ASC Report No. 33/2010

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Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-03-2

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ISBN 978-3-902627-03-2

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TU WIEN

CONVERGENCE OF ADAPTIVE FEM FOR SOME ELLIPTIC OBSTACLE PROBLEM WITH INHOMOGENEOUS DIRICHLET DATA

M. FEISCHL, M. PAGE, AND D. PRAETORIUS

Abstract. In this work, we show the convergence of adaptive lowest-order FEM (AFEM) for an elliptic obstacle problem with globally affine obstacle and non-homogeneous Dirichlet data. The adaptive loop is steered by some residual based error estimator introduced in Bartels, Carstensen & Hoppe (2007) that is extended to control oscillations of the Dirichlet data, as well. In the spirit of Cascon et al. (2008), we show that a weighted sum of energy error, estimator, and Dirichlet oscillations satisfies a contraction property up to certain vanishing energy contributions. This result extends the analysis of Bartels, Carstensen & Hoppe (2007) and Page & Praetorius (2009) to the case of non-homogeneous Dirichlet data and introduces some energy estimates to overcome the lack of nestedness of the discrete spaces. A short conclusion adresses AFEM for problems with non-affine obstacles.

1. Introduction

1.1. Comments on prior work. Adaptive finite element methods based on various types of a posteriori error estimators are a famous tool in science and engeneering and are used to deal with a wide range of problems. As far as elliptic boundary value problems are concerned, convergence and even quasi-optimality of the adaptive scheme is well understood and analyzed, see e.g. [4, 15, 18, 28, 29, 35, 36].

In recent years the analysis has been extended and adapted to cover more general applications, such as the p-Laplacian [38], mixed methods [12], non-conforming elements [13], and obstacle problems. The latter is a classic introductory example to study variational inequalities which represent a whole class of problems that often arise in physical context, such as fluid filtration in porous media, the elastic-plastic torsion problem, the Stefan problem (i.e. melting solids), optimal control, or mathematical finance. For a broader understanding of these problems, we refer to [20] and the references therein. As far as a posteriori error analysis is concerned, we refer to [5, 8, 7, 16, 25, 30, 37]. Convergence of an adaptive method for elliptic obstacle problems with globally affine obstacle was proven in [9, 32]. Both of these works, however, considered homogeneous Dirichlet boundary data only.

1.2. Contributions of current work. We follow the ideas from [32], i.e. adaptive P1-FEM for some elliptic obstacle problem with globally affine obstacle. Contrary to [32] and [9], however, we consider non-homogeneous Dirichlet boundary data $g \in H^1(\Gamma)$, which are approximated by g_ℓ via nodal interpolation within each step of the adaptive loop. In contrast to the aforementioned works, we thus do not have nestedness of the discrete ansatz sets, which is a crucial ingredient of the prior convergence proofs. In the spirit of [15] and in analogy

Date: December 22, 2010.

2000 *Mathematics Subject Classification.* 65N30, 65N50.

Key words and phrases. Adaptive finite element methods, Elliptic obstacle problems, Convergence analysis.

to [32], we show that our adaptive algorithm, steered by some estimator ϱ_ℓ, guarantees that the combined error quantity

$$\Delta_\ell := \mathcal{J}(U_\ell) - \mathcal{J}(u_\ell) + \gamma\,\mu_\ell^2 + \lambda\,\mathrm{apx}_\ell^2 \tag{1}$$

is a contraction up to some vanishing pertubations $\alpha_\ell \to 0$, i.e.

$$\Delta_{\ell+1} \leq \kappa\Delta_\ell + \alpha_\ell, \tag{2}$$

with $0 < \gamma, \kappa < 1, \lambda > 0$, and $\alpha_\ell \geq 0$. Here, the estimator μ_ℓ is an equivalent error estimator, i.e. $\varrho_\ell \simeq \mu_\ell$. The data oscillations on the Dirichlet boundary are controlled by the term apx_ℓ, and the quantity u_ℓ denotes the continuous solution subject to discrete boundary data g_ℓ, which is introduced to circumvent the lack of nestedness of the discrete spaces. Convergence then follows from the equivalence of the two involved error estimators and a certain weak reliability estimate of ϱ_ℓ, i.e.

$$|||u_\ell - U_\ell||| \lesssim \varrho_\ell \to 0 \quad \Rightarrow \quad \|u - U_\ell\|_{H^1(\Omega)} \to 0. \tag{3}$$

We point out that our convergence proof makes use of the so called *estimator reduction* and does therefore not need the *interior node property*, which makes the result fairly independent of the local mesh-refinement strategy.

1.3. Outline of current work. In Section 2, we formulate the continuous model problem and recall its unique solvability. In Section 3, the same is done for the discretized problem. Section 4 is a collection of the main results of this paper. Here, we introduce the two error estimators ϱ_ℓ and μ_ℓ, which are generalizations of the corresponding estimators from [9, 32]. We then state their weak reliability (Theorem 6) and our version of the adaptive algorithm (Algorithm 7). Finally (Theorem 8), we state that the sequence of discrete solutions indeed converges towards the continuous solution $u \in H^1(\Omega)$. The subsequent Sections 5–7 are then devoted to the proofs of the aforementioned results and numerical illustrations. Finally, some conclusions are drawn in Section 8.

2. Model Problem

2.1. Problem formulation. We consider an elliptic obstacle problem in $\mathbb{R}^2$ on a bounded domain Ω with polygonal boundary $\Gamma := \partial\Omega$. An obstacle on $\overline{\Omega}$ is defined by the affine function χ. Moreover, we consider inhomogeneous Dirichlet data $g \in H^{1/2}(\Gamma)$ and thus additionally require $\chi \leq g$ almost everywhere on Γ. By

$$\mathcal{A} := \{v \in H^1(\Omega) \,:\, v \geq \chi \text{ a.e. in } \Omega,\ v|_\Gamma = g \text{ a.e. on } \Gamma\}, \tag{4}$$

we denote the set of admissible functions. For given $f \in L^2(\Omega)$, we consider the energy functional

$$\mathcal{J}(v) = \frac{1}{2}\langle\!\langle v\,,\,v\rangle\!\rangle - (f, v), \tag{5}$$

with the bilinear form

$$\langle\!\langle u\,,\,v\rangle\!\rangle = \int_\Omega \nabla u \cdot \nabla v\,dx \quad \text{for all } u, v \in H^1(\Omega) \tag{6}$$

and with the L^2-scalar product

$$(f, v) = \int_\Omega f v\,dx. \tag{7}$$

By $|\!|\!|\cdot|\!|\!|$, we denote the energy norm on $H^1_0(\Omega)$ induced by $\langle\!\langle \cdot\,,\,\cdot\rangle\!\rangle$. The obstacle problem then reads as follows: *Find $u \in \mathcal{A}$ such that*

$$\mathcal{J}(u) = \min_{v\in\mathcal{A}} \mathcal{J}(v). \tag{8}$$

2.2. Unique solvability. For the sake of completeness and to collect the main arguments also needed below, we recall the proof that the obstacle problem (8) admits a unique solution.

