and Uniqueness of Solutions to Weakly Singular Integral-Algebraic and Integro-Differential Equations

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Abstract

In this article, we consider systems of integral-algebraic and integro-differential equations with weakly singular kernels. Although these problem classes are not in the focus of the main stream literature, they are interesting, not only in their own right, but also because they may arise from the analysis of certain classes of differential-algebraic systems of partial differential equations. In the first part of the paper, we deal with two-dimensional integral-algebraic equations. Next, we analyze Volterra integral equations of the first kind in which the determinant of the kernel matrix $k(t, x)$ vanishes when $t = x$. Finally, the third part of the work is devoted to the analysis of degenerate integro-differential systems. The aim of the paper is to specify conditions which are sufficient for the existence of a unique continuous solution to the above problems. Theoretical findings are illustrated by a number of examples.

Key words: weakly singular, two-dimensional Volterra integral-algebraic equations, integro-differential equations
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1. Introduction

The formulation of many mathematical problems includes sets of differential equations or integral Volterra equations, of first and second kind. By combining such equations one obtains systems of integro-differential equations whose leading coefficient matrix can become singular. Such systems are called degenerate integro-differential equations (DIDE). When they do not include integral or differential equations, they are called differential-algebraic equations (DAE) or integral-algebraic equations (IAE), respectively. While the qualitative theory of DAE is widely available (see [2], [17], [18], [14], [16]), there exist fewer results for IAE and DIDE (see, [3], [5], [6], [7], [15] and references therein).

In the present paper, we investigate the existence and uniqueness of solutions of some classes of IAE, DIDE, and systems of Volterra equations of the first kind with weakly singular kernel. This study is based on some special properties of matrix polynomials and pencils. As far as we are aware, such results have not been published yet. In Section 2, we analyze two-dimensional IAE with a weakly singular kernel. Similar equations with regular kernel were analyzed in [6]. Results for the one-dimensional case, were published in [4], in a paper which is unfortunately, not widely available. In Section 3, Volterra integral equations of the first kind (VIEFK) are considered, in which the determinant of the kernel matrix $k(t, x)$ vanishes as $t = x$. The case where the determinant of the kernel is not identically zero on the diagonal has been studied, for example, in [5], [20] and in the references therein. Here, we formulate sufficient conditions for the problem data, which guarantee the existence of a unique solution in the degenerate case.

In Section 4, we consider an initial value problem for an integro-differential system, where the coefficient matrix $A(t)$ in front of the derivative is singular (in the sense of rank-deficient) for all t and the kernel of the integral part is

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weakly singular. Here, we also derive sufficient conditions for the existence of a unique solution. The present investigation is based on the structure of the matrix polynomial $\lambda^2 A(t) + \lambda B(t) + K(t, t)$. A similar analysis for DIDE with regular kernel was carried out in [7]. Finally, conclusions complete the paper in Section 5.

We now introduce notation used throughout the paper which enables one to recast the different problem classes considered here into a unified framework.

For a given two-variable matrix function k, with $k(t, \tau)$ continuous for $0 \leq \tau \leq t < \infty$, we denote by K^α the Abel-type integral operators such that

$$K^\alpha u(t) = \int_0^t \frac{k(t, \tau) u(\tau)}{(t - \tau)^\alpha} d\tau. \tag{1}$$

Analogously, for $k(t, x, s, \tau)$ being a four-variable matrix function, we denote by $K^{\alpha, \beta}$ the integral operator such that

$$K^{\alpha, \beta} u(t, x) = \int_0^t \int_0^x \frac{k(t, x, \tau, s) u(s, \tau)}{(t - \tau)^\alpha (x - s)^\beta} ds\, d\tau. \tag{2}$$

In context of this notation, all systems considered in the present article, can be viewed as particular cases of the general equation,

$$A_0 \, Du + A_1 \, u + K^{\alpha, \beta} u = f, \tag{3}$$

where A_0, A_1 are given matrix functions and D represents a differential operator. The functions u and f are defined on a one- or two-dimensional domain. In Section 2 we consider the case where $A_0(t, x) \equiv 0$ and in Section 3, the case where $A_0(t, x) \equiv A_1(t, x) \equiv 0$. Section 4 is devoted to the case where all terms of equation (3) do not vanish.
Throughout the paper we assume that the solution u is continuous on the considered domain. The continuity assumptions for f are specified for each problem setting in the respective theorems.

2. Two-dimensional integral-algebraic equations with weakly singular kernel

Here, we consider the integral equation of the form

$$A(t, x) u(t, x) + K^{\alpha, \beta} u(t, x) = f(t, x), \tag{4}$$

defined on the rectangular domain

$$\Omega = \{0 \le \tau \le t \le a, \ \ 0 \le s \le x \le b\},$$

where $A(t, x)$ is a real valued $n \times n$ matrix, $f(t, x)$ is a given vector valued function and $K^{\alpha,\beta}$ is specified in (2), $0 < \alpha, \beta < 1$.
In the sequel, we analyze systems of the form (4) satisfying the condition

$$\det A(t, x) = 0, \quad \forall \, (t, x) \in \Omega.$$

Such problems are called two-dimensional Volterra integral-algebraic equations. Any continuous vector function $u(t, x)$, which satisfies (4) is a solution of the problem.

The considered problem class is essentially different from the systems of integral equations of the second kind with weak singularities. In the standard theory, if $\det A(t, x)$ vanishes at a certain point of Ω, such a point is called *singular* in the sense that there may be infinitely many solutions passing through this point. This is not the case here. In points where the rank of A changes, the solution may stay unique, as it is illustrated in the following example.

