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QUASI-OPTIMAL CONVERGENCE RATE FOR AN ADAPTIVE BOUNDARY ELEMENT METHOD

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Abstract. For the simple layer potential V that is associated with the 3D Laplacian, we consider the weakly singular integral equation $V\phi = f$. This equation is discretized by the lowest order Galerkin boundary element method. We prove convergence of an h-adaptive algorithm that is driven by a weighted residual error estimator. Moreover, we identify the approximation class for which the adaptive algorithm converges quasi-optimally with respect to the number of elements. In particular, we prove that adaptive mesh refinement is superior to uniform mesh refinement.

1. Introduction

For a surface $\Gamma \subset \mathbb{R}^3$, we consider the lowest order Galerkin boundary element method for the weakly singular integral equation

$$V\phi(\mathbf{x}) := \frac{1}{4\pi}\int_\Gamma \frac{\phi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\Gamma(\mathbf{y}) = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma. \tag{1}$$

We postpone a precise statement of our mathematical framework to Section 2. Problem (1) is an example of a boundary integral equation arising in elliptic partial differential equations; the analysis of such equations and the understanding of their numerical treatment has reached a certain level of maturity as witnessed by the recent monographs [31, 33, 38, 42]. However, adaptivity, which is the topic here, is not covered in these references.

We analyze a standard h-adaptive boundary element method (ABEM) of the form

$$\boxed{\texttt{solve}} \longrightarrow \boxed{\texttt{estimate}} \longrightarrow \boxed{\texttt{mark}} \longrightarrow \boxed{\texttt{refine}} \tag{2}$$

In the context of the finite element method on shape-regular meshes, h-adaptive algorithms of this type (AFEM) have been analyzed in the last 15 years and are by now fairly well understood. The works of Dörfler [20], Morin, Nochetto, and Siebert [34, 35], Binev, Dahmen, and DeVore [7], Stevenson [43], and Cascón, Kreuzer, Nochetto, and Siebert [19] were milestones for linear model problems and residual error estimates. The work [32] transferred these arguments to other error estimators by use of local equivalences. Very recently, also convergence and quasi-optimality for a nonlinear model problem have been proved [6].

The situation is considerably less developed for the adaptive boundary element method (ABEM). While several a posteriori error estimator for (1) are available in the literature, cf. [11, 13, 14, 17, 18, 22, 23, 24, 25, 39, 27, 30, 37, 41] and references therein, and numerous numerical studies indicate convergence of ABEM as well as superiority of ABEM over simple uniform mesh refinement, a rigorous mathematical justification of ABEM still appears to be missing. Such an analysis is the purpose of the present paper for the model equation (1).

The first work on convergence of ABEM known to the authors is [16], where convergence is guaranteed by use of a feedback control which occasionally leads to uniform refinements by a

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numerical check of the saturation assumption. This result is somewhat unsatisfactory, since the feedback control is computationally expensive and seemed to be unnecessary in practice. The second work on convergence of ABEM is [26]. Under the saturation assumption, [26] proved contraction of ABEM in each step of the adaptive loop, where an $(h - h/2)$-based estimator is used to mark certain elements for refinement. Their arguments have been relaxed in [2, 3], but still the analysis hinges on a weakened saturation assumption. Moreover, since the saturation assumption cannot be guaranteed mathematically and even fails to hold in general, cf. [8, 21], the results of [2, 3, 26] are still mathematically unsatisfactory. Besides this, all these works do not show that ABEM is superior to uniform mesh refinement.

Here, we consider the weigthed residual error estimator employed in [11, 17, 18] for 2D BEM and in [12] for 3D BEM. We prove linear convergence of ABEM (Theorem 3.1) and identify an approximation class for which the ABEM converges at the optimal rate (Theorem 4.1). Our procedure follows structurally [19]; however, given the non-local nature of the integral operator V and the Sobolev norms studied here, new tools such as the inverse estimate of Proposition 3.3 had to be developped. Very recently and independently of our work, Gantumur [28] presented an analogous convergence result, which relies on similar ideas and mathematical ingredients. In his work, however, the focus is rather on the hypersingular integral equation with a pseudo-differential operator of order $+1$, whereas the simple layer operator V studied here is of order -1. For the weakly singular integral equation (1), our result is stronger in two ways: First, the analysis of [28] is stated for $C^{1,1}$ surfaces, whereas the present work covers polyhedral geometries. Second, the error estimator considered in [28] is only reliable up to the weighted residual estimator used here, i.e. the algorithm proposed in [28] needs to compute two error estimators.

We close the introduction with some general remarks on the model problem (1) under consideration. First, the simple layer potential V of the 3D Laplacian is the prototype of an elliptic pseudo-differential operator of order -1. With appropriate modifications, our results can be transfered to the simple layer potentials of other elliptic equations such as the Lamé or Stokes equation. Second, although the paper is mainly concerned with the 3D problem, our analytical techniques also work for 2D, where certain proofs are even simpler.

For numerical experiments, the reader is referred to [11, 12, 17, 18], whereas this work provides the mathematical explanation for those empirical observations.

2. Setting

In this section, we introduce our model problem, its Galerkin discretization, and the adaptive algorithm that we analyze. The convergence of this adaptive algorithm is proved in Section 3; quasi-optimality of the algorithm is shown in Section 4.

2.1. Model problem. Let $\Omega \subset \mathbb{R}^3$ be a polyhedral Lipschitz domain, i.e. we have $\partial\Omega = \bigcup_{i=1}^n \Gamma_i$, where the Γ_i are plane surface pieces. The usual Sobolev spaces on Ω are denoted by $L_2(\Omega)$ and $H^1(\Omega)$. Sobolev spaces with noninteger order on $\partial\Omega$ or relatively open parts Γ of $\partial\Omega$ are defined by use of the Sobolev-Slobodeckij seminorm (see, e.g., [38, Def. 2.4.1]). The dual space of $H^{1/2}(\Gamma)$ is denoted by $\widetilde{H}^{-1/2}(\Gamma)$. We denote by $\langle \cdot\,, \cdot \rangle$ the L_2-scalar product which is extended to duality between $\widetilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, if the arguments of $\langle \cdot\,, \cdot \rangle$ are in these spaces. Let $\Gamma \subseteq \partial\Omega$ be a relatively open subset. Our model problem reads: For given $f \in H^{1/2}(\Gamma)$, find $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ such that

$$V\phi = f \quad \text{on } \Gamma. \tag{3}$$

Here, V denotes the simple-layer potential of the 3D Laplacian

$$V\phi(\mathbf{x}) = \frac{1}{4\pi}\int_\Gamma \frac{\phi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\Gamma(\mathbf{y}), \tag{4}$$

which is a continuous, elliptic, and symmetric isomorphism between the Sobolev spaces $\widetilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Throughout, $|\cdot|$ denotes the absolute value of a scalar as well as the Euclidean norm of a vector, and $\mathbf{n}(y)$ denotes the exterior normal vector on $\partial\Omega$ at some point $\mathbf{y} \in \Gamma$.

Since $V : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is an elliptic and symmetric isomorphism, it provides a scalar product defined by $\langle\!\langle \phi, \psi \rangle\!\rangle = \langle V\phi, \psi\rangle$. We denote by $|||\cdot|||^2 := \langle\!\langle \cdot, \cdot \rangle\!\rangle$ the induced energy norm, which is an equivalent norm on $\widetilde{H}^{-1/2}(\Gamma)$. In particular, the theorem of Riesz-Fischer proves the existence and uniqueness of the solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ of (3) stated in the variational form

$$\langle\!\langle \phi, \psi \rangle\!\rangle = \langle f, \psi\rangle \quad \text{for all } \psi \in \widetilde{H}^{-1/2}(\Gamma). \tag{5}$$

Remark 1. *The model problem* (3) *arises naturally in boundary integral equation methods. For example:*

(i) Let $\Gamma = \partial\Omega$ and let K be the double layer potential

$$Kg(\mathbf{x}) = \frac{1}{4\pi}\oint_\Gamma \frac{(\mathbf{y}-\mathbf{x})\cdot\mathbf{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,g(\mathbf{y})\,d\Gamma(\mathbf{y}).$$

Upon setting $f = (K+1/2)g$, where $g \in H^{1/2}(\Gamma)$ is given, the model problem (3) *can be used to solve the Laplace (Dirichlet) problem*

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega,\\ u &= g && \text{on } \Gamma = \partial\Omega. \end{aligned}$$

(ii) For $\Gamma \subset \partial\Omega$ an open screen, (3) *is equivalent to the Dirichlet screen problem*

$$\begin{aligned} -\Delta u &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{\Gamma},\\ u &= f && \text{on } \Gamma,\\ |u(\mathbf{x})| &= \mathcal{O}(|\mathbf{x}|^{-1}) && \text{as } |\mathbf{x}| \to \infty. \end{aligned}$$

□

2.2. Triangulation of Γ and Galerkin discretization. We suppose that we are given an initial regular triangulation $\mathcal{E}_0$ of Γ into flat compact triangles, see [9, Definition 3.3.11]. From this, we generate a sequence of regular triangulations $\mathcal{E}_\ell$ by local newest vertex bisection, see Section 4 below. The index ℓ will later on denote the ℓ-th step of the proposed h-adaptive algorithm. We consider the lowest order Galerkin discretization of (5). The space $\mathcal{P}^0(\mathcal{E}_\ell)$ is the space of piecewise constant functions on the triangulation $\mathcal{E}_\ell$. Again, the Riesz-Fischer theorem applies and guarantees existence and uniqueness of the solution $\Phi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$ of the Galerkin formulation:

$$\text{Find } \Phi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell) \text{ s.t.} \qquad \langle\!\langle \Phi_\ell, \Psi_\ell \rangle\!\rangle = \langle f, \Psi_\ell\rangle \quad \text{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell). \tag{6}$$

In the remainder of this paper, we use the following additional notation: The set of vertices of $\mathcal{E}_\ell$ is denoted by $\mathcal{N}_\ell$. Our analysis will rely on the notion of patches. The set $\omega_\ell(E) = \{E' \in \mathcal{E}_\ell : E \cap E' \neq \emptyset\} \subseteq \mathcal{E}_\ell$ denotes the usual element patch of an element $E \in \mathcal{E}_\ell$. Moreover, for a subset $\mathcal{R}_\ell \subseteq \mathcal{E}_\ell$, we define the patch $\omega_\ell(\mathcal{R}_\ell) = \{E' \in \mathcal{E}_\ell : \exists E \in \mathcal{R}_\ell \quad E' \in \omega_\ell(E)\} \subseteq \mathcal{E}_\ell$. Finally, for a set $\mathcal{R}_\ell \subseteq \mathcal{E}_\ell$ of elements, $\bigcup \mathcal{R}_\ell := \bigcup_{E\in\mathcal{R}_\ell} E \subseteq \overline{\Gamma}$ denotes the subregion of $\overline{\Gamma}$ covered by the elements in $\mathcal{R}_\ell$.