Proposition 1. *The obstacle problem* (8) *admits a unique solution $u \in \mathcal{A}$ which is equivalently characterized by the variational inequality*

$$\langle\!\langle u\,,\,v-u\rangle\!\rangle \geq (f, v-u) \tag{9}$$

for all $v \in \mathcal{A}$.

The following three lemmata provide the essential ingredients to prove Proposition 1. We start with a well-known abstract result, cf. e.g. [6, § 2.4 – 2.6].

Lemma 2. *Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle\!\langle \cdot\,,\,\cdot\rangle\!\rangle$ and $\mathcal{K} \subseteq \mathcal{H}$ be a closed, convex, and non-empty subset. Then, for given $L \in \mathcal{H}^*$, the variational problem*

$$\mathcal{J}(u) = \min_{v\in\mathcal{K}} \mathcal{J}(v) \quad \textit{with} \quad \mathcal{J}(v) = \frac{1}{2}\langle\!\langle v\,,\,v\rangle\!\rangle - \langle L\,,\,v\rangle \tag{10}$$

has a unique solution $u \in \mathcal{K}$, where $\langle\cdot\,,\,\cdot\rangle$ denotes the dual pairing between $\mathcal{H}$ and $\mathcal{H}^*$. In addition, this solution is equivalently characterized by the variational inequality

$$\langle\!\langle u\,,\,v-u\rangle\!\rangle \geq \langle L\,,\,v-u\rangle \tag{11}$$

for all $v \in \mathcal{K}$. □

To apply Lemma 2, we observe that the obstacle problem (8) can be shifted into a setting with homogeneous Dirichlet data and $\mathcal{H} = H^1_0(\Omega)$. This involves a standard lifting operator $\mathcal{L} : H^{1/2}(\Gamma) \to H^1(\Omega)$, see e.g. [26, Theorem 3.37], with the properties

$$(\mathcal{L}v)|_\Gamma = v \quad \text{and} \quad \|\mathcal{L}v\|_{H^1(\Omega)} \leq C_1\,\|v\|_{H^{1/2}(\Gamma)}, \tag{12}$$

where the constant $C_1 > 0$ depends only on Ω.

With elementary algebraic manipulations, see e.g. [24, Section II.6], one obtains the following well-known link between the obstacle problem (8) and the abstract minimization problem (10) from Lemma 2.

Lemma 3. *Let $\widehat{g} \in H^1(\Omega)$ be an arbitrary extension of $g \in H^{1/2}(\Gamma)$, e.g., $\widehat{g} = \mathcal{L}g$. For $u \in \mathcal{A}$ and $u - \widehat{g} = \widetilde{u} \in \mathcal{K} := \{\widetilde{v} \in H^1_0(\Omega) \,:\, \widetilde{v} \geq \chi - \widehat{g}\}$, the following statements are equivalent:*

(i) *u solves the obstacle problem* (8).
(ii) *u satisfies $\langle\!\langle u\,,\,v-u\rangle\!\rangle \geq (f, v-u)$ for all $v \in \mathcal{A}$.*
(iii) *$\widetilde{u}$ satisfies $\langle\!\langle \widetilde{u}\,,\,\widetilde{v}-\widetilde{u}\rangle\!\rangle \geq (f, \widetilde{v}-\widetilde{u}) - \langle\!\langle \widehat{g}\,,\,\widetilde{v}-\widetilde{u}\rangle\!\rangle$ for all $\widetilde{v} \in \mathcal{K}$.* □

Proof of Proposition 1. We aim to apply Lemma 2 with

$$\mathcal{K} \subseteq H^1_0(\Omega) =: \mathcal{H} \quad \text{and} \quad L \in \mathcal{H}^* \text{ with } \langle L\,,\,\widetilde{v}\rangle := (f, \widetilde{v}) - \langle\!\langle \widehat{g}\,,\,\widetilde{v}\rangle\!\rangle$$

to see that for fixed extension $\widehat{g} = \mathcal{L}g \in H^1(\Omega)$, the auxiliary problem in Lemma 3 (iii) has a unique solution $\widetilde{u} \in H^1_0(\Omega)$. To that end, we have to show that the set $\mathcal{K} \subset H^1_0(\Omega)$ is closed, convex, and non-empty:

Clearly, $\mathcal{K}$ is convex. To see that $\mathcal{K}$ is closed, let $(\widetilde{v}_k)$ be a convergent sequence in $\mathcal{K}$ with limit $\widetilde{v} \in H^1_0(\Omega)$. According to the Weyl theorem and L^2 convergence, there exists a subsequence $(\widetilde{v}_{k_\ell})$ which converges to $\widetilde{v}$ pointwise almost everywhere in Ω. From $(\widetilde{v}_{k_\ell}) \geq \chi - \widehat{g}$ almost everywhere, we thus infer $\widetilde{v} \geq \chi - \widehat{g}$ almost everywhere in Ω, i.e. $\widetilde{v} \in \mathcal{K}$.

To see that $\mathcal{K}$ is also non-empty, we recall that the maximum of two $H^1(\Omega)$-functions again belongs to $H^1(\Omega)$, see e.g. [21, Chapter II, Corollary 2.1]. From this, we infer $v := \max\{\widehat{g}, \chi\} \in H^1(\Omega)$ and by definition $v \geq \chi$ almost everywhere in Ω. Moreover, $\widehat{g}|_\Gamma = g \geq \chi|_\Gamma$ shows $v|_\Gamma = g$. Therefore, $\widetilde{v} := v - \widehat{g} \in \mathcal{K}$ so that $\mathcal{K}$ is non-empty.

Now, Lemma 2 applies and proves that the variational inequality of Lemma 3 (iii) has a unique solution $\widetilde{u} \in \mathcal{K}$. Moreover, $u := \widetilde{u} + \widehat{g} \in \mathcal{A}$ then is the unique solution of the obstacle problem (8). (In particular, this solution is thus independent of the special choice of $\widehat{g}$.) $\square$

Remark. *We stress that Proposition 1 applies to obstacle problems* (8) *with admissible set* $\mathcal{A}$ *from* (4)*, where the obstacle* $\chi \in H^1(\Omega)$ *and the Dirichlet data* $g \in H^{1/2}(\Gamma)$ *are arbitrary up to the constraint* $\chi|_\Gamma \leq g$ *almost everywhere on* Γ*.* $\square$

The next result is an elementary lemma for variational inequalities, see e.g. [32, Proposition 2].