Example 2.1. *Consider the system*

$$\begin{pmatrix} \phi(t, x) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix} +$$

$$\int_0^t \int_0^x (t - \tau)^{-\alpha} (x - s)^{-\beta} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} u_1(\tau, s) \\ u_2(\tau, s) \end{pmatrix} ds d\tau = \begin{pmatrix} 0 \\ f(t, x) \end{pmatrix},$$

(5)

Case (i): If $k_{21} = 0$, $k_{22} \ne 0$ and $f(0, x) \ne 0$ or $f(t, 0) \ne 0$ this problem doesn't possess any continuous solution, since in this case the second component satisfies the following integral equation of the first kind:

$$\int_0^t \int_0^x (t - \tau)^{-\alpha} (x - s)^{-\beta} u_2(\tau, s) ds d\tau = f(t, x),$$

which has no continuous solution.
If $k_{11} = k_{22} = 1$, $k_{21} = k_{12} = 0$ and $\phi(t, x)$ vanishes at certain points $(t, x) \in \Omega$, then at these points there may be more than one solution. For

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example, for $f(t, x) = 0$ and

$$\phi(t, \tau) = \frac{-t^{1-\alpha}x^{1-\beta}}{(1-\alpha)(1-\beta)},$$

the function u_1 satisfies the integral equation

$$\frac{-t^{1-\alpha}x^{1-\beta}}{(1-\alpha)(1-\beta)}u_1(t, x) + \int_0^t \int_0^x (t-\tau)^{-\alpha}(x-s)^{-\beta}u_2(\tau, s)dsd\tau = 0,$$

which has infinitely many solutions, $u_1(t, x) = c$, where c is an arbitrary constant. Therefore system (5) has the general solution $u_1(t, x) = c$, $u_2(t, x) = 0$. Case (ii): If $k_{11} = k_{22} = 0$, $k_{21} = k_{12} = 1$ and $\phi(t, x), f(t, x)$ are smooth functions, then system (5) has the form

$$\begin{pmatrix} \phi(t, x) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix} +$$

$$\int_0^t \int_0^x (t-\tau)^{-\alpha}(x-s)^{-\beta} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1(\tau, s) \\ u_2(\tau, s) \end{pmatrix} dsd\tau = \begin{pmatrix} 0 \\ f(t, x) \end{pmatrix}.$$

To the second equation of the above system, we now apply the formula for fractional order differentiation, see [20] (Chapter 5), and obtain

$$u_1(t, x) = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\partial}{\partial t} \frac{\partial}{\partial x} \int_0^t \int_0^x \frac{f(\tau, s)}{(t-\tau)^{1-\alpha}(x-s)^{1-\beta}}dsd\tau,$$

where we denote by $\Gamma(\cdot)$ the gamma function. The first equation reads

$$\phi(t, x)u_1(t, x) + \int_0^t \int_0^x (t-\tau)^{-\alpha}(x-s)^{-\beta}u_2(\tau, s)dsd\tau = 0,$$

and we again apply the formula for fractional order differentiation to deduce

$$u_2(t, x) = -\frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\partial}{\partial t} \frac{\partial}{\partial x} \int_0^t \int_0^x \frac{\phi(\tau, s)u_1(\tau, s)}{(t-\tau)^{1-\alpha}(x-s)^{1-\beta}}dsd\tau.$$

This means that the existence and uniqueness of solution of system (5) does not depend on whether $\phi(t, x)$ vanishes at certain points of Ω; it merely depends on the convergence of the involved improper integrals.

2.1. Auxiliary results

We first list definitions and auxiliary results used in the following sections.

Definition 2.1. *[13] The expression $\lambda A(t, x) + B(t, x)$, where $A(t, x)$ and $B(t, x)$ are square matrices of the same dimension and λ is a scalar, is called a matrix pencil.*

Definition 2.2. *[13]. The matrix pencil $\lambda A(t, x) + B(t, x)$ is said to be regular if*

$$\det(\lambda A(t, x) + B(t, x)) \neq 0, \quad \forall\, (t, x) \in \Omega.$$

Definition 2.3. *[12]. We say that the matrix pencil $\lambda A(t, x) + B(t, x)$ satisfies the rank-degree criterion (has index one or has a simple structure) on the domain Ω, if*

$$\operatorname{rank} A(t, x) = k = \operatorname{const}$$

and

$$\det\left(\lambda A(t, x) + B(t, x)\right) = a_0(t, x)\lambda^k + a_1(t, x)\lambda^{k-1} + \cdots + a_k(t, x),$$

where $a_0(t, x) \neq 0, \forall\, (t, x) \in \Omega$.

Lemma 2.1. *Let us denote by $C^p_{n \times n}([0, a] \times [0, b])$ the space of $n \times n$ matrices $A(t, x)$, where $(t, x) \in [0, a] \times [0, b]$, whose entries are p times continuously differentiable on $[0, a] \times [0, b]$. Provided that $A \in C^p_{n \times n}([0, a] \times [0, b])$ and $\operatorname{rank} A(t, x) = k = \operatorname{const}$, there exists a $n \times n$ matrix $P(t, x)$ such that $\det P(t, x) \neq 0, \forall\, (t, x) \in \Omega$, $P(x, t) \in C^p([0, a] \times [0, b])$ and*

$$P(t, x)A(t, x) = \begin{pmatrix} A_1(t, x) \\ 0 \end{pmatrix}, \tag{6}$$

where $A_1(t, x)$ is a $k \times n$ matrix, $\operatorname{rank} A_1(t, x) = k = \operatorname{const}, \forall\, (t, x) \in \Omega$, and 0 is a $(n - k) \times n$ zero matrix.