Some of our results will not depend on the fact that the sequence of meshes $(\mathcal{E}_\ell)_\ell$ is generated by newest vertex bisection but only relies on shape-regularity of the triangulation. To capture this, we recall the following standard notation: A mesh $\mathcal{E}_\ell$ is called γ-*shape regular* (or: γ-*isotropic*) if

$$\sigma(\mathcal{E}_\ell) := \max_{E\in\mathcal{E}_\ell} \frac{\mathrm{diam}(E)^2}{|E|} \le \gamma; \tag{7}$$

analogously, a sequence of meshes $(\mathcal{E}_\ell)_\ell$ is γ-shape regular if (7) is valid uniformly in ℓ.

2.3. Adaptive mesh refinement algorithm and main results. We study an h-adaptive algorithm that is driven by the local contributions of a weighted residual error estimator. In order to define this estimator in (9) below, we introduce with the surface area $|\cdot|$ the local mesh size function $h_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$ by

$$h_\ell|_E := h_\ell(E) := |E|^{1/2}, \tag{8}$$

where $\mathcal{E}_\ell$ is a regular triangulation Γ that is generated by the adaptive loop in Alg. 2.1 below. For the Galerkin solution Φ_ℓ of (6) with respect to $\mathcal{E}_\ell$, we define the error estimator μ_ℓ by

$$\mu_\ell := \|h_\ell^{1/2}\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(\Gamma)} = \Big(\sum_{E\in\mathcal{E}_\ell} \mu_\ell(E)^2\Big)^{1/2}, \text{ where } \mu_\ell(E)^2 = h_\ell(E)\|\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(E)}^2. \tag{9}$$

Here, $\nabla_\Gamma(\cdot)$ denotes the surface gradient. Note that the definition of μ_ℓ requires additional regularity $f \in H^1(\Gamma)$ of the given data, which is assumed from now on. Reliability of μ_ℓ has been proven in [12, Corollary 4.2], i.e. μ_ℓ provides a computable upper bound for the error in the energy norm $|||\cdot||| \simeq \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$:

$$|||\phi - \Phi_\ell||| \le C_{\mathrm{rel}}\,\mu_\ell, \tag{10}$$

where the constant C_{rel} depends only on the shape of the elements of $\mathcal{E}_\ell$ and on Γ.

Remark 2. *The constant C_{rel} depends on the shape of all possible vertex patches $\omega_{\ell,z} := \bigcup\big\{E \in \mathcal{E}_\ell : z \in E\big\}$, where z is a vertex of $\mathcal{E}_\ell$. Local refinement is done by use of newest vertex bisection, see Section 4 below. Among others, this ensures that only finitely many shapes of elements $E \in \mathcal{E}_\ell$ are generated over all steps $\ell \in \mathbb{N}_0$ of the adaptive loop. In particular, only finitely many shapes of node patches occur, and the constant C_{rel} is therefore known to be bounded for all $\ell \in \mathbb{N}_0$.* □

Using the Dörfler marking (11) introduced in [20] to single out the elements $E \in \mathcal{E}_\ell$ for refinement, our version of the adaptive loop (2) reads as follows:

Algorithm 2.1. *Input: Initial triangulation $\mathcal{E}_0$, parameter $0 < \theta < 1$, counter $\ell := 0$.*

(i) *Compute the Galerkin solution Φ_ℓ with respect to $\mathcal{E}_\ell$.*
(ii) *Compute refinement indicators $\mu_\ell(E)$ from (9) for all $E \in \mathcal{E}_\ell$.*
(iii) *Determine a set $\mathcal{M}_\ell \subset \mathcal{E}_\ell$ of minimal cardinality such that*

$$\sum_{E\in\mathcal{M}_\ell} \mu_\ell(E)^2 \ge \theta\mu_\ell^2. \tag{11}$$

(iv) *Refine at least marked elements $E \in \mathcal{M}_\ell$ by newest vertex bisection to obtain $\mathcal{E}_{\ell+1}$.*
(v) *Update counter $\ell \mapsto \ell + 1$ and goto* (i).

□

The main results of our work on the weakly singular integral equation (1) and the work [28] of Gantumur on the hypersingular integral equation read as follows: The usual version (2) of ABEM, which is stated in the algorithm above, leads to a sequence of adaptively generated triangulations and corresponding discrete solutions. A weighted sum of Galerkin error and weighted residual error estimator, called quasi-error in [19], then is contractive in each step of the adaptive loop

(Theorem 3.1). This proves linear convergence of ABEM independently of the adaptivity parameter $0 < \theta < 1$ chosen. Moreover, for sufficiently small adaptivity parameters $0 < \theta \le \theta_\star$, we prove that the convergence rate of ABEM is optimal with respect to the number of elements (Theorem 4.1). That is, the adaptive algorithm leads to the best possible convergence rate, constrained by the mesh refinement strategy and the use of the weighted residual error estimator.

While our proofs imitate the current convergence and quasi-optimality concepts for adaptive finite element schemes, cf. [19], our analysis has to circumvent additional difficulties which arise from the non-locality of the integral operator V and the non-local Sobolev norms involved. In particular, we prove a novel inverse estimate (Proposition 3.3) which might be of independent interest. Altogether, this work and [28] provide analytical results on ABEM for pseudo-differential operators of order ± 1 which are comparable to those on AFEM of [19].

3. Convergence

In this section, we prove convergence of Algorithm 2.1 in the spirit of [19]. We mention that the proof of Theorem 3.1 does not need the minimality condition on $\mathcal{M}_\ell$ in step (*iii*) of Algorithm 2.1. It is only necessary that the set of marked elements $\mathcal{M}_\ell$ satisfies the Dörfler Marking (11). In particular, the following theorem also holds for uniform mesh refinement, where $\mathcal{M}_\ell = \mathcal{E}_\ell$.

Theorem 3.1. *There are constants $0 < \lambda, \kappa < 1$ such that the quasi-error is contractive, i.e.*

$$\Delta_{\ell+1} \le \kappa\,\Delta_\ell \quad \textit{with} \quad \Delta_\ell := |||\phi - \Phi_\ell|||^2 + \lambda\mu_\ell^2. \tag{12}$$

The constants $\lambda, \kappa > 0$ depend only on the adaptivity parameter $0 < \theta < 1$, γ-shape regularity of the sequence $(\mathcal{E}_\ell)_\ell$, and on Γ. In particular, this implies convergence $\lim_{\ell\to\infty} |||\phi - \Phi_\ell||| = 0 = \lim_{\ell\to\infty} \mu_\ell$.

Remark 3. *Theorem 3.1 also holds for mesh refinement strategies other than newest vertex bisection. The essential ingredients are: First, the sequence of generated meshes $\mathcal{E}_\ell$ has to be uniformly γ-shape regular. Second, marked elements $E \in \mathcal{M}_\ell$ have to be refined such that the local mesh-size $h_{\ell+1}|_E \le q\,h_\ell|_E$ is uniformly decreased by some factor $0 < q < 1$. These assumptions are satisfied by each feasible mesh refinement strategy and, in particular, for newest vertex bisection, the popular red-green-blue refinement, and longest edge bisection, see e.g.* [45, Chapter 5]. □

The proof of Theorem 3.1 is carried out in Section 3.3. It relies on the estimator reduction found in Proposition 3.2 and some inverse estimate for the discrete space $V\mathcal{P}^0(\mathcal{E}_\ell)$ stated in Proposition 3.3.

3.1. Estimator reduction of residual error estimator μ_ℓ. The following proposition states that, in each step of the adaptive loop, the error estimator is contractive up to some perturbation which only depends on the difference of two successive Galerkin solutions. In fact, the contraction constant $\widetilde{\kappa}$ for the error estimator μ_ℓ essentially gives the contraction constant κ for the quasi-error Δ_ℓ of (12).

Proposition 3.2. *Algorithm 2.1 guarantees estimator reduction*

$$\mu_{\ell+1}^2 \le \widetilde{\kappa}\mu_\ell^2 + C_{\rm red}\,\|h_{\ell+1}^{1/2}\nabla_\Gamma V(\Phi_{\ell+1} - \Phi_\ell)\|_{L_2(\Gamma)}^2. \tag{13}$$

The constants $0 < \widetilde{\kappa} < 1$ and $C_{\rm red} > 0$ depend only on the adaptivity parameter $0 < \theta < 1$.

Proof. Using the triangle inequality and the Young inequality $(a+b)^2 \le (1+\delta)a^2 + (1+\delta^{-1})b^2$, we obtain for arbitrary $\delta > 0$

$$\begin{aligned}
\mu_{\ell+1}^2 &= \|h_{\ell+1}^{1/2}\nabla_\Gamma(V\Phi_{\ell+1} - f)\|_{L_2(\Gamma)}^2 \\
&\le \big(\|h_{\ell+1}^{1/2}\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(\Gamma)} + \|h_{\ell+1}^{1/2}\nabla_\Gamma V(\Phi_{\ell+1} - \Phi_\ell)\|_{L_2(\Gamma)}\big)^2 \\
&\le (1+\delta)\|h_{\ell+1}^{1/2}\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(\Gamma)}^2 + (1+\delta^{-1})\|h_{\ell+1}^{1/2}\nabla_\Gamma V(\Phi_{\ell+1} - \Phi_\ell)\|_{L_2(\Gamma)}^2.
\end{aligned} \tag{14}$$

We consider the first term elementwise and split it into marked and non-marked elements

$$\|h_{\ell+1}^{1/2}\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(\Gamma)}^2 = \sum_{E\in\mathcal{M}_\ell} \|h_{\ell+1}^{1/2}\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(E)}^2 + \sum_{E\in\mathcal{E}_\ell\setminus\mathcal{M}_\ell} \|h_{\ell+1}^{1/2}\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(E)}^2.$$

Marked elements $E \in \mathcal{M}_\ell$ are refined so that $h_{\ell+1}|_E \le q h_\ell|_E$ with $q = 2^{-1/2} < 1$, since the sons have half the area of E. Elements $E \in \mathcal{E}_\ell \setminus \mathcal{M}_\ell$ satisfy at least $h_{\ell+1}|_E \le h_\ell|_E$. Hence,

$$\|h_{\ell+1}^{1/2}\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(E)}^2 \le \begin{cases} q\mu_\ell(E)^2 & \text{for } E \in \mathcal{M}_\ell, \\ \mu_\ell(E)^2 & \text{for } E \in \mathcal{E}_\ell \setminus \mathcal{M}_\ell. \end{cases}$$

According to the Dörfler Marking (11), we see

$$\begin{aligned}
\|h_{\ell+1}^{1/2}\nabla_\Gamma(V\Phi_\ell - f)\|_{L_2(\Gamma)}^2 \le q\sum_{E\in\mathcal{M}_\ell}\mu_\ell(E)^2 + \sum_{E\in\mathcal{E}_\ell\setminus\mathcal{M}_\ell}\mu_\ell(E)^2 &= \mu_\ell^2 - (1-q)\sum_{E\in\mathcal{M}_\ell}\mu_\ell(E)^2 \\
&\le \big(1 - \theta(1-q)\big)\mu_\ell^2.
\end{aligned}$$

Combining this with (14), we obtain

$$\mu_{\ell+1}^2 \le (1+\delta)\big(1-\theta(1-q)\big)\mu_\ell^2 + (1+\delta^{-1})\|h_{\ell+1}^{1/2}\nabla_\Gamma V(\Phi_{\ell+1} - \Phi_\ell)\|_{L_2(\Gamma)}^2.$$

Finally, we choose $\delta > 0$ sufficiently small so that $\kappa = (1+\delta)\,(1-(1-q)\theta) < 1$. This concludes the proof. $\square$

3.2. Inverse estimate on $V\mathcal{P}^0(\mathcal{E}_\ell)$. The following proposition is the main result of this section. It allows us to estimate the perturbation term $\|h_{\ell+1}^{1/2}\nabla_\Gamma V(\Phi_{\ell+1} - \Phi_\ell)\|_{L_2(\Gamma)}$ in the estimator reduction (13) by the energy norm $|||\Phi_{\ell+1} - \Phi_\ell|||$.