Lemma 4. *Suppose that* $\mathcal{A}$ *is a convex subset of* $H^1(\Omega)$ *and* $f \in L^2(\Omega)$*. Let* $U \in \mathcal{A}$ *be the solution of the variational inequality*

$$\langle\!\langle U\,,\,V - U\rangle\!\rangle \geq (f, V - U) \quad \text{for all } V \in \mathcal{A}. \tag{13}$$

Then, for all $W \in \mathcal{A}$*, there holds*

$$\frac{1}{2}\,|\!|\!|U - W|\!|\!|^2 \leq \mathcal{J}(W) - \mathcal{J}(U), \tag{14}$$

i.e. the difference in the H^1*-seminorm is controlled in terms of the energy.* $\square$

3. Galerkin Discretization

3.1. Problem formulation. For the numerical solution of (8) by an adaptive finite element method, we consider conforming and in the sense of Ciarlet regular triangulations $\mathcal{T}_\ell$ of Ω and denote the standard P1-FEM space of globally continuous and piecewise affine functions by $\mathcal{S}^1(\mathcal{T}_\ell)$. Note that a discrete function $V_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ cannot satisfy the inhomogeneous Dirichlet conditions $g \in H^{1/2}(\Gamma)$ in general. We therefore have to approximate $g \approx g_\ell \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$, where the space $\mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$ denotes the space of globally continuous and piecewise affine functions on the boundary Γ. For this discretization, we assume additional regularity $g \in H^1(\Gamma)$ and consider the approximation $g_\ell \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$ which is derived by nodal interpolation of the boundary data. Note that nodal interpolation is well-defined since the Sobolev inequality on the 1D manifold Γ predicts the continuous inclusion $H^1(\Gamma) \subset C(\Gamma)$. Altogether, the set of discrete admissible functions $\mathcal{A}_\ell$ is given by

$$\mathcal{A}_\ell := \{V_\ell \in \mathcal{S}^1(\mathcal{T}_\ell) \,:\, V_\ell \geq \chi \text{ in } \Omega,\, V_\ell = g_\ell \text{ on } \Gamma\}, \tag{15}$$

and the discrete minimization problem reads: *Find* $U_\ell \in \mathcal{A}_\ell$ *such that*

$$\mathcal{J}(U_\ell) = \min_{V_\ell \in \mathcal{A}_\ell} \mathcal{J}(V_\ell). \tag{16}$$

Remark. *Note that* $g \in H^1(\Gamma) \subset C(\Gamma)$, $\chi \in \mathcal{P}^1(\Omega) \subset C(\overline{\Omega})$, *and* $\chi|_\Gamma \le g$ *imply that* $\chi(z) \le g(z) = g_\ell(z)$ *for all nodes* $z \in \Gamma$. *For each boundary edge* $E \subset \Gamma$, *the functions* $\chi|_E$ *and* $g_\ell|_E$ *are affine. Therefore, we conclude* $\chi|_\Gamma \le g_\ell$ *for the nodal interpolant* $g_\ell \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$. □

Remark. *Note that approximating the boundary data leads to a non-conforming discretization, i.e.* $\mathcal{A}_\ell \not\subseteq \mathcal{A}_{\ell+1}$, *which makes the analysis below much more involved than in the conforming case* [9, 32]. □

3.2. Notation. From now on, $\mathcal{N}_\ell$ denotes the set of nodes of the regular triangulation $\mathcal{T}_\ell$. The set of all interior edges $E = T^+ \cap T^-$ for certain elements $T^+, T^- \in \mathcal{T}_\ell$ is denoted by $\mathcal{E}_\ell^\Omega$. The set of all edges of $\mathcal{T}_\ell$ is denoted by $\mathcal{E}_\ell$. In particular, $\mathcal{E}_\ell|_\Gamma := \mathcal{E}_\ell^\Gamma := \mathcal{E}_\ell \backslash \mathcal{E}_\ell^\Omega$ contains all boundary edges and provides some partition of Γ. Finally, $\mathcal{T}_\ell^\Gamma := \{T \in \mathcal{T}_\ell \,:\, T \cap \Gamma \neq \emptyset\}$ is the set of all elements which touch the boundary Γ.

We recall that $\mathcal{S}^1(\mathcal{T}_\ell) = \{V_\ell \in C(\overline{\Omega}) \,:\, V_\ell|_T \text{ affine for all } T \in \mathcal{T}_\ell\}$ denotes the conforming P1-finite element space. Moreover, $\mathcal{S}_0^1(\mathcal{T}_\ell) = \mathcal{S}^1(\mathcal{T}_\ell) \cap H_0^1(\Omega) = \{V_\ell \in \mathcal{S}^1(\mathcal{T}_\ell) \,:\, V_\ell|_\Gamma = 0\}$.

3.3. Unique solvability. In this section, we recall that the discrete obstacle problem (16) admits a unique solution which is again characterized by a variational inequality.

Proposition 5. *The discrete obstacle problem* (16) *admits a unique solution* $U_\ell \in \mathcal{A}_\ell$, *which is equivalently characterized by the variational inequality*

$$\langle\!\langle U_\ell \,,\, V_\ell - U_\ell \rangle\!\rangle \ge \langle f \,,\, V_\ell - U_\ell \rangle \tag{17}$$

for all $V_\ell \in \mathcal{A}_\ell$.

The proof of Proposition 5 relies on the fact that discrete boundary data $g_\ell \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$ can be lifted to a discrete function $\widehat{g}_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ with $\widehat{g}_\ell|_\Gamma = g_\ell$. The proof of the latter is a consequence of (and even equivalent [17] to) the existence of the Scott-Zhang quasi-interpolation operator $P_\ell : H^1(\Omega) \to \mathcal{S}^1(\mathcal{T}_\ell)$, see [34], which is a linear and continuous projection onto $\mathcal{S}^1(\mathcal{T}_\ell)$, i.e.

$$P_\ell^2 = P_\ell \quad \text{and} \quad \|P_\ell v\|_{H^1(\Omega)} \le C_2 \|v\|_{H^1(\Omega)}, \quad \text{for all } v \in H^1(\Omega), \tag{18}$$

that preserves discrete boundary conditions, i.e.

$$(P_\ell v)|_\Gamma = v|_\Gamma \quad \text{for all } v \in H^1(\Omega) \text{ with } v|_\Gamma \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma). \tag{19}$$

The constant $C_2 > 0$ depends only on the shape regularity constant $\sigma(\mathcal{T}_\ell)$ and on the diameter of Ω. Moreover, P_ℓ has a local first-order approximation property which is, however, not used throughout.

Using the Scott-Zhang projection P_ℓ, we can now prove unique solvability of the discrete obstacle problem (16).

Proof or Proposition 5. First, we note that $\widehat{g}_\ell := P_\ell(\mathcal{L}g_\ell) \in \mathcal{S}^1(\mathcal{T}_\ell)$ provides a discrete extension of g_ℓ, i.e. $\widehat{g}_\ell|_\Gamma = g_\ell$. As in Lemma 3, it is an elementary observation that the discrete obstacle problem (16) is equivalent to the variational inequality (17) as well as to the variational inequality

$$\langle\!\langle \widetilde{U}_\ell \,,\, \widetilde{V}_\ell - \widetilde{U}_\ell \rangle\!\rangle \ge (f, \widetilde{V}_\ell - \widetilde{U}_\ell) - \langle\!\langle \widehat{g}_\ell \,,\, \widetilde{V}_\ell - \widetilde{U}_\ell \rangle\!\rangle \tag{20}$$