The proof of this lemma is given in [11]. Since this work is not easily available, we recall here the main ideas of this proof. In the first part, one shows that for any $c \in [0, a] \times [0, b]$ there exists a ball, $B(c, \delta) \subset [0, a] \times [0, b]$, in which a certain matrix $P(t, x)$ is defined, such that $\det P(t, x) \neq 0$ and (6) holds, where $A_1(t, x)$ is a $k \times n$ matrix and $\operatorname{rank} A_1(t, x) = k = \operatorname{const}$ in $B(b, \delta)$. Then, since this result is of a local nature, we prove that there exists a matrix $P(t, x) \in C^p([0, a] \times [0, b])$, satisfying the prescribed conditions.

Lemma 2.2. *[10] If the matrix pencil*

$$\lambda \begin{pmatrix} A_1(t,x) \\ 0 \end{pmatrix} + \begin{pmatrix} B_1(t,x) \\ B_2(t,x) \end{pmatrix}, \tag{7}$$

where $A_1(t,x), B_1(t,x)$ are $k \times n$ matrices, $B_2(t,x)$ is a $(n-k) \times n$ matrix, satisfies the rank-degree criterion, then

$$\det \begin{pmatrix} cA_1(t,x) \\ dB_2(t,x) \end{pmatrix} \neq 0, \ \forall \, (t,x) \in \Omega, \ c \neq 0, \ d \neq 0.$$

Proof. We have to show that

$$\det \begin{pmatrix} A_1(t,x) \\ B_2(t,x) \end{pmatrix} \neq 0, \quad \forall (t,x) \in \Omega.$$

To this aim, we represent the matrix pencil (7) in the form

$$\lambda \begin{pmatrix} A_{11}(t,x) & A_{12}(t,x) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_{11}(t,x) & B_{12}(t,x) \\ B_{21}(t,x) & B_{22}(t,x) \end{pmatrix}, \tag{8}$$

where A_{11}, A_{12} are $k \times k$ and $k \times (n-k)$ submatrices of A_1 ; B_{11}, B_{12} are $k \times k$ and $k \times (n-k)$ submatrices of B_1; B_{21} and B_{22} are $(n-k) \times k$ and $(n-k) \times (n-k)$ submatrices of B_2, respectively. Now let us multiply (8) from the left by the matrix

$$\lambda \mathcal{I}_1 + \mathcal{I}_2 = \lambda \begin{pmatrix} 0 & 0 \\ 0 & I_{n-k} \end{pmatrix} + \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

This yields

$$\lambda \mathcal{D}(t,x) + \mathcal{G}(t,x) = \lambda \begin{pmatrix} A_{11}(t,x) & A_{12}(t,x) \\ B_{21}(t,x) & B_{22}(t,x) \end{pmatrix} + \begin{pmatrix} B_{11}(t,x) & B_{12}(t,x) \\ 0 & 0 \end{pmatrix}. \tag{10}$$

Therefore,

$$\det(\lambda \mathcal{D}(t,x) + \mathcal{G}(t,x)) =$$

$$\det(\lambda \mathcal{I}_1 + \mathcal{I}_2) \det \left(\lambda \begin{pmatrix} A_1(t,x) \\ 0 \end{pmatrix} + \begin{pmatrix} B_1(t,x) \\ B_2(t,x) \end{pmatrix} \right) = \bar{a}_0(t,x)\lambda^n + \cdots .$$

From the assumption and (10), it follows

$$\bar{a}_0(t, x) \neq 0, \quad \forall (t, x) \in \Omega,$$

where

$$\bar{a}_0(t, x) = \det \begin{pmatrix} A_1(t, x) \\ B_2(t, x) \end{pmatrix}$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The proofs of the following two theorems, on the solvability of the integral equations of the second kind with weakly singular kernels, can be found in [20].

Theorem 2.1. *The equation*

$$u(t) + K^{\alpha} u(t) = f(t), \quad t \in [0, a],$$

where $f(t)$ is continuous on $[0, a]$ and K^{α} is defined by (1), $0 < \alpha < 1$, has a unique continuous solution on $[0, a]$.

Theorem 2.2. *Consider the system*

$$u(t, x) + K_1 u(t, x) + K_2 u(t, x) + K^{\alpha, \beta} u(t, x) = f(t, x), \qquad (11)$$

where K_1 and K_2 are integral operators defined by

$$K_1 u(t, x) = \int_0^t k_1(t, x, \tau) u(\tau, x) d\tau,$$

$$K_2 u(t, x) = \int_0^x k_2(t, x, s) u(t, s) ds,$$

$(t, x) \in \Omega$, and $K^{\alpha, \beta}$ is given by (2), $0 < \alpha, \beta < 1$. If $f(t, x)$, $k_1(t, x, \tau)$, and $k_2(t, x, s)$ are continuous, then system (11) has a unique continuous solution in Ω.

Definition 2.4. *[8]. The matrix polynomial $\lambda^2 A(t, x) + \lambda B(t, x) + C(t, x)$, where $A(t, x)$, $B(t, x)$, and $C(t, x)$ are $n \times n$ matrices, has a simple structure if*

   1. rank $A(t) = k = \text{const}, \forall t \in [0, 1]$,

<div align="center">8</div>

2. rank $[A(t)|\,B(t)] = k + l = \text{const}, \forall\, t \in [0, 1]$,
3. $\det\,(\lambda^2 A(t) + \lambda B(t) + C(t)) = a_0(t)\lambda^{2k+l} + a_1(t)\lambda^{2k+l-1} + \ldots + a_{2k+l}(t)$, $a_0(t) \neq 0, \forall\, t \in [0, 1]$.