Proposition 3.3. *There is some constant $C_{\rm inv} > 0$ such that*

$$\|h_\ell^{1/2}\nabla_\Gamma V\Psi_\ell\|_{L_2(\Gamma)} \le C_{\rm inv}\,|||\Psi_\ell||| \quad \textit{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell). \tag{15}$$

The constant $C_{\rm inv} > 0$ depends only on Γ and the γ-shape regularity of $\mathcal{E}_\ell$.

Remark 4. *Under the assumption that Γ is $C^{1,1}$, Gantumur [28] proves the inverse estimate (15) by wavelet-based techniques. Moreover, for $C^{1,1}$-surfaces Γ, his work provides similar estimates for the double layer potential K, its adjoint K', and the hypersingular integral operator W. Our analysis leads to the same estimates for polyhedral Γ. While the far-field bound in Lemma 3.6 follows from general potential estimates, the proof of near-field bound in Lemma 3.7 has to be modified to the integral operator at hand.* $\square$

The proof of Proposition 3.3 is given below and depends on several technical lemmata, which are proven in the following. We start with some trace inequality.

Lemma 3.4 (trace inequality). *For $\Gamma \subset \mathbb{R}^3$ a hyperplane and $B \subset \mathbb{R}^3$ a ball with center in Γ and radius $r_B > 0$, there is a constant $C_{\rm trace} > 0$ such that*

$$C_{\rm trace}^{-1} \|u\|^2_{L_2(\Gamma\cap B)} \le \frac{1}{r_B}\|u\|^2_{L_2(B)} + \|u\|_{L_2(B)}\|\nabla u\|_{L_2(B)} \quad \textit{for all } u \in H^1(B). \tag{16}$$

The constant $C_{\rm trace} > 0$ does neither depend on Γ, B, nor on u or on r_B.

Proof. We consider the reference configuration $\widehat{\Gamma} = \mathbb{R}^2$ and the ball $\widehat{B} = B_1(\mathbf{0}) \subset \mathbb{R}^3$ centered at the origin with radius 1. From the multiplicative trace inequality [9, Theorem 1.6.6], we infer

$$\|\widehat{u}\|^2_{L_2(\widehat{\Gamma}\cap\widehat{B})} \lesssim \|\widehat{u}\|_{L_2(\widehat{B}_+)}\|\widehat{u}\|_{H^1(\widehat{B}_+)} \le \|\widehat{u}\|_{L_2(\widehat{B})}\|\widehat{u}\|_{H^1(\widehat{B})} \quad \text{for all } \widehat{u} \in H^1(\widehat{B}),$$

where $\widehat{B}_+ := \big\{x \in \widehat{B} \,:\, \mathbf{x}_3 > 0\big\}$. Next,

$$\|\widehat{u}\|_{L_2(\widehat{B})}\|\widehat{u}\|_{H^1(\widehat{B})} = \big(\|\widehat{u}\|^4_{L_2(\widehat{B})} + \|\widehat{u}\|^2_{L_2(\widehat{B})}\|\nabla\widehat{u}\|^2_{L_2(\widehat{B})}\big)^{1/2} \le \|\widehat{u}\|^2_{L_2(\widehat{B})} + \|\widehat{u}\|_{L_2(\widehat{B})}\|\nabla\widehat{u}\|_{L_2(\widehat{B})}.$$

The statement now follows by use of common scaling arguments. $\square$

For the remainder of this section, we use the following notation. For $r > 0$ and $\mathbf{x} \in \mathbb{R}^3$, we denote by $B_r(\mathbf{x})$ the open ball in $\mathbb{R}^3$ centered at $\mathbf{x}$ with radius r. For $\delta > 0$, we define a neighborhood U_E of an element E by

$$E \subset U_E := \bigcup_{\mathbf{x}\in E} B_{2\delta h_\ell(E)}(\mathbf{x}).$$

By the γ-shape regularity of the mesh, we can select $\delta > 0$ and $M \in \mathbb{N}$, which depend solely on γ and the polyhedral boundary Γ, such that $U_E \cap \Gamma$ is contained in the patch of E

$$U_E \cap \Gamma \subset \bigcup \omega_\ell(E) \tag{17a}$$

and that the covering $\Gamma \subset \bigcup_{E\in\mathcal{E}_\ell} U_E$ is locally finite

$$\#\big\{U_E \,:\, E \in \mathcal{E}_\ell \text{ and } \mathbf{x} \in U_E\big\} \le M \text{ for all } \mathbf{x} \in \mathbb{R}^3. \tag{17b}$$

In fact, M may be taken to be the maximum number of elements in a node patch, i.e. $M = \max\limits_{z\in\mathcal{N}_\ell} \#\big\{E \in \mathcal{E}_\ell \,:\, z \in E\big\}$ for an appropriate choice of $\delta > 0$.

Lemma 3.5 (Caccioppoli-type inequality). *Let $\psi \in L_2(\Gamma)$. For each $E \in \mathcal{E}_\ell$, define $u_E^{\rm far} = V(\psi\chi_{\Gamma\setminus U_E})$ as the far-field of $u = V\psi$ with respect to E. With $\Omega^{\rm ext} = \mathbb{R}^3\backslash\overline{\Omega}$, it then holds that $u_E^{\rm far}|_\Omega \in C^\infty(\Omega)$, $u_E^{\rm far}|_{\Omega^{\rm ext}} \in C^\infty(\Omega^{\rm ext})$, and $u_E^{\rm far}|_{U_E} \in C^\infty(U_E)$. Furthermore,*

$$\|D^2 u_E^{\rm far}\|_{L_2(B_{\delta h_\ell(E)}(\mathbf{x}))} \le C_{\rm cacc}\,\frac{1}{h_\ell(E)}\|\nabla u_E^{\rm far}\|_{L_2(B_{2\delta h_\ell(E)}(\mathbf{x}))} \quad \textit{for all } \mathbf{x} \in E. \tag{18}$$

The constant $C_{\rm cacc} > 0$ depends only on the γ-shape regularity of $\mathcal{E}_\ell$.

Proof. The far-field $u_E^{\rm far} \in H^1(\Omega) \times H^1_{\ell oc}(\Omega^{\rm ext})$ solves the transmission problem

$$\begin{array}{rcll} -\Delta u_E^{\rm far} &=& 0 & \text{in } \Omega \cup \Omega^{\rm ext} \\ {[u_E^{\rm far}]} &=& 0 & \text{on } \Gamma \\ {[\partial_{\mathbf{n}} u_E^{\rm far}]} &=& -\psi\chi_{\Gamma\setminus U_E} & \text{on } \Gamma \\ u_E^{\rm far}(\mathbf{x}) &=& \mathcal{O}(1/|\mathbf{x}|) & \text{as } |\mathbf{x}| \to \infty, \end{array} \tag{19}$$

where $[\cdot]$ denotes the jump across Γ, see e.g. [38]. Moreover, since u_E^{far} is (weakly) harmonic, it holds that $u_E^{\mathrm{far}}|_\Omega \in C^\infty(\Omega)$ and $u_E^{\mathrm{far}}|_{\Omega^{\mathrm{ext}}} \in C^\infty(\Omega^{\mathrm{ext}})$. Fix $\mathbf{x}' \in E$. For a test function $v \in \mathcal{D}(B_{2\delta h_\ell(E)}(\mathbf{x}'))$, we use integration by parts to see

$$\begin{aligned}
\langle u_E^{\mathrm{far}}, -\Delta v\rangle &= -\int_{B_{2\delta h_\ell(E)}(\mathbf{x}')} u_E^{\mathrm{far}} \Delta v \, d\mathbf{x} \\
&= -\int_{B_{2\delta h_\ell(E)}(\mathbf{x}')\cap\Omega} u_E^{\mathrm{far}} \Delta v \, d\mathbf{x} - \int_{B_{2\delta h_\ell(E)}(\mathbf{x}')\cap\Omega^{\mathrm{ext}}} u_E^{\mathrm{far}} \Delta v \, d\mathbf{x} \\
&= -\int_{B_{2\delta h_\ell(E)}(\mathbf{x}')\cap\Omega} \Delta u_E^{\mathrm{far}} v \, d\mathbf{x} + \int_{\Gamma\cap B_{2\delta h_\ell(E)}(\mathbf{x}')} \partial_{\mathbf{n}}^{\mathrm{ext}} u_E^{\mathrm{far}} \, v \, d\Gamma - \int_{\Gamma\cap B_{2\delta h_\ell(E)}(\mathbf{x}')} u_E^{\mathrm{far}} \partial_{\mathbf{n}}^{\mathrm{ext}} v \, d\Gamma \\
&\quad - \int_{B_{2\delta h_\ell(E)}(\mathbf{x}')\cap\Omega^{\mathrm{ext}}} \Delta u_E^{\mathrm{far}} v \, d\mathbf{x} + \int_{\Gamma\cap B_{2\delta h_\ell(E)}(\mathbf{x}')} \partial_{\mathbf{n}} u_E^{\mathrm{far}} \, v \, d\Gamma - \int_{\Gamma\cap B_{2\delta h_\ell(E)}(\mathbf{x}')} u_E^{\mathrm{far}} \, \partial_{\mathbf{n}} v \, d\Gamma \\
&= -\int_{B_{2\delta h_\ell(E)}(\mathbf{x}')\cap(\Omega\cup\Omega^{\mathrm{ext}})} \Delta u_E^{\mathrm{far}} v \, d\mathbf{x} + \int_{\Gamma\cap B_{2\delta h_\ell(E)}(\mathbf{x}')} [\partial_{\mathbf{n}} u_E^{\mathrm{far}}] v \, d\Gamma - \int_{\Gamma\cap B_{2\delta h_\ell(E)}(\mathbf{x}')} [u_E^{\mathrm{far}}] \, \partial_{\mathbf{n}} v \, d\Gamma.
\end{aligned}$$

According to (19), the volume integral and the second boundary integral vanish. Moreover, $\Gamma \cap B_{2\delta h_\ell(E)}(\mathbf{x}') \subseteq U_E$. Therefore, also the first boundary integral vanishes. This means that $-\Delta u_E^{\mathrm{far}} = 0$ on $B_{2\delta h_\ell(E)}(\mathbf{x}')$ in the sense of distributions. Weyl's Lemma (see, e.g., [36, Theorem 2.3.1]) reveals $u_E^{\mathrm{far}} \in C^\infty(B_{2\delta h_\ell(E)}(\mathbf{x}'))$. Integration by parts yields

$$\begin{aligned}
0 &= -\int_{B_{2\delta h_\ell(E)}(\mathbf{x}')} \Delta u_E^{\mathrm{far}} \, v \, d\mathbf{x} = \int_{B_{2\delta h_\ell(E)}(\mathbf{x}')} \nabla u_E^{\mathrm{far}} \cdot \nabla v \, d\mathbf{x} - \int_{\partial B_{2\delta h_\ell(E)}(\mathbf{x}')} \partial_{\mathbf{n}} u_E^{\mathrm{far}} \, v \, d\Gamma \\
&= \int_{B_{2\delta h_\ell(E)}(\mathbf{x}')} \nabla u_E^{\mathrm{far}} \cdot \nabla v \, d\mathbf{x},
\end{aligned} \tag{20}$$

since v vanishes on the boundary of $B_{2\delta h_\ell(E)}(\mathbf{x}')$.