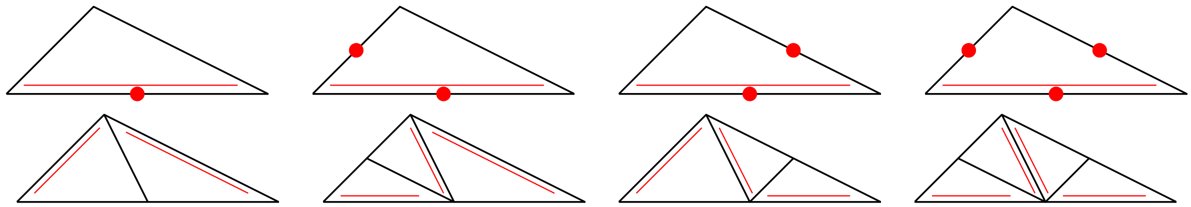

FIGURE 1. For each triangle $T \in \mathcal{T}$, there is one fixed *reference edge*, indicated by the double line (left, top). Refinement of T is done by bisecting the reference edge, where its midpoint becomes a new node. The reference edges of the son triangles are opposite to this newest vertex (left, bottom). To avoid hanging nodes, one proceeds as follows: We assume that certain edges of T, but at least the reference edge, are marked for refinement (top). Using iterated newest vertex bisection, the element is then split into 2, 3, or 4 son triangles (bottom).

for all $\widetilde{V}_\ell \in \mathcal{K}_\ell := \{\widetilde{V}_\ell \in \mathcal{S}^1_0(\mathcal{T}_\ell) \,:\, \widetilde{V}_\ell \geq \chi - \widehat{g}_\ell\}$. In order to apply Lemma 2 for

$$\mathcal{K}_\ell \subset \mathcal{S}^1_0(\mathcal{T}_\ell) =: \mathcal{H} \quad \text{and} \quad L \in \mathcal{H}^* \text{ with } \langle L\,,\,\widetilde{V}_\ell\rangle := (f, \widetilde{V}_\ell) - \langle\!\langle \widehat{g}_\ell\,,\,\widetilde{V}_\ell\rangle\!\rangle,$$

we thus have to show that $\mathcal{K}_\ell$ is a closed, convex, and non-empty subset of $\mathcal{S}^1_0(\mathcal{T}_\ell)$. Arguing as in the proof of Proposition 1, we see that $\mathcal{K}_\ell$ is closed and convex. To see that $\mathcal{K}_\ell$ is also non-empty, we define

$$\widetilde{V}_\ell \in \mathcal{S}^1(\mathcal{T}_\ell) \quad \text{by } \widetilde{V}_\ell(z) := \max\{\chi(z) - \widehat{g}_\ell(z), 0\} \text{ for all } z \in \mathcal{N}_\ell.$$

From $\chi, \widehat{g}_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$, we infer that this definition guarantees $\widetilde{V}_\ell \geq \chi - \widehat{g}_\ell$ in Ω. Moreover, for $z \in \mathcal{N}_\ell \cap \Gamma$ holds $\chi(z) - \widehat{g}_\ell(z) = \chi(z) - g_\ell(z) \leq 0$, whence $\widetilde{V}_\ell|_\Gamma = 0$, i.e. $\widetilde{V}_\ell \in \mathcal{K}_\ell$. Application of Lemma 2 thus provides a unique solution $\widetilde{U}_\ell \in \mathcal{K}_\ell$ of (20). According to Lemma 3, $U_\ell = \widetilde{U}_\ell + \widehat{g}_\ell \in \mathcal{A}_\ell$ is the unique solution of (17). $\square$

Remark. *We stress that Proposition 5 applies to discrete obstacle problems* (16) *with admissible set $\mathcal{A}_\ell \subset \mathcal{S}^1(\mathcal{T}_\ell)$ from* (15), *where the obstacle $\chi \in \mathcal{S}^1(\mathcal{T}_\ell)$ and the Dirichlet data $g_\ell \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$ are arbitrary up to $\chi|_\Gamma \leq g_\ell$ on Γ.* $\square$

4. Adaptive Mesh-Refining Algorithm and Main Results

4.1. Newest vertex bisection. For the mesh-refinement, we use newest-vertex bisection. Assume that $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \cup \mathcal{E}_\ell$ is a set of edges and triangles which have to be refined. We then mark all edges $E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell$ for bisection. For all elements $T \in \mathcal{T}_\ell \cap \mathcal{M}_\ell$, we mark their reference edge. The refinement rules are shown in Figure 1, and the reader is also referred to [36, Chapter 4]. We stress a certain decay of the mesh-widths:

- Marked edges $E \in \mathcal{M}_\ell$ are split into two edges $E', E'' \in \mathcal{E}_{\ell+1}$ of half length.
- If at least one edge E of an element $T \in \mathcal{T}_\ell$ is marked, T is refined into up to four son elements $T' \in \mathcal{T}_{\ell+1}$ with area $|T|/4 \leq |T'| \leq |T|/2$, cf. Figure 1.

Moreover, given an initial mesh $\mathcal{T}_0$, newest vertex bisection only leads to at most $4 \cdot \#\mathcal{T}_0$ similarity classes of triangles. In particular, the generated meshes are uniformly shape regular

$$\sup_{\ell\in\mathbb{N}} \sigma(\mathcal{T}_\ell) < \infty \quad \text{with} \quad \sigma(\mathcal{T}_\ell) = \max_{T\in\mathcal{T}_\ell} \frac{\mathrm{diam}(T)^2}{|T|}. \tag{21}$$

Furthermore, the number of different shapes of e.g. node patches that can occur, is finite. These observations will be necessary in the scaling arguments below.

4.2. Weakly reliable error estimator. Let $u \in \mathcal{A}$ denote the continuous solution of (8) and $U_\ell \in \mathcal{A}_\ell$ be the discrete solution of (16) for some fixed triangulation $\mathcal{T}_\ell$. To steer the adaptive mesh-refinement, we use some residual-based error estimator

$$\varrho_\ell^2 := \sum_{E\in\mathcal{E}_\ell^\Omega} \big(\eta_\ell(E)^2 + \mathrm{osc}_\ell(E)^2\big) + \sum_{T\in\mathcal{T}_\ell^\Gamma} \mathrm{osc}_\ell(T)^2 + \sum_{E\in\mathcal{E}_\ell^\Gamma} \mathrm{apx}_\ell(E)^2 \tag{22}$$

which has essentially been introduced in [9] for homogeneous Dirichlet data $g = 0$. First, $\eta_\ell(E)^2$ denotes the weighted L^2-norms of the normal jump

$$\eta_\ell(E)^2 := h_E \,\|[\partial_n U_\ell]\|_{L^2(E)}^2 \quad \text{for } E \in \mathcal{E}_\ell^\Omega \tag{23}$$

with $h_E = \mathrm{diam}(E)$ the length of E and $[\cdot]$ the jump over an interior edge $E = T^+\cap T^- \in \mathcal{E}_\ell^\Omega$. Second, $\mathrm{osc}_\ell(E)^2$ denotes the data oscillations of f over E

$$\mathrm{osc}_\ell(E)^2 := |\Omega_{\ell,E}|\, \|f - f_{\Omega_{\ell,E}}\|_{L^2(\Omega_{\ell,E})}^2 \quad \text{for } E \in \mathcal{E}_\ell^\Omega \tag{24}$$

with $\Omega_{\ell,E} = T^+ \cup T^-$ the patch associated with E and $f_{\Omega_{\ell,E}} = (1/|\Omega_{\ell,E}|) \int_{\Omega_{\ell,E}} f\,dx$ the corresponding integral mean of f. Third, for elements $T \in \mathcal{T}_\ell^\Gamma$ at the boundary, ϱ_ℓ involves the weighted element residuals

$$\mathrm{osc}_\ell(T)^2 := |T|\|f\|_{L^2(T)}^2 \quad \text{for } T \in \mathcal{T}_\ell. \tag{25}$$

Finally and in order to control the approximation of the nonhomogeneous Dirichlet data g by its nodal interpolant g_ℓ, we apply an idea from [3] and use

$$\mathrm{apx}_\ell(E)^2 := h_E \,\|(g - g_\ell)'\|_{L^2(E)}^2 \quad \text{for } E \in \mathcal{E}_\ell^\Gamma, \tag{26}$$

where $(\cdot)'$ denotes the arclength derivative.