Definition 2.5. *[1], [19]. The matrix A^+ is said to be the pseudoinverse of A if it satisfies the following requirements (known as Penrose equations):*

1. $A\,A^+\,A = A$,
2. $A^+\,A\,A^+ = A$,
3. $(A\,A^+)^T = A\,A^+$,
4. $(A^+\,A)^T = A^+\,A$.

For every matrix A there exists a unique pseudoinverse A^+. In particular, for a regular $n \times n$ matrix, we have $A^+ = A^{-1}$.

Lemma 2.3. *[8] For matrices $A(t)$, $B(t)$, $C(t) \in C^p_{n\times n}([0, 1])$ such that the corresponding matrix polynomial has a simple structure, there exist $n \times n$ matrices $P(t)$ and $Q(t)$ which are nonsingular for all $t \in [0, 1]$, $P(t)$, $Q(t) \in C^p_{n\times n}([0, 1])$, such that*

$$P(t)(\lambda^2 A(t) + \lambda B(t) + C(t))Q(t)$$

$$= \lambda^2 \begin{pmatrix} I_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} B_1(t) & 0 & B_2(t) \\ 0 & I_l & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} C_1(t) & C_2(t) & 0 \\ C_3(t) & C_4(t) & 0 \\ 0 & 0 & I_{n-k-l} \end{pmatrix},$$

where B_1, C_1 are $k \times k$ matrices, B_2 is a $k \times (n - k - l)$ matrix, C_2 is a $k \times l$ matrix, C_3 is $l \times k$ matrix, and C_4 is a $l \times l$ matrix.

2.2. Existence and uniqueness of solution

In the next theorem, we formulate sufficient conditions for the existence of a unique continuous solution to problem (4).

Theorem 2.3. *Let us assume that for problem (4) the following conditions hold:*

1. *The entries of $K(t, x, \tau, s)$, $A(t, x)$, and $f(t, x)$ are two times continuously differentiable with respect to all variables on Ω for all $t \in [0, a]$.*

2. rank $A(t, 0) = \text{rank}[A(t, 0)|\, f(t, 0)]$, rank $A(0, x) = \text{rank}[A(0, x)|\, f(0, x)]$, *for all $t \in [0, b]$.*

3. *The matrix pencil $\lambda A(t, x) + K(t, x, t, x)$ satisfies the rank-degree criterion.*

Then, problem (4) has a unique continuous solution.

Proof. By substituting $t = 0$ and $x = 0$ into (4) we obtain

$$A(0, x)u(0, x) = f(0, x), \tag{12}$$

$$A(t, 0)u(t, 0) = f(t, 0), \tag{13}$$

respectively. The second condition of the theorem guarantees the solvability of the linear systems (12) and (13). Let us multiply (4) by the matrix $P(t, x)$ (nonsingular on Ω) such that

$$P(t, x)A(t, x) = \begin{pmatrix} A_1(t, x) \\ 0 \end{pmatrix},$$

where $A_1(t, x)$ is a $k \times n$ matrix with rank $A_1(t, x) = k = \text{const}, \forall\, (t, x) \in \Omega$. The existence of such a matrix follows from the third assumption and Lemma 2.1. This results in two systems of integral equations,

$$A_1(t, x)u(t, x) + \int_0^t \int_0^x (t - \tau)^{-\alpha}(x - s)^{-\beta} \mathcal{K}_1(t, x, \tau, s)u(\tau, s)dsd\tau = f_1(t, x), \tag{14}$$

and

$$\int_0^t \int_0^x (t - \tau)^{-\alpha}(x - s)^{-\beta} \mathcal{K}_2(t, x, \tau, s)u(\tau, s)dsd\tau = f_2(t, x), \tag{15}$$

where

$$\begin{pmatrix} \mathcal{K}_1(t, x, \tau, s) \\ \mathcal{K}_2(t, x, \tau, s) \end{pmatrix} = P(t, x)K(t, x, \tau, s), \quad \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \end{pmatrix} = P(t, x)f(t, x).$$

We now use the standard technique developed in [20]. We apply the operator $V_{\alpha-1,\beta-1}$ given by

$$V_{\alpha-1,\beta-1}\psi(t, x) = \int_0^t \int_0^x (t - \tau)^{\alpha-1}(x - s)^{\beta-1}\psi(\tau, s)dsd\tau$$

to the system (15) and have

$$\int_0^t \int_0^x L(t,x,\tau,s)u(\tau,s)dsd\tau = F(t,x), \qquad (16)$$

where

$$L(t,x,\tau,s) = \int_\tau^t \int_s^x (t-v)^{-\alpha}(v-\tau)^{\alpha-1}(x-w)^{-\beta}(w-s)^{\beta-1}\mathcal{K}_2(v,w,\tau,s)dwdv,$$

$$(17)$$

and

$$F(t,x) = \int_0^t \int_0^x (t-\tau)^{\alpha-1}(v-\tau)^{\alpha-1}(x-s)^{\beta-1}f_2(\tau,s)dsd\tau. \qquad (18)$$

Next, the variable substitutions $v = \tau + q(t-\tau)$ and $w = s + r(x-s)$ are carried out in (17). This yields,

$$L(t,x,\tau,s) = \int_0^1 \int_0^1 q^{-\alpha}(1-q)^{1-\alpha}r^{-\beta}(1-r)^{1-\beta}\mathcal{K}_2(\tau+q(t-\tau),s+r(x-s),\tau,s)drdq.$$

$$(19)$$

The coefficients of L are two times continuously differentiable with respect to all variables and

$$L(t,x,t,x) = B(2-\alpha,\alpha)B(2-\beta,\beta)\mathcal{K}_2(t,x,t,x), \qquad (20)$$

where by

$$B(\gamma,\delta) = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)},$$

we denote the Beta function [20].