The Caccioppoli inequality (18) now is a manifestation of interior regularity. According to [36, Lemma 5.7.1], it holds that

$$\|D^2 u\|_{L^2(B_r)} \lesssim \|f\|_{L^2(B_{r+h})} + \frac{1}{h} \|\nabla u\|_{L^2(B_{r+h})} + \frac{1}{h^2}\|u\|_{L^2(B_{r+h})}$$

for each $u \in H^1(B_{r+h})$ such that $u \in H^2(B_r)$ and $\Delta u = f$ on B_{r+h} with balls $B_r \subset B_{r+h}$ with radii $0 < r < r+h$ and some $f \in L^2(B_{r+h})$. The constant hidden in the $\lesssim$ notation depends only on the space dimension, but is independent of $r, h > 0$ and u, f. We use the last estimate for $u = u_E^{\mathrm{far}} - c_E$, where $c_E = \fint_{B_{2\delta h_\ell(E)}(\mathbf{x})} u_E^{\mathrm{far}}(\mathbf{y}) \, d\mathbf{y}$. According to (20), $f = 0$. With this and a Poincaré inequality, we conclude the proof. □

Lemma 3.6 (far-field bound). *With the assumptions and notations of Lemma 3.5, it holds that*

$$\|h_\ell^{1/2} \nabla_\Gamma u_E^{\mathrm{far}}\|^2_{L_2(E)} \le C_{\mathrm{far}} \|\nabla u_E^{\mathrm{far}}\|^2_{L_2(U_E)}. \tag{21}$$

The constant $C_{\mathrm{far}} > 0$ depends only on Γ and the γ-shape regularity of $\mathcal{E}_\ell$.

Proof. Without loss of generality, we may assume that $E \subseteq \mathbb{R}^2$ and that the center of mass of E is $c_E = 0$. We now consider the scaled element $\widehat{E} = h_\ell^{-1} E$ so that $|\widehat{E}| = 1$. All angles of $\widehat{E}$ are controlled in terms of the γ-shape regularity of $\mathcal{E}_\ell$. Consequently, there is a constant $c > 0$ which depends only on γ such that $\widehat{E} \subseteq [-c,c]^2$. In particular, there are finitely many $\widetilde{\mathbf{x}}_1, \dots, \widetilde{\mathbf{x}}_N \in [-c,c]^2$ with $|\widetilde{\mathbf{x}}_j - \widetilde{\mathbf{x}}_k| \le \delta/2$ if $B_\delta(\widetilde{\mathbf{x}}_j) \cap B_\delta(\widetilde{\mathbf{x}}_k) \neq \emptyset$ such that

$$[-c,c]^2 \subseteq \bigcup_{j=1}^N B_\delta(\widetilde{\mathbf{x}}_j).$$

Note that the number $N \in \mathbb{N}$ depends only on the radius $\delta > 0$, whence on γ and on Γ. Consequently, there are also $\widehat{\mathbf{x}}_1, \dots, \widehat{\mathbf{x}}_N \in \widehat{E}$ such that

$$\widehat{E} \subseteq \bigcup_{j=1}^N B_\delta(\widehat{\mathbf{x}}_j)$$

and hence by scaling $\mathbf{x}_1, \dots, \mathbf{x}_N \in E$ such that

$$E \subseteq \bigcup_{j=1}^N B_{\delta h_\ell(E)}(\mathbf{x}_j).$$

Now, let $B_i := B_{\delta h_\ell(E)}(\mathbf{x}_i)$ and $\widehat{B}_i := B_{2\delta h_\ell(E)}(\mathbf{x}_i)$. Using the trace inequality (16) and Caccioppoli's inequality (18), we infer for all i that

$$\begin{aligned}
\|\nabla_\Gamma u_E^{\mathrm{far}}\|_{L_2(B_i \cap E)}^2 &\lesssim \frac{1}{h_\ell(E)} \|\nabla u_E^{\mathrm{far}}\|_{L_2(B_i)}^2 + \|\nabla u_E^{\mathrm{far}}\|_{L_2(B_i)} \|D^2 u_E^{\mathrm{far}}\|_{L_2(B_i)} \\
&\lesssim \frac{1}{h_\ell(E)} \Big(\|\nabla u_E^{\mathrm{far}}\|_{L_2(B_i)}^2 + \|\nabla u_E^{\mathrm{far}}\|_{L_2(\widehat{B}_i)}^2 \Big).
\end{aligned}$$

We use the last estimate to get

$$\|\nabla_\Gamma u_E^{\mathrm{far}}\|_{L_2(E)}^2 \le \sum_{i=1}^N \|\nabla_\Gamma u_E^{\mathrm{far}}\|_{L_2(B_i \cap E)}^2 \lesssim \frac{1}{h_\ell(E)} \sum_{i=1}^N \|\nabla u_E^{\mathrm{far}}\|_{L_2(\widehat{B}_i)}^2 \lesssim \frac{1}{h_\ell(E)} \|\nabla u_E^{\mathrm{far}}\|_{L_2(U_E)}^2.$$

This concludes the proof of (21). $\square$

Lemma 3.7 (near-field bound). *Let $\psi \in L_2(\Gamma)$. For each $E \in \mathcal{E}_\ell$, define $u_E^{\mathrm{near}} = V(\psi \chi_{U_E \cap \Gamma})$ as the near-field of $u = V\psi$ with respect to E. Then, $u_E^{\mathrm{near}}|_\Gamma \in H^1(\Gamma)$, $u_E^{\mathrm{near}}|_\Omega \in H^1(\Omega)$ and*

$$\sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma u_E^{\mathrm{near}}\|_{L_2(E)}^2 \le C_{\mathrm{near}} \|h_\ell^{1/2} \psi\|_{L_2(\Gamma)}^2. \tag{22}$$

Moreover, for discrete $\psi = \Psi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$, it holds that

$$\sum_{E \in \mathcal{E}_\ell} \|\nabla u_E^{\mathrm{near}}\|_{L_2(U_E)}^2 \le C_{\mathrm{near}} \|h_\ell^{1/2} \Psi_\ell\|_{L_2(\Gamma)}^2. \tag{23}$$

The constant $C_{\mathrm{near}} > 0$ depends only on Γ and the γ-shape regularity of $\mathcal{E}_\ell$.

Proof. According to [44], $V : L_2(\Gamma) \to H^1(\Gamma)$ is a bounded linear operator. Therefore, for each $E \in \mathcal{E}_\ell$

$$\|\nabla_\Gamma u_E^{\mathrm{near}}\|_{L_2(E)} \le \|V(\psi \chi_{U_E \cap \Gamma})\|_{H^1(\Gamma)} \lesssim \|\psi \chi_{U_E \cap \Gamma}\|_{L_2(\Gamma)} = \|\psi\|_{L_2(U_E \cap \Gamma)}.$$

Summing the last estimate over $\mathcal{E}_\ell$ and using the finite overlap property (17b) of the set U_E, we thus see

$$\sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma u_E^{\mathrm{near}}\|_{L_2(E)}^2 = \sum_{E \in \mathcal{E}_\ell} |E| \|\nabla_\Gamma u_E^{\mathrm{near}}\|_{L_2(E)}^2 \lesssim \sum_{E \in \mathcal{E}_\ell} |E| \|\psi\|_{L_2(U_E \cap \Gamma)}^2 \simeq \|h_\ell^{1/2} \psi\|_{L_2(\Gamma)}^2,$$

where all estimates hold with constants which depend only on the γ-shape regularity. This concludes the proof of (22).

To prove (23), we fix an element $E \in \mathcal{E}_\ell$. Taking into account (17a) and applying the gradient, we see

$$\nabla u_E^{\rm near}(\mathbf{x}) = \frac{1}{4\pi} \sum_{E' \in \omega_\ell(E)} \Psi_\ell(E') \int_{E'} \nabla_{\mathbf{x}} \frac{\chi_{U_E \cap \Gamma}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \, d\Gamma(\mathbf{y}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^3 \setminus \Gamma,$$

where $\Psi_\ell(E')$ denotes the value of $\Psi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$ on the element E'. The number of elements E' in the patch $\omega_\ell(E)$ is bounded in terms of the γ-shape regularity and thus

$$|\nabla u_E^{\rm near}(\mathbf{x})|^2 \lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell(E')|^2 \Big(\int_{E'} \Big| \nabla_{\mathbf{x}} \frac{1}{|\mathbf{x}-\mathbf{y}|} \Big| \, d\Gamma(\mathbf{y}) \Big)^2.$$

Since the mesh $\mathcal{E}_\ell$ is γ-shape regular, we can select a constant $c > 0$, which depends solely on the shape regularity constant γ, such that for each E' we have $U_E \subset B_{ch_\ell(E)}(b_{E'})$, where $B_{ch_\ell(E)}(b_{E'})$ is the ball of radius $ch_\ell(E)$ centered at the barycenter $b_{E'}$ of E'. We integrate over U_E and estimate the remaining integral

$$\begin{aligned} \int_{U_E} |\nabla u_E^{\rm near}(\mathbf{x})|^2 \, d\mathbf{x} &\lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell(E')|^2 \int_{U_E} \Big(\int_{E'} \Big| \nabla_{\mathbf{x}} \frac{1}{|\mathbf{x}-\mathbf{y}|} \Big| \, d\Gamma(\mathbf{y}) \Big)^2 \, d\mathbf{x} \\ &\lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell(E')|^2 \int_{B_{ch_\ell(E)}(b_{E'})} \Big(\int_{B_{ch_\ell(E)}(b_{E'}) \cap \Gamma_{E'}} \frac{1}{|\mathbf{x}-\mathbf{y}|^2} \, d\Gamma(\mathbf{y}) \Big)^2 \, d\mathbf{x}, \end{aligned}$$

where we denote by $\Gamma_{E'}$ the plane that is spanned by E'. Scaling arguments then yield

$$\int_{U_E} |\nabla u_E^{\rm near}(\mathbf{x})|^2 \, d\mathbf{x} \lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell(E')|^2 h_\ell(E)^3 \int_{B_1(\mathbf{0})} \Big(\int_{B_1(\mathbf{0}) \cap \mathbb{R}^2} \frac{1}{|\mathbf{x}-\mathbf{y}|^2} \, d\Gamma(\mathbf{y}) \Big)^2 \, d\mathbf{x}$$