To state a reliability result for the proposed error estimator ϱ_ℓ, we need to introduce a continuous auxiliary problem. Given the discrete Dirichlet data $g_\ell \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$, we define

$$\mathcal{A}_\ell^\star := \{v_\ell \in H^1(\Omega) \,:\, v_\ell \geq \chi \text{ a.e. in } \Omega,\, v_\ell|_\Gamma = g_\ell \text{ a.e. on } \Gamma\}. \tag{27}$$

Applying Proposition 1 for the data $(\mathcal{A}_\ell^\star, g_\ell)$, we see that the auxiliary problem

$$\mathcal{J}(u_\ell) = \min_{v_\ell\in\mathcal{A}_\ell^\star} \mathcal{J}(v_\ell) \tag{28}$$

admits a unique solution $u_\ell \in \mathcal{A}_\ell^\star$. The following theorem now states weak reliability of ϱ_ℓ in the sense that $\varrho_\ell \to 0$ implies $\|u - U_\ell\|_{H^1(\Omega)} \to 0$ as $\ell \to \infty$. The proof is given in Section 5 below.

Theorem 6. *With $u_\ell \in \mathcal{A}_\ell^\star$ the solution of the auxiliary problem* (28)*, the error estimator ϱ_ℓ from* (22) *satisfies*

$$\frac{1}{2}|||u_\ell - U_\ell|||^2 \leq \mathcal{J}(U_\ell) - \mathcal{J}(u_\ell) \leq C_3^2\, \widetilde{\varrho}_\ell^{\,2}, \tag{29}$$

with

$$\widetilde{\varrho}_\ell^2 := \sum_{E\in\mathcal{E}_\ell^\Omega} \big(\eta_\ell(E)^2 + \mathrm{osc}_\ell(E)^2\big) + \sum_{T\in\mathcal{T}_\ell^\Gamma} \mathrm{osc}_\ell(T)^2, \tag{30}$$

as well as

$$\|u - U_\ell\|_{H^1(\Omega)} \le |||u - u_\ell||| + \sqrt{2}C_3\,\varrho_\ell, \tag{31}$$

where the constant $C_3 > 0$ depends only on $\sigma(\mathcal{T}_\ell)$ and on Ω. Moreover, there holds

$$\varrho_\ell \xrightarrow{\ell\to\infty} 0 \quad\Longrightarrow\quad \|u - U_\ell\|_{H^1(\Omega)} \xrightarrow{\ell\to\infty} 0. \tag{32}$$

Remark. *We stress that the reliability estimate* (29) *depends on the use of newest vertex bisection in the sense that only finitely many shapes of node patches and edge patches can occur. For red-green-blue refinement* [36, Chapter 4], *the equivalence of edge and node oscillations is open. The entire analysis, however, applies for a coarser error estimator, where edge oscillations are bounded in terms of the element residuals $\|h_\ell f\|_{L^2(T)}$.* □

4.3. Convergent adaptive mesh-refining algorithm. We can now state our version of the adaptive algorithm in the usual form:

$$\boxed{\texttt{Solve}} \longmapsto \boxed{\texttt{Estimate}} \longmapsto \boxed{\texttt{Mark}} \longmapsto \boxed{\texttt{Refine}}$$

Throughout, we assume that the Galerkin solution $U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ is computed exactly. For marking, we use the strategy proposed by Dörfler [18].

Algorithm 7. *Fix an adaptivity parameter $0 < \theta < 1$, let $\mathcal{T}_\ell$ with $\ell = 0$ be the initial triangulation, and fix a reference edge for each element $T \in \mathcal{T}_0$. For each $\ell = 0, 1, 2, \dots$ do:*

(i) *Compute discrete solution $U_\ell \in \mathcal{A}_\ell$.*
(ii) *Compute $\eta_\ell(E)^2, \mathrm{osc}_\ell(E)^2, \mathrm{osc}_\ell(T)^2$, and $\mathrm{apx}_\ell(E)^2$ for all $E \in \mathcal{E}_\ell$ and all $T \in \mathcal{T}_\ell^\Gamma$.*
(iii) *Determine (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell \cup \mathcal{T}_\ell^\Gamma$ which satisfies*

$$\theta\,\varrho_\ell^2 \le \sum_{E\in\mathcal{E}_\ell^\Omega\cap\mathcal{M}_\ell} \big(\eta_\ell(E)^2 + \mathrm{osc}_\ell(E)^2\big) + \sum_{T\in\mathcal{T}_\ell^\Gamma\cap\mathcal{M}_\ell} \mathrm{osc}_\ell(T)^2 + \sum_{E\in\mathcal{E}_\ell^\Gamma\cap\mathcal{M}_\ell} \mathrm{apx}_\ell(E)^2. \tag{33}$$

(iv) *Mark all edges $E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell$ and all reference edges of $T \in \mathcal{T}_\ell^\Gamma \cap \mathcal{M}_\ell$ for refinement.*
(v) *Obtain new mesh $\mathcal{T}_{\ell+1}$ by newest vertex bisection, increase counter $\ell \mapsto \ell + 1$, and iterate.* □

To state our convergence result, we need a slight modification of the error estimator ϱ_ℓ introduced above. Recall that $\mathcal{E}_\ell$ is the set of all edges of $\mathcal{T}_\ell$. For the interior edges $E \in \mathcal{E}_\ell^\Omega$, the edge data oscillations $\mathrm{osc}_\ell(E)$ are defined in (24). For the boundary edges, we define

$$\mathrm{osc}_\ell(E)^2 := \mathrm{osc}_\ell(T)^2 \quad \text{for all } E \in \mathcal{E}_\ell^\Gamma := \mathcal{E}_\ell \backslash \mathcal{E}_\ell^\Omega \text{ and } T \in \mathcal{T}_\ell^\Gamma \text{ with } E \subset \partial T, \tag{34}$$

where $\mathrm{osc}_\ell(T)$ is defined in (25). We now consider

$$\mu_\ell^2 := \sum_{E\in\mathcal{E}_\ell^\Omega} \eta_\ell(E)^2 + \sum_{E\in\mathcal{E}_\ell} \mathrm{osc}_\ell(E)^2 + \sum_{T\in\mathcal{T}_\ell^\Gamma} \mathrm{osc}_\ell(T)^2 + \sum_{E\in\mathcal{E}_\ell^\Gamma} \mathrm{apx}_\ell(E)^2. \tag{35}$$

By definition, there holds

$$\varrho_\ell^2 \le \mu_\ell^2 \le 4\,\varrho_\ell^2, \tag{36}$$

since one element $T \in \mathcal{T}_\ell^\Gamma$ may have at most three boundary edges. In particular, this proves that Theorem 6 also holds with ϱ_ℓ replaced by μ_ℓ. The proof of the following convergence theorem is given in Section 6 below.