Finally, we differentiate[1] (16) with respect to t and x and obtain

$$F''_{tx}(t,x) = c\mathcal{K}_2(v,w,\tau,s) = \int_0^t \mathcal{K}(t,x,\tau,x)u(\tau,x)d\tau$$

$$(21)$$

$$+ \int_0^x \mathcal{L}(t,x,t,s)u(t,s)ds + \int_0^t \int_0^x \mathcal{M}(t,x,\tau,s)u(\tau,s)dsd\tau,$$

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[1]This is possible, according to the smoothness assumption of the theorem.

where $c := B(2 - \alpha, \alpha)B(2 - \beta, \beta)$ and $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ are certain matrices with continuous coefficients.

By combining equations (14) and (21), we obtain the following system:

$$\begin{pmatrix} A_1(t, x) \\ c\mathcal{K}_2(t, x, t, x) \end{pmatrix} u(t, x) + \int_0^t \begin{pmatrix} 0 \\ \mathcal{K}(t, x, \tau, x) \end{pmatrix} u(\tau, x)d\tau + \int_0^x \begin{pmatrix} 0 \\ \mathcal{L}(t, x, t, s)u(t, s)ds \end{pmatrix}$$
$$+ \int_0^t \int_0^x \begin{pmatrix} (t - \tau)^{-\alpha}(x - s)^{-\beta}\mathcal{K}_1(t, x, \tau, s) \\ \mathcal{M}(t, x, \tau, s) \end{pmatrix} u(\tau, s)d\tau ds = \begin{pmatrix} f_1(t, x) \\ F''_{tx}(t, x) \end{pmatrix}.$$
$$\tag{22}$$

According to the third assumption and Lemma 2.2, the matrix

$$\begin{pmatrix} A_1(t, x) \\ c\mathcal{K}_2(t, x, t, x) \end{pmatrix} \tag{23}$$

is nonsingular for any $(t, x) \in \Omega$. Multiplying (22) by the inverse of (23) yields a system which satisfies the conditions of Theorem 2.2 and therefore, it has a continuous solution. This completes the proof and the result follows. $\square$

Remark 2.1. *In Example 2.1, in case (i), the third condition of Theorem 2.3 is not satisfied in points where $t = 0$ or $x = 0$. In this case rank of A satisfies rank $A(t, x) = 0$ for $t = 0$ or $x = 0$, while rank $A(t, x) = 1$ for $t \neq 0$ and $x \neq 0$. The degree of the characteristic polynomial $\det(\lambda A(t, x) + K(t, x, t, x)) = \lambda\phi(t, x) + 1$ is one for $t \neq 0$ and $x \neq 0$ and zero otherwise. In case (ii), the third condition of Theorem 2.3 is satisfied in all points of Ω, because the degree of the characteristic polynomial $\det(\lambda A(t, x) + K(t, x, t, x)) = -1$ is equal to zero independently on ϕ.*

Note that the result of Theorem 2.3 can be extended to multidimensional equations of the form

$$A(x)u(x) + K^\alpha u(x) = f(x), \tag{24}$$

defined on the domain

$$\Omega = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_p],$$

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where $A(x)$ is a $n \times n$ matrix, such that $\det A(x) = 0$, $\forall\, x \in \Omega$; u and f are n dimensional vector functions, $x = (x_1, x_2, \ldots, x_p)$.

In this case the integral operator K^α is multidimensional,

$$K^\alpha u(x) = \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_p} \frac{k(x,s)u(s)}{\prod_{i=1}^{p}(x_j - s_j)^{\alpha_j}} ds_1 ds_2 \ldots ds_p.$$

Moreover, $s = (s_1, s_2, \ldots, s_p)$, $0 \le s_i \le x_i \le a_i$, $i = 1, \ldots, p$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$, $0 < \alpha_i < 1$, $i = 1, \ldots, p$.

We complete this section by a theorem, analogous to Theorem 2.3, which specifies sufficient conditions for the existence of a unique continuous solution of problem (24). The respective proof is very similar to the proof of Theorem 2.3.

Theorem 2.4. *Assume that for problem (24) the following assumptions hold:*

1. *The entries of $K(x,s)$, $A(x)$ and $f(x)$ have continuous second derivatives with respect to all variables in Ω.*
2. $\operatorname{rank} A(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p) =$
   $\operatorname{rank} [A(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p)|\, f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p)]$, $i = 0, 1, \ldots, p$.
3. *The matrix pencil $\lambda A(x) + K(x,x)$ satisfies the rank-degree criterion.*

Then, problem (24) has a unique continuous solution.

3. Systems of integral equations of the first kind with weakly singular kernel

In this section, we consider a system of integral equations of the form

$$K^\alpha u(t)dt = f(t), \quad t \in [0,1], \quad 0 < \alpha < 1, \tag{25}$$

where K^α is defined by (1), and $f(t)$, $u(t)$ are n-dimensional vector functions. If $\det K(t,t) \ne 0, \forall\, t \in [0,1]$, then system (25) can be reduced to a system of Volterra equations of the first kind whose kernel is nondegenerate at the diagonal and has no singularities. This can be done by a technique, which was designed for the case where $k(t,\tau)$ is a scalar function and $k(t,t) \ne 0$,

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$\forall\, t \in [0,1]$ [20], [5].