The use of polar coordinates in $\mathbb{R}^2$ for the inner integral and $\mathbb{R}^3$ for the outer integral shows

$$\int_{B_1(\mathbf{0})} \Big(\int_{B_1(\mathbf{0}) \cap \mathbb{R}^2} \frac{1}{|\mathbf{x}-\mathbf{y}|^2} \, d\Gamma(\mathbf{y}) \Big)^2 \, d\mathbf{x} < \infty.$$

Finally, we are thus led to

$$\int_{U_E} |\nabla u_E^{\rm near}(\mathbf{x})|^2 \, d\mathbf{x} \lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell(E')|^2 h_\ell(E)^3 \lesssim \|h_\ell^{1/2} \Psi_\ell\|^2_{L_2(\omega_\ell(E))}.$$

Summing this last estimate over all $E \in \mathcal{E}_\ell$ gives (23). $\square$

Proof of Proposition 3.3. We adopt the notation of Lemma 3.5 and Lemma 3.7 for $\psi = \Psi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$ and note that $V\Psi_\ell = u_E^{\rm near} + u_E^{\rm far}$ for each $E \in \mathcal{E}_\ell$. We now write

(24)
$$\|h^{1/2} \nabla_\Gamma V \Psi_\ell\|^2_{L_2(\Gamma)} = \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma V \Psi_\ell\|^2_{L_2(E)} \lesssim \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma u_E^{\rm far}\|^2_{L_2(E)} + \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma u_E^{\rm near}\|^2_{L_2(E)}$$

The far-field contribution on the right hand side is bounded by use of the far-field bound (21) of Lemma 3.6 and a triangle inequality for $u_E^{\rm far} = V\Psi_\ell - u_E^{\rm near}$,

$$\sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma u_E^{\rm far}\|^2_{L_2(E)} \lesssim \sum_{E \in \mathcal{E}_\ell} \|\nabla u_E^{\rm far}\|^2_{L_2(U_E)} \lesssim \sum_{E \in \mathcal{E}_\ell} \|\nabla V \Psi_\ell\|^2_{L_2(U_E)} + \sum_{E \in \mathcal{E}_\ell} \|\nabla u_E^{\rm near}\|^2_{L_2(U_E)}. \tag{25}$$

The first term in (25) is estimated by stability of V and the finite overlap property (17b)

$$\sum_{E\in\mathcal{E}_\ell} \|\nabla V\Psi_\ell\|_{L_2(U_E)}^2 \lesssim \|\nabla V\Psi_\ell\|_{L_2(U)}^2 \lesssim \|\Psi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}^2.$$

Here, $U \subset \mathbb{R}^3$ is an arbitrary Lipschitz domain with $\bigcup_{E\in\mathcal{E}_\ell} U_E \subseteq U$ which may thus be chosen independently of ℓ. The second term in (25) is bounded with the help of the discrete near-field bound (23). Finally, the near-field contribution in on the right hand side of (24) is bounded by the near-field bound (22). Up to now, this proves

$$\|h_\ell^{1/2}\nabla_\Gamma V\Psi_\ell\|_{L_2(\Gamma)}^2 \lesssim \|\Psi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 + \|h_\ell^{1/2}\Psi_\ell\|_{L_2(\Gamma)}^2.$$

Finally, an inverse estimate from [29, Theorem 3.6] for the $\widetilde{H}^{-1/2}$-norm states

$$\|h_\ell^{1/2}\Psi_\ell\|_{L_2(\Gamma)}^2 \lesssim \|\Psi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 \simeq |||\Psi_\ell|||, \tag{26}$$

where the hidden constant depends only on Γ and the γ-shape regularity of $\mathcal{E}_\ell$. This and norm equivalence $|||\cdot||| \simeq \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$ conclude the proof of (15). □

3.3. Proof of contraction theorem (Theorem 3.1). We first note that due to nestedness $\mathcal{P}^0(\mathcal{E}_\ell) \subseteq \mathcal{P}^0(\mathcal{E}_{\ell+1})$ and Galerkin orthogonality

$$\langle\!\langle \phi - \Phi_{\ell+1}\,, \Psi_{\ell+1}\rangle\!\rangle = 0 \quad \text{for all } \Psi_{\ell+1} \in \mathcal{P}^0(\mathcal{E}_{\ell+1}),$$

the Pythagoras theorem states

$$|||\phi - \Phi_{\ell+1}|||^2 = |||\phi - \Phi_\ell|||^2 - |||\Phi_{\ell+1} - \Phi_\ell|||^2.$$

Combining the estimator reduction (13) and the inverse estimate (15), we see

$$\mu_{\ell+1}^2 \le \widetilde{\kappa}\,\mu_\ell^2 + C_{\rm red}\|h_{\ell+1}^{1/2}\nabla_\Gamma V(\Phi_{\ell+1} - \Phi_\ell)\|_{L_2(\Gamma)}^2 \le \widetilde{\kappa}\,\mu_\ell^2 + C_{\rm red}C_{\rm inv}^2\,|||\Phi_{\ell+1} - \Phi_\ell|||^2. \tag{27}$$

For sufficiently small $\lambda > 0$, the weighted sum of the last two estimates gives

$$\begin{aligned}\Delta_{\ell+1} = |||\phi - \Phi_{\ell+1}|||^2 + \lambda\mu_{\ell+1}^2 &\le |||\phi - \Phi_\ell|||^2 + \lambda\widetilde{\kappa}\,\mu_\ell^2 + (\lambda C_{\rm red}C_{\rm inv}^2 - 1)\,|||\Phi_{\ell+1} - \Phi_\ell|||^2\\ &\le |||\phi - \Phi_\ell|||^2 + \lambda\widetilde{\kappa}\,\mu_\ell^2.\end{aligned}$$

We introduce an additional parameter $\varepsilon > 0$ and use the reliability (10) of μ_ℓ to see

$$|||\phi - \Phi_\ell|||^2 + \lambda\widetilde{\kappa}\,\mu_\ell^2 \le (1 - \lambda\varepsilon)\,|||\phi - \Phi_\ell|||^2 + \lambda(\widetilde{\kappa} + C_{\rm rel}^2\varepsilon)\,\mu_\ell^2 \le \kappa\,\Delta_\ell,$$

where $\kappa := \max\{1 - \lambda\varepsilon\,, \widetilde{\kappa} + C_{\rm rel}^2\varepsilon\}$. Recall that $\lambda > 0$ and $0 < \widetilde{\kappa} < 1$. For sufficiently small $\varepsilon > 0$, we obtain $0 < \kappa < 1$ and conclude the proof. □

Remark 5. *Following the seminal work* [5], *it can be shown that adaptive boundary or finite element methods converge a-priori, i.e.:* $\lim_{\ell\to\infty} |||\Phi_{\ell+1} - \Phi_\ell||| = 0$ *in the context of the preceding proof. We stress that estimate* (27) *is then already sufficient to obtain estimator convergence* $\lim_{\ell\to\infty}\mu_\ell^2 = 0$. *This concept of estimator reduction is followed in* [3] *to prove convergence of* $(h - h/2)$*-based adaptive mesh-refinement. In* [2], *this result is extended to include the adaptive resolution of the given data so that the overall scheme only deals with discrete integral operators, i.e. matrices.* □

4. Quasi-optimality

In this section, we prove a quasi-optimality result for Algorithm 2.1. To that end, let $\mathbb{T}$ denote the set of all regular triangulations that can be obtained from the initial mesh $\mathcal{E}_0$ by arbitrary steps of newest vertex bisection. We assume that the initial triangulation $\mathcal{E}_0$ satisfies the admissibility condition of [7], i.e. each interior edge $e = E_+ \cap E_-$ with $E_+, E_- \in \mathcal{E}_0$ is either the reference edge of both E_+ and E_- or of none of them. Under this assumption, it has been shown in [7] that the closure step in newest vertex bisection, which avoids hanging nodes, has optimal complexity in the sense of

$$\#\mathcal{E}_\ell - \#\mathcal{E}_0 \le C_{\mathrm{nvb}} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j, \tag{28}$$

where $C_{\mathrm{nvb}} > 0$ depends only on $\mathcal{E}_0$.

Now, let $\mathbb{T}_N = \big\{\mathcal{E}_\star \in \mathbb{T} \,:\, \#\mathcal{E}_\star - \#\mathcal{E} \le N\big\}$ denote the set of all triangulations that have at most N elements more than the initial triangulation $\mathcal{E}_0$. For given $s > 0$, we write $\phi \in \mathbb{A}_s$ if and only if

$$|\phi|_s := \sup_{N>0} \big(N^s \inf_{\mathcal{E}_\star \in \mathbb{T}_N} \mu_\star\big) < \infty. \tag{29}$$

Here, $\mu_\star$ denotes the weighted residual error estimator with respect to $\mathcal{E}_\star$. The idea behind the definition of $\mathbb{A}_s$ is that it characterizes the best possible convergence behavior $\mu_\star = \mathcal{O}(N^{-s})$. The main theorem states that the triangulations $\mathcal{E}_\ell$ generated by Algorithm 2.1 satisfy $\mu_\ell = \mathcal{O}(N^{-s})$ for the corresponding error estimators, i.e. the sequence of estimators have a quasi-optimal decay. The proof of this theorem is carried out in Section 4.4.

Theorem 4.1. *There is a constant $0 < \theta_\star < 1$ such that for all adaptivity parameters $0 < \theta \le \theta_\star$, the discrete solutions Φ_ℓ generated by Algorithm 2.1 satisfy*

$$C_{\mathrm{rel}}^{-1} |||\phi - \Phi_\ell||| \le \mu_\ell \le C_{\mathrm{opt}} \, (\#\mathcal{E}_\ell - \#\mathcal{E}_0)^{-s} \tag{30}$$

provided that $\phi \in \mathbb{A}_s$ for some $s > 0$. The constant $C_{\mathrm{opt}} > 0$ then depends on $|\phi|_s$, the use of newest vertex bisection, and the appropriate labeling of the reference edges to ensure (28).