Theorem 8. *Let μ_ℓ denote the modified error estimator from* (35) *and suppose that, nevertheless, ϱ_ℓ is used for marking* (33) *in Algorithm 7. Then, there are constants $0 < \kappa, \gamma < 1$ and $\lambda > 0$ and a sequence $\alpha_\ell \geq 0$ with $\lim_{\ell\to\infty} \alpha_\ell = 0$ such that the combined error quantity*

$$\Delta_\ell := \mathcal{J}(U_\ell) - \mathcal{J}(u_\ell) + \gamma\mu_\ell^2 + \lambda\,\mathrm{apx}_\ell^2 \geq 0 \tag{37}$$

satisfies contraction up to the zero sequence α_ℓ, i.e.

$$\Delta_{\ell+1} \leq \kappa\,\Delta_\ell + \alpha_\ell \quad \textit{for all } \ell \in \mathbb{N}. \tag{38}$$

In particular, this implies

$$0 = \lim_{\ell\to\infty} \Delta_\ell = \lim_{\ell\to\infty} \mu_\ell = \lim_{\ell\to\infty} \varrho_\ell = \lim_{\ell\to\infty} \|u - U_\ell\|_{H^1(\Omega)} \tag{39}$$

as well as convergence of the energies

$$\mathcal{J}(u) = \lim_{\ell\to\infty} \mathcal{J}(u_\ell) = \lim_{\ell\to\infty} \mathcal{J}(U_\ell). \tag{40}$$

Remark. *For homogeneous Dirichlet data $g = 0$, the main result of* [32] *states that Theorem 8 holds with vanishing $\alpha_\ell = 0$ in* (38)*, i.e. Δ_ℓ is contractive.* $\square$

5. Proof of Theorem 6 (Weak Reliability of Error Estimator)

5.1. Stability of continuous problem. For the finite element discretization, we have to replace the continuous Dirichlet data $g \in H^{1/2}(\Gamma)$ by appropriate discrete functions g_ℓ. To make this procedure feasible, we have to prove that the solution of the obstacle problem (8) depends continuously on the given data.

In the following, let $\mathcal{H}$ be a Hilbert space with scalar product $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ and $\mathcal{K}, \mathcal{K}_\ell^\star \subset \mathcal{H}$ be closed, convex, and non-empty subsets of $\mathcal{H}$. We say, that $\mathcal{K}_\ell^\star$ converges to $\mathcal{K}$ *in the sense of Mosco* if and only if

$$\forall v \in \mathcal{K}\,\exists v_\ell \in \mathcal{K}_\ell^\star : \quad v_\ell \to v \text{ strongly in } \mathcal{H} \text{ as } \ell \to \infty \tag{41}$$

and

$$\forall v \in \mathcal{H}\,\forall v_\ell \in \mathcal{K}_\ell^\star : \quad \big(v_\ell \rightharpoonup v \text{ weakly in } \mathcal{H} \text{ as } \ell \to \infty \quad \Rightarrow \quad v \in \mathcal{K}\big). \tag{42}$$

We can now state an abstract stability result of Mosco:

Lemma 9 ([27, Theorem A])**.** *Assume the sets $\mathcal{K}, \mathcal{K}_\ell^\star$ satisfy* (41)–(42)*. Let $L, L_\ell \in \mathcal{H}^*$ be functionals and assume $L_\ell \to L$ in $\mathcal{H}^*$ as $\ell \to \infty$. Finally, let $u \in \mathcal{K}$ and $u_\ell \in \mathcal{K}_\ell^\star$ be the unique solutions of* (10) *with respect to the data $(\mathcal{K}, L)$ and $(\mathcal{K}_\ell^\star, L_\ell)$, respectively. Then there holds strong convergence*

$$u_\ell \to u \textit{ in } \mathcal{H} \textit{ as } \ell \to \infty, \tag{43}$$

i.e. the solution of the variational problem (10) *depends continuously on the given data.* $\square$

Remark. *We remark that we have stated* [27, Theorem A] *only in a simplified form. In general, the preceding lemma of Mosco includes the approximation of the bilinear form* $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ *as well, and it also holds for nonlinear variational inequalities, where the underlying operators* $A_\ell, A : \mathcal{H} \to \mathcal{H}^*$*, e.g.* $Av := \langle\!\langle v\,,\,\cdot \rangle\!\rangle$*, are assumed to be Lipschitz continuous and strictly monotone with constants independent of* ℓ*.* □

We are now in the position to show that the solution $u \in \mathcal{A}$ of the obstacle problem (8) continuously depends on the boundary data $g \in H^{1/2}(\Gamma)$.

Proposition 10. *Recall that* $u_\ell \in \mathcal{A}_\ell^\star$ *denotes the solution of the auxiliary problem* (28)*. Provided convergence* $g_\ell \to g$ *in* $H^{1/2}(\Gamma)$ *of the Dirichlet data, there also holds convergence* $u_\ell \to u$ *in* $H^1(\Omega)$ *of the (continuous) solutions as* $\ell \to \infty$*.*

Proof. As in the proof of Proposition 1, we define the sets

$$\mathcal{K} = \{\widetilde{v} \in H^1_0(\Omega) \,:\, \widetilde{v} \geq \chi - \mathcal{L}g\} \quad \text{and} \quad \mathcal{K}_\ell^\star = \{\widetilde{v}_\ell \in H^1_0(\Omega) \,:\, \widetilde{v}_\ell \geq \chi - \mathcal{L}g_\ell\}.$$

We stress that both sets are convex, closed, and non-empty subsets of $H^1_0(\Omega)$. As usual, let $H^{-1}(\Omega) := H^1_0(\Omega)^*$ denote the dual space of $H^1_0(\Omega)$. With respect to the abstract notation, we have $L, L_\ell \in H^{-1}(\Omega)$ with

$$\langle L\,,\,\widetilde{v}\rangle = (f, \widetilde{v}) - \langle\!\langle \mathcal{L}g\,,\,\widetilde{v}\rangle\!\rangle \quad \text{and} \quad \langle L_\ell\,,\,\widetilde{v}\rangle = (f, \widetilde{v}) - \langle\!\langle \mathcal{L}g_\ell\,,\,\widetilde{v}\rangle\!\rangle.$$

Note that this implies

$$\|L - L_\ell\|_{H^{-1}(\Omega)} \leq \|\mathcal{L}g - \mathcal{L}g_\ell\|_{H^1(\Omega)} \lesssim \|g - g_\ell\|_{H^{1/2}(\Gamma)} \xrightarrow{\ell\to\infty} 0.$$

Next, we prove that $\mathcal{K}_\ell^\star \to \mathcal{K}$ in the sense of Mosco (41)–(42).