In this section we shall investigate the existence and uniqueness of continuous solutions to problem (25) under the condition

$$\det K(t,t) = 0, \forall\, t \in [0,1].$$

We illustrate the difference between the above situation and the case where $\det K(t,t) \neq 0, \forall\, t \in [0,1]$, in the example following Theorem 3.1 below. First, we formulate sufficient conditions for the existence of a unique solution of system (25).

Theorem 3.1. *Let the following assumptions hold:*

1. *The entries of $K(t,\tau)$, $K_t'(t,\tau)$, and $f(t)$ are two times continuously differentiable on their definition domains.*
2. *$f(0) = 0$, $\operatorname{rank} K(0,0) = \operatorname{rank}[K(0,0)|\,\psi(0)]$, where*

$$\psi(t) = \frac{d}{dt} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

3. *The matrix pencil $\lambda K(t,t) + K_t'(t,\tau)|_{\tau=t}$ satisfies the rank-degree criterion, for any $x \in \Omega$.*

Then problem (25) has a unique continuous solution.

Proof. We first substitute $t = 0$ into (25) and conclude that $f(0) = 0$ holds, which coincides with the second assumption. As in Theorem 2.3, let us apply the integral operator $V_{\alpha-1}$, defined by

$$V_{\alpha-1}\omega(t) = \int_0^t \int_0^x (t-\tau)^{\alpha-1}\omega(\tau)d\tau \qquad (26)$$

to (25). Then we obtain

$$\int_0^t L(t,\tau)u(\tau)d\tau = \xi(t), \quad t \in [0,1], \qquad (27)$$

where

$$L(t,\tau) = \int_\tau^t (t-v)^{-\alpha}(v-\tau)^{1-\alpha}K(v,\tau)dv, \qquad (28)$$

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and

$$\xi(t) = \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau. \tag{29}$$

Introducing new variable q in (28), $v = \tau + q(t - \tau)$, cf. [5], [20], yields

$$L(t, \tau) = \int_0^1 q^{-\alpha}(1 - q)^{1-\alpha} K(\tau + q(t - \tau), \tau) dq. \tag{30}$$

After differentiating (27) with respect to t, we have

$$L(t, t)u(t) + \int_0^t L_t'(t, \tau)u(\tau)d\tau = \psi(t), \tag{31}$$

where $\psi(t) = \xi'(t)$. Due to results presented in [5], [20], we conclude that

$$L(t, t) = \pi \sin(\alpha\pi) K(t, t)$$

holds. We now substitute $t = 0$ into (31) and obtain the following system of linear equations:

$$\pi \sin(\alpha\pi) K(0, 0)u(0) = \psi(0),$$

which is solvable, according to the second assumption of the theorem.

From the third assumption of the theorem and Lemma 2.1 it follows that there exists a matrix $P(t)$, nonsingular for any $t \in [0, 1]$, whose components are two times continuously differentiable, such that

$$P(t)K(t, t) = \begin{pmatrix} K_1(t) \\ 0 \end{pmatrix}, \tag{32}$$

where $K_1(t)$ is a $k \times n$ matrix and $\text{rank} K_1(t) = k = \text{const}, \forall t \in [0, 1]$. Multiplying (31) by $P(t)$ results in

$$\pi \sin(\alpha\pi) K_1(t)u(t) + \int_0^t L_1(t, \tau)u(\tau)d\tau = \psi_1(t), \quad t \in [0, 1], \tag{33}$$

and

$$\int_0^t L_2(t, \tau)u(\tau)d\tau = \psi_2(t), \quad t \in [0, 1], \tag{34}$$

where

$$\begin{pmatrix} L_1(t, \tau) \\ L_2(t, \tau) \end{pmatrix} = P(t)L_t'(t, \tau), \quad \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = P(t)\psi(t).$$

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The matrix pencil

$$\lambda \begin{pmatrix} K_1(t,\tau) \\ 0 \end{pmatrix} + \begin{pmatrix} L_1(t,\tau) \\ L_2(t,\tau) \end{pmatrix}$$

satisfies the rank-degree criterion, according to the third assumption. Moreover, from the first two assumptions and (30) it follows that $L_2(t,\tau)$ and $\psi_2(t)$ are continuously differentiable. Hence, differentiating (34) and combining the result with (33) finally yields

$$\begin{pmatrix} \pi \sin(\alpha\pi)K_1(t) \\ L_2(t,t) \end{pmatrix} u(t) + \int_0^t \begin{pmatrix} L_1(t,\tau) \\ L'_{2,t}(t,\tau) \end{pmatrix} u(\tau)d\tau = \begin{pmatrix} \psi_1(t) \\ \psi'_2(t) \end{pmatrix}. \quad (35)$$

According to the third assumption and Lemma 2.2,

$$\det \begin{pmatrix} \pi \sin(\alpha\pi)K_1(t) \\ L_2(t,t) \end{pmatrix} \neq 0, \quad \forall t \in [0,1]$$

and therefore system (35) is a system of Volterra equations of the second kind with continuous initial data due to the first assumption. Finally, due to Theorem 2.2, this system has a unique continuous solution and the result follows. $\square$.