Remark 6. *For quasi-uniform meshes $\mathcal{E}_\ell$, it has been shown in* [10] *that the weighted residual error estimator μ_ℓ is not only reliable, see* (10)*, but also efficient, i.e.*

$$\mu_\ell \le C_{\mathrm{eff}} \, |||\phi - \Phi_\ell||| \tag{31}$$

for some $C_{\mathrm{eff}} > 0$. An immediate consequence of Theorem 4.1 is that the convergence order for adaptive mesh refinement is at least as good as for uniform mesh-refinement. To see this, we denote by $\big(\Phi_\ell^{\mathrm{unif}}\big)_{\ell \in \mathbb{N}}$ a sequence of Galerkin solutions computed with uniform mesh refinement. If $|||\phi - \Phi_\ell^{\mathrm{unif}}||| = \mathcal{O}(N^{-t})$, we conclude from (31) *that $\phi \in \mathbb{A}_t$. With Theorem 4.1, we infer that adaptive mesh-refinement has at least the same convergence order t.* □

Remark 7. *Theorem 4.1 is stated for 3D and newest vertex bisection. For 2D, the constants in all estimates depend on the uniform boundedness of the K-mesh constant (or: local mesh-ratio)*

$$\kappa(\mathcal{E}_\ell) = \max\big\{h_\ell(E)/h_\ell(E') \,:\, E, E' \in \mathcal{E}_\ell \text{ with } E \cap E' \neq \emptyset\big\} \tag{32}$$

instead of the γ-shape regularity. The work [4, Section 2.2] *provides a local mesh-refinement for 2D BEM, which guarantees optimality of the closure* (28) *as well as $\kappa(\mathcal{E}_\ell) \le 2\,\kappa(\mathcal{E}_0)$. With this mesh refinement, Theorem 4.1 holds for 2D accordingly.* □

Remark 8. *The characterization of the approximation class $\mathbb{A}_s$ remains an open issue for future research. For the 2D Dirichlet problem $V\phi = (K+1/2)g$, see Remark 1, it is shown in* [1] *that*

$$|\phi|_s \simeq \sup_{N>0}\Big(N^s \inf_{\mathcal{E}_\star\in\mathbb{T}_N}\ \inf_{\Psi_\star\in\mathcal{P}^0(\mathcal{E}_\star)} |||\phi-\Psi_\star||| \Big),$$

i.e., μ_ℓ-adaptive mesh refinement recovers in fact the best possible convergence rate for the error in $\widetilde{H}^{-1/2}(\Gamma)$. □

4.1. Weak efficiency of weighted residual estimator. Although the efficiency estimate (31) is observed in numerical experiments, see e.g. [12], it has not been shown mathematically, yet. However, for the proof of Theorem 4.1 it suffices to provide some weaker efficiency estimate of the type

$$C_{\rm eff}^{-1}\mu_\ell \le |||\phi-\Phi_\ell||| + {\rm osc}_\ell, \tag{33}$$

where the data oscillations ${\rm osc}_\ell$ are locally dominated by the error estimator, i.e.

$${\rm osc}_\ell^2 = \sum_{E\in\mathcal{E}_\ell}{\rm osc}_\ell(E)^2 \quad\text{with}\quad {\rm osc}_\ell(E)^2 \le C_{\rm osc}\sum_{E'\in\omega_\ell(E)}\mu_\ell(E)^2 \quad \text{for all } E\in\mathcal{E}_\ell. \tag{34}$$

Both (33) and (34) are proven in the following proposition.

Proposition 4.2. *Let $\mathbb{P}_\ell$ denote the Scott-Zhang projection onto the space of continuous, $\mathcal{E}_\ell$-piecewise affine functions $\mathcal{S}^1(\mathcal{E}_\ell)$, cf.* [40], *which is defined in a way that $\mathbb{P}_\ell$ is a continuous operator with respect to both L_2-norm and H^1-norm on Γ. Then,* (33)*–*(34) *hold with*

$${\rm osc}_\ell(E) = \|h_\ell^{1/2}\nabla_\Gamma(1-\mathbb{P}_\ell)(V\Phi_\ell - f)\|_{L_2(E)} \quad \textit{for all } E\in\mathcal{E}_\ell. \tag{35}$$

The constants $C_{\rm eff}, C_{\rm osc}>0$ depend only on γ-shape regularity of $\mathcal{E}_\ell$.

Proof. Let $R_\ell := V\Phi_\ell - f \in H^1(\Gamma)$ abbreviate the residual. By the triangle inequality and the inverse estimate

$$\|h_\ell^{1/2}\nabla_\Gamma V_\ell\|_{L_2(\Gamma)} \lesssim \|V_\ell\|_{H^{1/2}(\Gamma)} \quad \text{for all } V_\ell\in\mathcal{S}^1(\mathcal{E}_\ell)$$

from [15, Proposition 3.1] for the $H^{1/2}$-norm, we see

$$\begin{aligned}\mu_\ell = \|h_\ell^{1/2}\nabla_\Gamma R_\ell\|_{L_2(\Gamma)} &\le \|h_\ell^{1/2}\nabla_\Gamma\mathbb{P}_\ell R_\ell\|_{L_2(\Gamma)} + \|h_\ell^{1/2}\nabla_\Gamma(1-\mathbb{P}_\ell)R_\ell\|_{L_2(\Gamma)}\\ &\lesssim \|\mathbb{P}_\ell R_\ell\|_{H^{1/2}(\Gamma)} + {\rm osc}_\ell.\end{aligned}$$

The L_2- and H^1-stability of $\mathbb{P}_\ell$ imply the $H^{1/2}$-stability. Together with the stability of $V : \widetilde{H}^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$, we see

$$\|\mathbb{P}_\ell R_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|R_\ell\|_{H^{1/2}(\Gamma)} = \|V(\Phi_\ell-\phi)\|_{H^{1/2}(\Gamma)} \lesssim \|\Phi_\ell-\phi\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

Norm equivalence $|||\cdot||| \simeq \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$ thus proves (33).

Finally, (34) follows from the local H^1-stability of $\mathbb{P}_\ell$, i.e.

$$\|\nabla_\Gamma\mathbb{P}_\ell v\|_{L_2(E)} \lesssim \|\nabla_\Gamma v\|_{L_2(\omega_\ell(E))} \quad \text{for all } v\in H^1(\Gamma).$$

Since $\mu_\ell(E) = \|h_\ell^{1/2}\nabla_\Gamma R_\ell\|_{L^2(E)}$, this gives

$${\rm osc}_\ell(E) \le \mu_\ell(E) + \|h_\ell^{1/2}\nabla_\Gamma\mathbb{P}_\ell R_\ell\|_{L_2(E)} \lesssim \mu_\ell(E) + \|h_\ell^{1/2}\nabla_\Gamma R_\ell\|_{L_2(\omega_\ell(E))} \lesssim \Big(\sum_{E'\subseteq\omega_\ell(E)}\mu_\ell(E')^2\Big)^{1/2}$$

and concludes the proof. □

4.2. Discrete local reliability. In this subsection, we prove that the energy error of two Galerkin solutions is bounded by the estimator contributions on the refined elements. Such a result is needed to prove the optimality of the Dörfler marking in Proposition 4.4.

Proposition 4.3. *Let $\mathcal{E}_\star \in \mathbb{T}$ be an arbitrary refinement of $\mathcal{E}_\ell$ with corresponding Galerkin solution $\Phi_\star \in \mathcal{P}^0(\mathcal{E}_\star)$. Define the set*

$$\mathcal{R}_\ell := \omega_\ell(\mathcal{E}_\ell \setminus \mathcal{E}_\star) := \big\{E \in \mathcal{E}_\ell \,:\, \exists E' \in \mathcal{E}_\ell \setminus \mathcal{E}_\star : \quad E \in \omega_\ell(E')\big\} \tag{36}$$

of elements $E \in \mathcal{E}_\ell$ that belong to the patch of a refined element $E' \in \mathcal{E}_\ell \setminus \mathcal{E}_\star$. Then, there holds

$$\#\mathcal{R}_\ell \le \#\omega_\ell(\mathcal{R}_\ell) \le C'_{\rm dlr}\#\big(\mathcal{E}_\ell \setminus \mathcal{E}_\star\big)$$

as well as

$$|||\Phi_\star - \Phi_\ell||| \le C_{\rm dlr}\Big(\sum_{E\in\mathcal{R}_\ell}\mu_\ell(E)^2\Big)^{1/2}. \tag{37}$$

The constant $C_{\rm dlr} > 0$ depends only on Γ and the use of newest vertex bisection. The constant $C'_{\rm dlr} > 0$ depends only on γ-shape regularity.

Proof. The bounds on the cardinality of $\mathcal{R}_\ell$ follow from the fact that for γ-shape regular meshes, the number of patches meeting at one element is uniformly bounded by a constant depending solely on γ.

We now show (37). As $\langle\!\langle \cdot\,,\cdot\rangle\!\rangle = \langle V\cdot\,,\cdot\rangle$ induces an equivalent scalar product on the space $\widetilde{H}^{-1/2}(\Gamma)$, we have

$$|||\Phi_* - \Phi_\ell|||^2 = \langle V(\Phi_* - \Phi_\ell)\,, \Phi_* - \Phi_\ell\rangle = \langle f - V\Phi_\ell\,, \Phi_* - \Phi_\ell\rangle.$$

The last equality follows from Galerkin orthogonality since $\Phi_* - \Phi_\ell \in \mathcal{P}^0(\mathcal{E}_*)$.

Let $\eta_z \in \mathcal{S}^1(\mathcal{E}_\ell)$ denote the hat function associated with a node $z \in \mathcal{N}_\ell$. The set of nodes lying in a refined element is denoted by $\mathcal{N}_\ell^{\mathcal{R}} := \mathcal{N}_\ell \cap \Big(\bigcup\big(\mathcal{E}_\ell \setminus \mathcal{E}_\star\big)\Big)$. Additionally, we introduce the *layer* around the refined elements, i.e., $S_\ell := \mathcal{R}_\ell \setminus \big(\mathcal{E}_\ell \setminus \mathcal{E}_\star\big)$. Note that this gives us a disjoint decomposition

$$\mathcal{R}_\ell = (\mathcal{E}_\ell \setminus \mathcal{E}_\star)\,\dot\cup S_\ell \quad \text{ as well as } \quad \mathcal{E}_\ell = (\mathcal{E}_\ell \setminus \mathcal{E}_\star)\,\dot\cup\,(\mathcal{E}_\ell \setminus \mathcal{R}_\ell)\,\dot\cup S_\ell.$$

We define an operator $\pi_\ell : \mathcal{P}^0(\mathcal{E}_\star) \to \mathcal{P}^0(\mathcal{E}_\ell)$ $\mathcal{E}_\ell$-elementwise by

$$\pi_\ell(\Psi_\star)(E) := \begin{cases} 0 & \text{for } E \in \mathcal{E}_\ell \setminus \mathcal{E}_\star,\\ \Psi_\star(E) & \text{elsewhere.}\end{cases}$$

Recall that $\chi := \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}}\eta_z$ satisfies $\chi \in \mathcal{S}^1(\mathcal{E}_\ell)$ with $\chi_{\bigcup\mathcal{E}_\ell\setminus\mathcal{E}_\star} \equiv 1$. For any $\Psi_\star \in \mathcal{P}^0(\mathcal{E}_\star)$, the Galerkin orthogonality on $\mathcal{P}^0(\mathcal{E}_\ell)$ and the above definition of π_ℓ prove

$$\begin{aligned}\langle f - V\Phi_\ell\,, \Psi_*\rangle_\Gamma &= \langle f - V\Phi_\ell\,, (1-\pi_\ell)\Psi_*\rangle_\Gamma\\ &= \Big\langle \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}}\eta_z(f - V\Phi_\ell)\,, (1-\pi_\ell)\Psi_*\Big\rangle_\Gamma\\ &= \Big\langle \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}}\eta_z(f - V\Phi_\ell)\,, \Psi_*\Big\rangle_\Gamma - \Big\langle \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}}\eta_z(f - V\Phi_\ell)\,, \Psi_*|_{\bigcup S_\ell}\Big\rangle_\Gamma.\end{aligned}$$

The first equality follows from Galerkin orthogonality, the second from $\mathrm{supp}((1-\pi_\ell)\Psi_\star) = \mathcal{E}_\ell \setminus \mathcal{E}_\star$. The third one follows from taking into account the supports of η_z for $z \in \mathcal{N}_\ell^{\mathcal{R}}$ and the definition of