To verify (41), let $\widetilde{v} \in \mathcal{K}$ be arbitrary. Define

$$v := \widetilde{v} + \mathcal{L}g \in \mathcal{A}$$

and let $w_\ell \in H^1(\Omega)$ be the unique solution of the (linear) auxiliary problem

$$\Delta w_\ell = 0 \text{ in } \Omega \quad \text{with Dirichlet conditions} \quad w_\ell = g_\ell - g \text{ on } \Gamma.$$

In the next step, we consider

$$\overline{v}_\ell := v + w_\ell \in H^1(\Omega).$$

Then, there holds $\overline{v}_\ell|_\Gamma = g_\ell$ as well as

$$\|v - \overline{v}_\ell\|_{H^1(\Omega)} = \|w_\ell\|_{H^1(\Omega)} \lesssim \|g_\ell - g\|_{H^{1/2}(\Gamma)} \xrightarrow{\ell\to\infty} 0.$$

Since the maximum of H^1-functions belongs to H^1, we obtain

$$v_\ell := \max\{\chi, \overline{v}_\ell\} \in H^1(\Omega).$$

By definition, there holds $v_\ell \geq \chi$ almost everywhere in Ω as well as $v_\ell|_\Gamma = g_\ell$ since $\overline{v}_\ell|_\Gamma = g_\ell \geq \chi|_\Gamma$ almost everywhere on Γ. Altogether, this proves $v_\ell \in \mathcal{A}_\ell$. Moreover, since the function $\max\{\chi, \cdot\} : H^1(\Omega) \to H^1(\Omega)$ is well-defined and continuous (see [21, Chapter II, Corollary 2.1]), there holds

$$v = \max\{\chi, v\} = \lim_{\ell\to\infty} \max\{\chi, \overline{v}_\ell\} = \lim_{\ell\to\infty} v_\ell \text{ in } H^1(\Omega).$$

Finally, we observe

$$\widetilde{v}_\ell := v_\ell - \mathcal{L}g_\ell \xrightarrow{\ell\to\infty} v - \mathcal{L}g = \widetilde{v} \text{ in } H^1(\Omega)$$

and $\widetilde{v}_\ell \in \mathcal{K}_\ell^\star$.

To verify (42), let $v_\ell \in \mathcal{K}_\ell^\star$ and assume that the weak limit $v_\ell \rightharpoonup v \in H_0^1(\Omega)$ exists as $\ell \to \infty$. According to the Rellich compactness theorem, there is a subsequence (v_{ℓ_k}) which converges strongly to v in $L^2(\Omega)$. According to the Weyl theorem, we may thus extract a further subsequence $(v_{\ell_{k_j}})$ which converges to v pointwise almost everywhere in Ω. Moreover, by continuity of the lifting operator, there holds $\mathcal{L}g_{\ell_{k_j}} \to \mathcal{L}g$ in $H^1(\Omega)$ as $j \to \infty$. Since this implies L^2-convergence and again according to the Weyl theorem, we may choose a subsequence $(\mathcal{L}g_{\ell_{k_{j_i}}})$ which converges to $\mathcal{L}g$ pointwise almost everywhere in Ω. Altogether, the estimate

$$v_{\ell_{k_{j_i}}} \geq \chi - \mathcal{L}g_{\ell_{k_{j_i}}} \quad \text{a.e. in } \Omega$$

and monotonicity of the pointwise limit imply

$$v \geq \chi - \mathcal{L}g \quad \text{a.e. in } \Omega,$$

whence $v \in \mathcal{K}$.

Now, we may apply Lemma 9 to see that the unique solutions $\widetilde{u} \in \mathcal{K}$ and $\widetilde{u}_\ell \in \mathcal{K}_\ell^\star$ of the minimization problems (10) with respect to the data $(\mathcal{K}, L)$ and $(\mathcal{K}_\ell^\star, L_\ell)$, respectively, guarantee

$$\|\widetilde{u} - \widetilde{u}_\ell\|_{H^1(\Omega)} \xrightarrow{\ell\to\infty} 0.$$

With Lemma 3, there holds $u = \widetilde{u} + \mathcal{L}g$ as well as $u_\ell = \widetilde{u}_\ell + \mathcal{L}g_\ell$. Altogether, we thus obtain

$$\|u - u_\ell\|_{H^1(\Omega)} \lesssim \|\widetilde{u} - \widetilde{u}_\ell\|_{H^1(\Omega)} + \|g - g_\ell\|_{H^{1/2}(\Gamma)} \xrightarrow{\ell\to\infty} 0$$

and conclude the proof. $\square$

5.2. Proof of Theorem 6. We start with estimate (29), where the lower estimate clearly holds due to Lemma 4. We thus only need to show the upper estimate. To that end, we define $\sigma_\ell \in H^{-1}(\Omega)$ by

$$\langle \sigma_\ell \,,\, v\rangle := (f, v) - \langle\!\langle u_\ell \,,\, v\rangle\!\rangle \quad \text{for all } v \in H_0^1(\Omega).$$

Since $u_\ell|_\Gamma = U_\ell|_\Gamma$, we may argue as in the proof of [9, Theorem 1] to see

$$|\!|\!| u_\ell - U_\ell |\!|\!|^2 + \langle \sigma \,,\, u_\ell - U_\ell\rangle \lesssim \widetilde{\varrho}_\ell^2,$$

where the hidden constant depends only on the shape regularity constant $\sigma(\mathcal{T}_\ell)$. A direct calculation finally shows

$$\begin{aligned}
\mathcal{J}(U_\ell) - \mathcal{J}(u_\ell) &= \frac{1}{2}\langle\!\langle U_\ell \,,\, U_\ell\rangle\!\rangle - (f, U_\ell) - \Big(\frac{1}{2}\langle\!\langle u_\ell \,,\, u_\ell\rangle\!\rangle - (f, u_\ell)\Big) \\
&\quad + \big(\langle\!\langle u_\ell \,,\, u_\ell - U_\ell\rangle\!\rangle - (f, u_\ell - U_\ell) + \langle \sigma \,,\, u_\ell - U_\ell\rangle\big) \\
&= \frac{1}{2}|\!|\!| u_\ell - U_\ell |\!|\!|^2 + \langle \sigma \,,\, u_\ell - U_\ell\rangle,
\end{aligned}$$

whence $\mathcal{J}(U_\ell) - \mathcal{J}(u_\ell) \lesssim \widetilde{\varrho}_\ell^2$.