If the second assumption of Theorem 3.1 does not hold, system (25) may not have any continuous solution. If the third condition is not satisfied at some point of $[0,1]$, then there may exist several solutions of (25) passing through this point, or no solution at all. Such points are said to be singular. If the third assumption is violated at any point of $[0,1]$, then nothing can be said about the existence of a unique solution, even if the matrix pencil $\lambda K(t,t) + K'_t(t,\tau)|_{\tau=t}$ is regular. This situation is illustrated by the following example.

Example 3.1. *Consider the system*

$$\int_0^t (t-\tau)^{-\alpha} \begin{pmatrix} 1-\alpha & t-\tau+r \\ (2-\alpha)(t-\tau) & (t-\tau)^2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} d\tau = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (36)$$

where r is a scalar parameter. Here the matrix pencil

$$\lambda K(t,t) + K'_t(t,\tau)|_{\tau=t} = \lambda \begin{pmatrix} 1-\alpha & r \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2-\alpha & 0 \end{pmatrix}$$

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is regular and does not depend on t. However, $\operatorname{rank} K(t,t) = 1$ *and the degree of the polynomial* $\det(\lambda K(t,t) + K'_t(t,t)) = -\lambda r(2 - \alpha) - (2 - \alpha)$ *is one for* $r \neq 0$ *and zero otherwise. So, in case when* $r = 0$, *the third condition is not satisfied and it is not difficult to verify that system (36) has an infinite set of solutions of the form*

$$u(t) = (ct, -c)^T,$$

where c is an arbitrary constant. In case when $r \neq 0$, *all conditions of Theorem 3.1 are satisfied and the problem has a unique trivial solution.*

4. Degenerate integro-differential systems with a weak singularity

We now study the following initial value problem:

$$A(t)x'(t) + B(t)x(t) + K^\alpha x(t) = f(t), \quad t \in [0,1], \tag{37}$$

$$x(0) = x_0, \tag{38}$$

where $A(t), B(t)$ are $n \times n$ matrices, $f(t)$ and $x(t)$ are n-dimensional vector functions, K^α is defined by (1) and $x_0 \in \mathbb{R}^n$.

Here, we consider the case where

$$\det A(t) \equiv 0.$$

Note, that the case

$$\det(\lambda A(t) + B(t)) \equiv 0$$

is also covered by the analysis. We shall call such problems degenerate integro-differential systems [7], [9]. They are also referred to as integro-algebraic-differential equations [3].

Again, we formulate sufficient conditions for the existence of a unique continuous solution of such problem and show that under properly stated assumptions the above problem can be split, by means of nonsingular transformations, into three systems, an integro-differential system and two Volterra integral systems, one of the second kind and the other of the first kind.

Before formulating the next theorem, let us introduce additional notations,

$$\mathcal{B}(t) = (I - A(t)A^+(t))B(t), \quad \mathcal{K}(t,\tau) = (I - A(t)A^+(t))k(t,\tau), \tag{39}$$

$$\zeta(t) = (I - A(t)A^+(t))f(t), \quad \chi(t) = \frac{d}{dt}\int_0^t (t-\tau)^{\alpha-1}(I - \mathcal{B}(\tau)\mathcal{B}^-(\tau))\zeta(\tau)d\tau.$$

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Theorem 4.1. *Assume that for problem (37)–(38) the following conditions hold:*

1. *The components of $A(t), B(t), K(t, \tau), f(t)$ are two times continuously differentiable on their definition domains.*
2. *$(I - A(0)A^+(0))B(0)x_0 = (I - A(0)A^+(0))\zeta(0)$, where A^+ is the pseudoinverse of A, cf. Definition 2.5.*
3. *$\pi \sin(\alpha\pi)(I - \mathcal{B}(0)\mathcal{B}^+(0))\mathcal{K}(0, 0)x_0 = (I - \mathcal{B}(0)\mathcal{B}^+(0))\chi(0)$, where $\mathcal{B}^+$ is the pseudoinverse of $\mathcal{B}$.*
4. *The matrix polynomial $\lambda^2 A(t) + \lambda B(t) + K(t, t)$ has a simple structure, cf. Definition 2.4.*

Then problem (37)–(38) has a unique continuously differentiable solution.

Proof. Note that the second and third condition of the theorem mean that the initial condition (38) is compatible with the right-hand side of (37). Multiplying both sides of (37) by $(I - A(t)A^+(t))$, results in

$$(I - A(t)A^+(t))B(t)x(t) + (I - A(t)A^+(t)) \int_0^t (t - \tau)^{-\alpha} K(t, \tau)x(\tau)d\tau = \zeta(t).$$
(40)

Evaluating this system at $t = 0$, yields the following linear system of algebraic equations:

$$(I - A(0)A^+(0))B(0)x(0) = \zeta(0),$$

which is solvable according to the second condition of the theorem.
Using the notation (39) in (40), we obtain

$$\mathcal{B}(t)x(t) + \int_0^t (t - \tau)^{-\alpha} \mathcal{K}(t, \tau)x(\tau)d\tau = \zeta(t).$$
(41)

Multiplying both sides of (41) by $I - \mathcal{B}(t)\mathcal{B}^+(t)$, yields

$$\int_0^t (t - \tau)^{-\alpha}(I - \mathcal{B}(t)\mathcal{B}^+(t))\mathcal{K}(t, \tau)x(\tau)d\tau = (I - \mathcal{B}(t)\mathcal{B}^+(t))\zeta(t).$$
(42)

By applying the integral operator $V_{\alpha-1}$ to (42), differentiating and evaluating at $t = 0$ (as was done in the proof of Theorem 3.1), we finally derive the system

$$\pi \sin(\alpha\pi)(I - \mathcal{B}(0)\mathcal{B}^+(0))\mathcal{K}(0, 0)x(0) = (I - \mathcal{B}(0)\mathcal{B}^+(0))\chi(0),$$

whose solvability follows from the third assumption.