π_ℓ. We conclude

$$
\begin{aligned}
\langle f - V\Phi_\ell\,, \Psi_*\rangle_\Gamma &\le \Big\| \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} \eta_z(f - V\Phi_\ell)\Big\|_{H^{1/2}(\Gamma)} \|\Psi_*\|_{\widetilde{H}^{-1/2}(\Gamma)} \\
&\qquad + \Big\| h_\ell^{-1/2} \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} \eta_z(f - V\Phi_\ell)\Big\|_{L^2(\Gamma)} \|h_*^{1/2}\Psi_*\|_{L^2(\cup S_\ell)} \\
&\le \Big\| \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} \eta_z(f - V\Phi_\ell)\Big\|_{H^{1/2}(\Gamma)} \|\Psi_*\|_{\widetilde{H}^{-1/2}(\Gamma)} \\
&\qquad + \Big\| h_\ell^{-1/2} \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} \eta_z(f - V\Phi_\ell)\Big\|_{L^2(\Gamma)} \|h_*^{1/2}\Psi_*\|_{L^2(\Gamma)},
\end{aligned} \tag{38}
$$

where we used $h_\ell = h_*$ on S_ℓ. The inverse inequality (26) from [29, Theorem 3.6] for the $\widetilde{H}^{-1/2}(\Gamma)$-norm then yields

$$
\langle f - V\Phi_\ell\,, \Psi_*\rangle_\Gamma \lesssim \Big(\Big\| \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} \eta_z(f - V\Phi_\ell)\Big\|_{H^{1/2}(\Gamma)} + \Big\| h_\ell^{-1/2} \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} \eta_z(f - V\Phi_\ell)\Big\|_{L^2(\Gamma)} \Big) \|\Psi_*\|_{H^{-1/2}(\Gamma)}. \tag{39}
$$

Now we estimate the term inside the parentheses analogously as in [12]. We choose a decomposition $\mathcal{N}_\ell = \bigcup_{i=1}^m \mathcal{N}_\ell^i$, where the $\mathcal{N}_\ell^i$ are pairwise disjoint and $\mathrm{supp}(\eta_{z_1}) \cap \mathrm{supp}(\eta_{z_2}) = \emptyset$ for $z_1 \in \mathcal{N}_\ell^i$, $z_2 \in \mathcal{N}_\ell^j$ with $i \ne j$. With $\omega_z = \mathrm{supp}(\eta_z)$ and $h_z = \mathrm{diam}(\eta_z)$, [12, Lemma 2.1] gives

$$
\begin{aligned}
\Big\| \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} \eta_z(f - V\Phi_\ell)\Big\|_{H^{1/2}(\Gamma)}^2 &\lesssim \sum_{i=1}^m \Big\| \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}\cap\mathcal{N}_\ell^i} \eta_z(f - V\Phi_\ell)\Big\|_{H^{1/2}(\Gamma)}^2 \\
&\lesssim \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} \|\eta_z(f - V\Phi_\ell)\|_{H^{1/2}(\omega_z)}^2.
\end{aligned} \tag{40}
$$

We now argue as in [12, Theorem 3.2] and use the interpolation estimate $\|\cdot\|_{H^{1/2}(\Gamma)}^2 \lesssim \|\cdot\|_{L^2(\Gamma)} \|\cdot\|_{H^1(\Gamma)}$ to see

$$
\begin{aligned}
\|\eta_z(f - V\Phi_\ell)\|_{H^{1/2}(\omega_z)}^2 &\le \|\eta_z(f - V\Phi_\ell)\|_{H^{1/2}(\Gamma)}^2 \\
&\lesssim \|\eta_z(f - V\Phi_\ell)\|_{L^2(\omega_z)} \|\eta_z(f - V\Phi_\ell)\|_{H^1(\omega_z)} \\
&\lesssim h_z(1 + h_z^2)^{1/2} \|\nabla_\Gamma(\eta_z(f - V\Phi_\ell))\|_{L^2(\omega_z)}^2,
\end{aligned} \tag{41}
$$

where the last estimate follows from Friedrichs' inequality. Using $\langle V(\Phi_* - \Phi_\ell)\,, \chi_\ell\rangle = 0$ for every characteristic function $\chi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$ and Poincaré's inequality we get

$$
\begin{aligned}
\|\nabla_\Gamma(\eta_z(f - V\Phi_\ell))\|_{L^2(\omega_z)} &\le \|(\nabla_\Gamma\eta_z)(f - V\Phi_\ell))\|_{L^2(\omega_z)} + \|\eta_z\nabla_\Gamma(f - V\Phi_\ell)\|_{L^2(\omega_z)} \\
&\lesssim \|h_\ell^{-1}(f - V\Phi_\ell)\|_{L^2(\omega_z)} + \|\nabla_\Gamma(f - V\Phi_\ell)\|_{L^2(\omega_z)} \\
&\lesssim \|\nabla_\Gamma(f - V\Phi_\ell)\|_{L^2(\omega_z)}.
\end{aligned} \tag{42}
$$

We combine the estimates (40)–(42) to see

$$
\Big\| \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} \eta_z(f - V\Phi_\ell)\Big\|_{H^{1/2}(\Gamma)}^2 \lesssim \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} h_z \|\nabla_\Gamma(f - V\Phi_\ell)\|_{L^2(\omega_z)}^2 \lesssim \|h_\ell^{1/2}\nabla_\Gamma(f - V\Phi_\ell)\|_{L_2(\mathcal{R}_\ell)}^2. \tag{43}
$$

The second term in parentheses in (39) is estimated analogously

$$\begin{aligned}\Big\|h_\ell^{-1/2}\sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}}\eta_z(f-V\Phi_\ell)\Big\|_{L^2(\Gamma)}^2 &\lesssim \sum_{i=1}^m\Big\|\sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}\cap\mathcal{N}_\ell^i} h_\ell^{-1/2}\eta_z(f-V\Phi_\ell)\Big\|_{L^2(\Gamma)}^2\\ &\lesssim \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}} h_z^{-1}\|\eta_z(f-V\Phi_\ell)\|_{L^2(\omega_z)}^2\\ &\lesssim \sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}}\|h_\ell^{1/2}\nabla_\Gamma(\eta_z(f-V\Phi_\ell)\|_{L^2(\omega_z)}^2.\end{aligned}$$

From (42) we conclude

$$\Big\|h_\ell^{-1/2}\sum_{z\in\mathcal{N}_\ell^{\mathcal{R}}}\eta_z(f-V\Phi_\ell)\Big\|_{L^2(\Gamma)}\lesssim\|h_\ell^{1/2}\nabla_\Gamma(f-V\Phi_\ell)\|_{L^2(\mathcal{R}_\ell)}. \tag{44}$$

Plugging (43)–(44) into (39) and setting $\Psi_\star=\Phi_\star-\Phi_\ell$ finally gives

$$|||\Phi_*-\Phi_\ell|||^2=\langle f-V\Phi_\ell\,,\Psi_*\rangle\lesssim\|h_\ell^{1/2}\nabla_\Gamma(f-V\Phi_\ell)\|_{L^2(\mathcal{R}_\ell)}\,\|\Phi_*-\Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

Norm equivalence $|||\cdot|||\simeq\|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$ concludes the proof. We stress that $C_{\mathrm{dlr}}>0$ depends formally on the shape of node patches ω_z, where Friedrichs' and Poincares' inequalities are used. Since newest vertex bisection only leads to finitely many shapes of these patches, C_{dlr} is independent of $\mathcal{E}_\ell$ but depends on the use of newest vertex bisection. □

Remark 9. *We stress that the reliability* (10) *of μ_ℓ is a consequence of the discrete local reliability* (37) *and quasi-optimality of the Galerkin method: We may assume $\phi\neq\Phi_\ell$. Galerkin orthogonality and discrete local reliability* (37) *give*

$$|||\phi-\Phi_\ell|||^2=|||\phi-\Phi_\star|||^2+|||\Phi_\star-\Phi_\ell|||^2\le|||\phi-\Phi_\star|||^2+C_{\mathrm{dlr}}^2\mu_\ell^2.$$

for each refinement $\mathcal{E}_\star$ of $\mathcal{E}_\ell$. Now, we exploit the best approximation property of the Galerkin method and its consequence that uniform mesh refinement always leads to convergence. Given $\varepsilon>0$, we find a uniform refinement $\mathcal{E}_\star$ of $\mathcal{E}_\ell$ such that

$$|||\phi-\Phi_\star|||^2\le\varepsilon|||\phi-\Phi_\ell|||^2.$$

We conclude

$$|||\phi-\Phi_\ell|||^2\le\frac{C_{\mathrm{dlr}}^2}{1-\varepsilon}\mu_\ell^2\quad\textit{for all }\varepsilon>0,$$

i.e., the reliability (10) *of μ_ℓ even with $C_{\mathrm{rel}}=C_{\mathrm{dlr}}$.* □

4.3. Optimality of Dörfler marking. So far, we have proven that Dörfler marking (11) guarantees contraction (12) of the quasi-error Δ_ℓ defined in (12). In this section, we prove the converse estimate, i.e. a certain contraction of the quasi-error implies that the set of refined elements satisfies the Dörfler marking.

Proposition 4.4. *There are constants $0<\theta_\star,\kappa_\star<1$ such that for all refinements $\mathcal{E}_\star\in\mathbb{T}$ of $\mathcal{E}_\ell$ which satisfy $\Delta_\star\le\kappa_\star\Delta_\ell$ for the corresponding quasi-errors of* (12)*, the set $\mathcal{R}_\ell\subseteq\mathcal{E}_\ell$ of* (36) *satisfies the Dörfler marking*

$$\theta\,\mu_\ell^2\le\sum_{E\in\omega_\ell(\mathcal{R}_\ell)}\mu_\ell(E)^2 \tag{45}$$

for all $0 < \theta \le \theta_\star$. Here, $\omega_\ell(\mathcal{R}_\ell) = \{E' \in \mathcal{E}_\ell : \exists E \in \mathcal{R}_\ell : E' \in \omega_\ell(E)\}$ is the set of refined elements $\mathcal{E}_\ell \setminus \mathcal{E}_\star$ plus two additional layers of elements. The constants $\theta_\star$ and $\kappa_\star$ depend on Γ and the use of newest vertex bisection.

The proof needs an estimate for the oscillation terms.

Lemma 4.5. *Let $\mathcal{E}_\star$ be an arbitrary refinement of $\mathcal{E}_\ell$ with corresponding oscillations*

$$\mathrm{osc}_\star^2 = \sum_{E\in\mathcal{E}_\star} \|h_\star^{1/2}\nabla_\Gamma(1-\mathbb{P}_\star)(V\Phi_\star - f)\|_{L_2(E)}^2$$

and $\mathcal{R}_\ell := \omega_\ell(\mathcal{E}_\ell \setminus \mathcal{E}_\star)$ be defined in (36). Then,

$$\sum_{E\in\mathcal{E}_\ell\setminus\mathcal{R}_\ell} \mathrm{osc}_\ell(E)^2 \le 2 \sum_{E\in\mathcal{E}_\ell\setminus\mathcal{R}_\ell} \mathrm{osc}_\star(E)^2 + C_3 \sum_{E\in\mathcal{R}_\ell} \mu_\ell(E)^2 \tag{46}$$

with some constant $C_3 > 0$ that depends solely on Γ and the γ-shape regularity of $\mathcal{E}_\ell$.