Let us now multiply both sides of (37) by $P(t)$ and carry out the variable substitution $x(t) = Q(t)y(t)$, where $P(t)$ and $Q(t)$ are matrices specified in Lemma 2.3. Then we have

$$(P(t)A(t)Q(t))y'(t) + (P(t)A(t)Q'(t) + P(t)B(t)Q(t))y(t)$$

$$+ \int_0^t (t-\tau)^{-\alpha} P(t)K(t,\tau)Q(\tau)y(\tau)d\tau = P(t)f(t).$$

According to the first and third conditions and Lemma 2.3, one can guarantee that the matrix $P(t)A(t)Q(t)$ has the form

$$P(t)A(t)Q(t) = \begin{pmatrix} I_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{43}$$

Moreover, from Lemma 2.3, we have

$$P(t)A(t)Q'(t) + P(t)B(t)Q(t) = (P(t)A(t)Q(t))Q^{-1}(t)Q'(t) + P(t)B(t)Q(t)$$

$$= \begin{pmatrix} B_1(t) & B_2(t) & B_3(t) \\ 0 & I_l & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

where $B_1(t), B_2(t), B_3(t)$ are matrices of the proper dimensions. From Lemma 2.3 it also follows that

$$P(t)K(t,\tau)Q(\tau)$$

$$= \begin{pmatrix} K_1(t,\tau) & K_2(t,\tau) & K_3(t,\tau) \\ K_4(t,\tau) & K_5(t,\tau) & K_6(t,\tau) \\ K_7(t,\tau) & K_8(t,\tau) & K_9(t,\tau) \end{pmatrix}\Bigg|_{\tau=t} = \begin{pmatrix} K_1(t,t) & K_2(t,t) & 0 \\ K_4(t,t) & K_5(t,t) & 0 \\ 0 & 0 & I_{n-k-l} \end{pmatrix} \tag{45}$$

Therefore, taking equalities (43)–(45) into account and introducing $y := (y_1^T, y_2^T, y_3^T)^T$, system (43) can be rewritten in form of the following three

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systems:

$$y'_1(t) + B_1(t)y_1(t) + B_2(t)y_2(t) + B_3(t)y_3(t)$$

$$+ \int_0^t (t-\tau)^{-\alpha}(K_1(t,\tau)y_1(\tau) + +K_2(t,\tau)y_2(\tau) + K_3(t,\tau)y_3(\tau))d\tau = f_1(t), \quad (46)$$

$$y_1(0) = [I_k|\,0|\,0]Q^{-1}(0)x_0,$$

$$y_2(t) + \int_0^t (t-\tau)^{-\alpha}(K_4(t,\tau)y_1(\tau) + K_5(t,\tau)y_2(\tau) + K_6(t,\tau)y_3(\tau))d\tau = f_2(t), \quad (47)$$

$$\int_0^t (t-\tau)^{-\alpha}(K_7(t,\tau)y_1(\tau) + K_8(t,\tau)y_2(\tau) + K_9(t,\tau)y_3(\tau))d\tau = f_3(t). \quad (48)$$

By integrating (46), we obtain

$$y_1(t) + \int_0^t (\mathcal{K}_1(t,\tau)y_1(\tau) + \mathcal{K}_2(t,\tau)y_2(\tau) + \mathcal{K}_3(t,\tau)y_3(\tau))d\tau = Q_1(t), \quad (49)$$

where $\mathcal{K}_j(t,\tau) = B_j(\tau) + \int_v^t (v-\tau)^{-\alpha}K_j(v,\tau)dv$, $j = 1,2,3$, are matrices with continuous components and

$$Q_1(t) = \int_0^t f_1(\tau)d\tau + y_1(0).$$

We now deal with (48) as shown in the proof of Theorem 3.1. First, the operator $V_{\alpha-1}$ specified in (26) is applied to both sides of (48). Then, carrying out the steps described in (27)–(31) results in a system of the form

$$y_3(t) + \int_0^t (\mathcal{K}_7(t,\tau)y_1(\tau) + \mathcal{K}_8(t,\tau)y_2(\tau) + \mathcal{K}_9(t,\tau)y_3(\tau))d\tau = Q_3(t), \quad (50)$$

where $\mathcal{K}_j(t,\tau)$, $j = 7,8,9$ are matrices with continuously differentiable components, and

$$Q_3(t) = \frac{1}{2\sin\alpha\pi}\frac{d}{dt}\int_0^t (t-\tau)^{\alpha-1}f_3(\tau)d\tau$$

is a continuously differentiable vector function. Finally, combining (47), (49) and (50) results in one system,

$$y(t) + \int_0^t (t-\tau)^{-\alpha}M(t,\tau)y(\tau)d\tau = \psi(t), \quad (51)$$

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where the components of $M(t, \tau)$ and $\psi(t)$ are continuously differentiable. This follows from (49), (50) and the first assumption of the theorem. Hence the system (51) has a unique continuously differentiable solution, according to Theorem 2.1. Finally, since $x(t) = Q(t)y(t)$, the original problem (37)–(38) also has a unique continuously differentiable solution. □

5. Conclusions

In this paper we have analyzed systems of integral-algebraic and integro-differential equations with weakly singular kernels. We have formulated sufficient conditions for the existence and uniqueness of solution to such problems. In the future, we plan to propose numerical approaches for the considered problem classes based on the product integration formula introduced in [5].

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