Proof. Take an element $E \in \mathcal{E}_\ell \setminus \mathcal{R}_\ell$. For a function v we have $(\mathbb{P}_\ell v)|_E = (\mathbb{P}_\star v)|_E$ due to the (local) definition of the Scott-Zhang projection. Using stability of $\mathbb{P}_\ell$ after a triangle inequality yields

$$\begin{aligned}\mathrm{osc}_\ell(E)^2 &= h_\ell(E)\|\nabla_\Gamma(1-\mathbb{P}_\ell)(V\Phi_\ell - f)\|_{L_2(E)}^2\\ &\le 2h_\ell(E)\|\nabla_\Gamma(1-\mathbb{P}_\star)(V\Phi_\star - f)\|_{L_2(E)}^2\\ &\quad + 2h_\ell(E)\|\nabla_\Gamma(1-\mathbb{P}_\ell)V(\Phi_\ell - \Phi_\star)\|_{L_2(E)}^2\\ &\le 2\mathrm{osc}_\star(E)^2\\ &\quad + 2C_{\mathrm{apx}}h_\star(E)\|\nabla_\Gamma V(\Phi_\ell - \Phi_\star)\|_{L_2(\omega(E))}^2,\end{aligned}$$

with some constant $C_{\mathrm{apx}} > 0$. In the last step, we used $h_\ell = h_\star$ on $\omega(E)$. Taking the sum of this last estimate over all $E \in \mathcal{E}_\ell \setminus \mathcal{R}_\ell$ gives

$$\sum_{E\in\mathcal{E}_\ell\setminus\mathcal{R}_\ell} \mathrm{osc}_{\mathcal{E}}(E)^2 \le 2 \sum_{E\in\mathcal{E}_\ell\setminus\mathcal{R}_\ell} \mathrm{osc}_\star(E)^2 + 2C_{\mathrm{overlap}}C_{\mathrm{apx}}\|h_\star^{1/2}\nabla_\Gamma V(\Phi_\ell - \Phi_\star)\|_{L_2(\Gamma)}^2$$

with some constant $C_{\mathrm{overlap}} > 0$ that depends solely on γ-shape regularity, since the number of patches $\omega(E')$ the element E can be part of is bounded uniformly in ℓ. The inverse estimate (15) and the discrete local reliability of Proposition 4.3 then imply

$$\|h_\star^{1/2}\nabla_\Gamma V(\Phi_\ell - \Phi_\star)\|_{L_2(\Gamma)}^2 \lesssim |||\Phi_\ell - \Phi_\star|||^2 \lesssim \sum_{E\in\mathcal{R}_\ell} \mu_\ell(E)^2$$

Combining the last two estimates shows (46). □

Proof of Proposition 4.4. We note that the quasi-error from (12) is equivalent to the sum of error and oscillations: First, weak efficiency (33) yields

$$\Delta_\ell = |||\phi - \Phi_\ell|||^2 + \lambda\mu_\ell^2 \lesssim |||\phi - \Phi_\ell|||^2 + \mathrm{osc}_\ell^2. \tag{47}$$

Conversely, the fact that oscillations are bounded by μ_ℓ, cf. (34), gives

$$|||\phi - \Phi_\ell|||^2 + \mathrm{osc}_\ell^2 \lesssim |||\phi - \Phi_\ell|||^2 + \lambda\mu_\ell^2 = \Delta_\ell. \tag{48}$$

Assume that $\Delta_\star \le \kappa_\star\Delta_\ell$, where $\kappa_\star$ will be chosen below. In view of (47)–(48), this can equivalently be written as

$$|||\phi - \Phi_\star|||^2 + \mathrm{osc}_\star^2 \le \widehat{\kappa}_\star\left(|||\phi - \Phi_\ell|||^2 + \mathrm{osc}_\ell^2\right) \tag{49}$$

with a certain $0 < \widehat{\kappa}_\star < 1$ if $\kappa_\star$ is sufficiently small. Now, we choose $\kappa_\star$ sufficiently small to ensure $\widehat{\kappa}_\star < 1/2$. The efficiency estimate (33) together with (49) gives

$$
\begin{aligned}
(1-2\widehat{\kappa}_\star)C_{\rm eff}^{-2}\mu_\ell^2 &\le (1-2\widehat{\kappa}_\star)\left(|||\phi-\Phi_\ell|||^2+\operatorname{osc}_\ell^2\right)\\
&\le |||\phi-\Phi_\ell|||^2+\operatorname{osc}_\ell^2 - 2\left(|||\phi-\Phi_\star|||^2+\operatorname{osc}_\star^2\right)\\
&\le |||\Phi_\ell-\Phi_\star|||^2+\operatorname{osc}_\ell^2 - 2\operatorname{osc}_\star^2.
\end{aligned}
$$

The last two terms can be estimated by (34) and (46):

$$
\begin{aligned}
\operatorname{osc}_\ell^2 - 2\operatorname{osc}_\star^2 &\le \sum_{E\in\mathcal{R}_\ell}\operatorname{osc}_\ell(E)^2 + \sum_{E\in\mathcal{E}_\ell\setminus\mathcal{R}_\ell}\operatorname{osc}_\ell(E)^2 - 2\sum_{E\in\mathcal{E}_\ell\setminus\mathcal{R}_\ell}\operatorname{osc}_\star(E)^2\\
&\le (C_{\rm osc}+C_3)\sum_{E\in\omega(\mathcal{R}_\ell)}\mu_\ell(E)^2.
\end{aligned}
$$

Combining the last two estimates and using the discrete local reliability, we obtain

$$(1-2\widehat{\kappa}_\star)C_{\rm eff}^{-2}\mu_\ell^2 \le \left(C_3+C_{\rm osc}+C_{\rm dlr}^2\right)\sum_{E\in\omega(\mathcal{R}_\ell)}\mu_\ell(E)^2.$$

Choosing $\theta_\star := (1-2\widehat{\kappa}_\star)C_{\rm eff}^{-2}\left(C_3+C_{\rm osc}+C_{\rm dlr}^2\right)^{-1}$ we conclude the proof. □

4.4. Proof of quasi-optimality theorem (Theorem 4.1). The proof of Theorem 4.1 is carried out as in [19, Theorem 5.11] and follows the lines of [43]. For completeness' sake, we recall it briefly.

Proof of Theorem 4.1. Choose $\varepsilon^2 := \delta\Delta_\ell$ for some $0<\delta<1$ which is determined later. Because of $\phi\in\mathbb{A}_s$, we can choose a mesh $\mathcal{E}_\varepsilon$ with

$$\#\mathcal{E}_\varepsilon - \#\mathcal{E}_0 \lesssim \varepsilon^{-1/s} \quad\text{and}\quad \mu_\varepsilon^2\le\varepsilon^2. \tag{50}$$

As in (47) we infer $\Delta_\varepsilon \lesssim \mu_\varepsilon^2 \le \varepsilon^2 = \delta\Delta_\ell$. We claim that the overlay $\mathcal{E}_\star := \mathcal{E}_\ell\oplus\mathcal{E}_\varepsilon$ fulfils

$$\#\mathcal{E}_\star \le \#\mathcal{E}_\ell + \#\mathcal{E}_\varepsilon - \#\mathcal{E}_0, \tag{51}$$

$$\Delta_\star \lesssim \Delta_\varepsilon \lesssim \delta\Delta_\ell. \tag{52}$$

The first bound on the cardinality of $\mathcal{E}_\star$ follows according to [43] and hinges on the use of newest vertex bisection. We stress that the quasi-error is not monotone (as the error estimator is not), and therefore the first estimate in (52) has to be shown: First, the triangle inequality, the inverse estimate (15), and $h_\star\le h_\varepsilon$ yield

$$
\begin{aligned}
\mu_\star^2 &= \|h_\star^{1/2}\nabla_\Gamma(V\Phi_\star - f)\|_{L_2(\Gamma)}^2\\
&\lesssim \|h_\star^{1/2}\nabla_\Gamma(V\Phi_\varepsilon - f)\|_{L_2(\Gamma)}^2 + \|h_\star^{1/2}\nabla_\Gamma V(\Phi_\star-\Phi_\varepsilon)\|_{L_2(\Gamma)}^2\\
&\lesssim \mu_\varepsilon^2 + |||\Phi_\star-\Phi_\varepsilon|||^2.
\end{aligned}
$$

By use of the Galerkin orthogonality $|||\phi-\Phi_\star|||^2+|||\Phi_\star-\Phi_\varepsilon|||^2 = |||\phi-\Phi_\varepsilon|||^2$, we conclude

$$\Delta_\star = |||\phi-\Phi_\star|||^2+\lambda\mu_\star^2 \lesssim |||\phi-\Phi_\varepsilon|||^2+\lambda\mu_\varepsilon^2 = \Delta_\varepsilon.$$

Choosing $\delta>0$ sufficiently small in (52), we obtain $\Delta_\star\le\kappa_\star\Delta_\ell$ and may use Proposition 4.4 to see

$$\theta\mu_\ell^2 \le \sum_{E\in\omega_\ell(\mathcal{R}_\ell)}\mu_\ell(E)^2.$$

According to Algorithm 2.1, the set of marked elements has minimal cardinality, whence follows $\#\mathcal{M}_\ell\le\#\omega_\ell(\mathcal{R}_\ell)$. We conclude

$$\#\mathcal{M}_\ell \le \#\omega_\ell(\mathcal{R}_\ell) \lesssim \#\left(\mathcal{E}_\ell\setminus\mathcal{E}_\star\right) \le \#\mathcal{E}_\star - \#\mathcal{E}_\ell \le \#\mathcal{E}_\varepsilon - \#\mathcal{E}_0.$$

From (50) and the choice of ε we conclude

$$\#\mathcal{M}_\ell \lesssim \varepsilon^{-1/s} \simeq \Delta_\ell^{-1/(2s)}. \tag{53}$$

We use the optimality (28) of the mesh closure to obtain

$$\#\mathcal{E}_\ell - \#\mathcal{E}_0 \lesssim \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \lesssim \sum_{j=0}^{\ell-1} \Delta_j^{-1/(2s)}$$

Using the contraction property (12) inductively we see $\Delta_\ell \le \kappa^{\ell-j}\Delta_j$, whence $\Delta_j^{-1/(2s)} \le \kappa^{(\ell-j)/(2s)}\Delta_\ell^{-1/(2s)}$, with $0 < \kappa < 1$ from Theorem 3.1. We thus end up with

$$\#\mathcal{E}_\ell - \#\mathcal{E}_0 \lesssim \Delta_\ell^{-1/(2s)} \sum_{j=0}^{\ell-1} \kappa^{(\ell-j)/(2s)} \lesssim \Delta_\ell^{-1/(2s)},$$

where the last step follows from the convergence of the geometric series. Raising this estimate to the power $-2s$ and using $\mu_\ell^2 \lesssim \Delta_\ell$ finally gives

$$\mu_\ell^2 \lesssim \Delta_\ell \lesssim (\#\mathcal{E}_\ell - \#\mathcal{E}_0)^{-2s}.$$

This concludes the proof. $\square$

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