Next we prove (31). According to the Rellich compactness theorem and the triangle inequality, there holds

$$\begin{aligned}
\|u - U_\ell\|_{H^1(\Omega)} &\simeq |\!|\!| u - U_\ell |\!|\!| + \|g - g_\ell\|_{H^{1/2}(\Gamma)} \\
&\leq |\!|\!| u - u_\ell |\!|\!| + |\!|\!| u_\ell - U_\ell |\!|\!| + \|g - g_\ell\|_{H^{1/2}(\Gamma)}.
\end{aligned}$$

From (29), we infer

$$|||u_\ell - U_\ell|||^2 \lesssim \widetilde{\varrho}_\ell^2 \le \varrho_\ell^2.$$

From [3, Lemma 3.3], we additionally infer that

$$\|g - g_\ell\|_{H^{1/2}(\Gamma)}^2 \lesssim \sum_{E\in\mathcal{E}_\ell^\Gamma} \mathrm{apx}_\ell(E)^2 \le \varrho_\ell^2,$$

and the constant depends only on Γ and the local mesh ratio of $\mathcal{E}_\ell^\Gamma = \mathcal{T}_\ell|_\Gamma$, whence on Ω and $\sigma(\mathcal{T}_\ell)$. The combination of the last three estimates yields (31). Moreover, according to the last estimate, the convergence $\varrho_\ell \to 0$ implies $g_\ell \to g$ in $H^{1/2}(\Gamma)$ as $\ell \to \infty$. According to the stability result from Proposition 10, this yields convergence $u_\ell \to u$ in $H^1(\Omega)$, and we also conclude the proof of (32). □

6. Proof of Theorem 8 (Convergence of AFEM)

The idea of our convergence proof is roughly sketched as follows: Dörfler marking (33) with (θ, ϱ_ℓ) implies Dörfler marking with $(\theta/4, \mu_\ell)$. The latter yields that μ_ℓ is contractive up to $|||U_{\ell+1} - U_\ell|||$, see Section 6.1. Since there seems to be no estimate for this term to be available (recall nonconformity $\mathcal{A}_\ell \not\subseteq \mathcal{A}_{\ell+1}$), we introduce a discrete auxiliary problem with solution $U_{\ell+1,\ell}$ which allows to control

$$|||U_{\ell+1} - U_\ell||| \le |||U_{\ell+1} - U_{\ell+1,\ell}||| + |||U_{\ell+1,\ell} - U_\ell|||$$

in terms of the energy $\mathcal{J}(U_\ell) - \mathcal{J}(U_{\ell+1,\ell})$, see Section 6.3. In Section 6.4, we finally follow the concept of [15] to prove Theorem 8.

6.1. Estimator reduction. For the moment, let us use the local contributions of μ_ℓ in the Dörfler marking (33) of Algorithm 7, i.e. given some parameter $0 < \theta < 1$, let $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell \cup \mathcal{T}_\ell^\Gamma$ satisfy

$$\theta\, \mu_\ell^2 \le \sum_{E\in\mathcal{E}_\ell^\Omega\cap\mathcal{M}_\ell} \eta_\ell(E)^2 + \sum_{E\in\mathcal{E}_\ell\cap\mathcal{M}_\ell} \mathrm{osc}_\ell(E)^2 + \sum_{T\in\mathcal{T}_\ell^\Gamma\cap\mathcal{M}_\ell} \mathrm{osc}_\ell(T)^2 + \sum_{E\in\mathcal{E}_\ell^\Gamma\cap\mathcal{M}_\ell} \mathrm{apx}_\ell(E)^2. \tag{44}$$

Then, the following estimate is called *estimator reduction* in [2, 15].

Proposition 11. *Suppose that the set $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell \cup \mathcal{T}_\ell^\Gamma$ satisfies (44) and that marked edges and marked elements are refined as stated in Section 4.1. Then, there holds*

$$\mu_{\ell+1}^2 \le q\, \mu_\ell^2 + C_4 |||U_{\ell+1} - U_\ell|||^2 \tag{45}$$

with some contraction constant $q \in (0,1)$ which depends only on $\theta \in (0,1)$. The constant $C_4 > 0$ additionally depends only on the initial mesh $\mathcal{T}_0$.

Since the proof of Proposition 11 follows along the lines of the proof of [32, Proposition 3], we only sketch it for brevity.

Sketch of proof. First, the Young inequality proves for arbitrary $\delta > 0$

$$\begin{aligned}\mu_{\ell+1}^2 \le &\sum_{T'\in\mathcal{T}_{\ell+1}^\Gamma} \mathrm{osc}_{\ell+1}(T')^2 + \sum_{E'\in\mathcal{E}_{\ell+1}} \mathrm{osc}_{\ell+1}(E')^2 + \sum_{E'\in\mathcal{E}_{\ell+1}^\Gamma} \mathrm{apx}_{\ell+1}(E')^2 \\ &+ (1+\delta) \sum_{E'\in\mathcal{E}_{\ell+1}^\Omega} h_{E'}\, \|[\partial_n U_\ell]\|_{L^2(E')}^2 + (1+\delta^{-1}) \sum_{E'\in\mathcal{E}_{\ell+1}^\Omega} h_{E'}\, \|[\partial_n (U_{\ell+1} - U_\ell)]\|_{L^2(E')}^2.\end{aligned} \tag{46}$$

A scaling argument allows to estimate the last term by

$$\sum_{E' \in \mathcal{E}^\Omega_{\ell+1}} h_{E'} \, \|[\partial_n(U_{\ell+1} - U_\ell)]\|^2_{L^2(E')} \le C \, |||U_{\ell+1} - U_\ell|||^2, \tag{47}$$

and the constant $C > 0$ depends only on $\sigma(\mathcal{T}_\ell)$. Next, one investigates the reduction of the other four terms if the mesh is refined locally: According to [32, Lemma 4], there holds

$$\sum_{T' \in \mathcal{T}^\Gamma_{\ell+1}} \mathrm{osc}_{\ell+1}(T')^2 \le \sum_{T \in \mathcal{T}^\Gamma_\ell} \mathrm{osc}_\ell(T)^2 - \frac{1}{2} \sum_{T \in \mathcal{T}^\Gamma_\ell \cap \mathcal{M}_\ell} \mathrm{osc}_\ell(T)^2. \tag{48}$$

Next, [32, Lemma 5] states

$$\sum_{E' \in \mathcal{E}^\Omega_{\ell+1}} h_{E'} \, \|[\partial_n U_\ell]\|^2_{L^2(E')} \le \sum_{E \in \mathcal{E}^\Omega_\ell} \eta_\ell(E)^2 - \frac{1}{2} \sum_{E \in \mathcal{E}^\Omega_\ell \cap \mathcal{M}_\ell} \eta_\ell(E)^2. \tag{49}$$

Third, [32, Lemma 6] yields

$$\sum_{E' \in \mathcal{E}_{\ell+1}} \mathrm{osc}_{\ell+1}(E')^2 \le \sum_{E \in \mathcal{E}_\ell} \mathrm{osc}_\ell(E)^2 - \frac{1}{4} \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} \mathrm{osc}_\ell(E)^2, \tag{50}$$

and we stress that this estimate enforces us to work with μ_ℓ instead of ϱ_ℓ. (For details the reader is referred to [32, Section 3.1].) Moreover, it is part of the proof of [3, Theorem 4.2] that

$$\sum_{E' \in \mathcal{E}^\Gamma_{\ell+1}} \mathrm{apx}_{\ell+1}(E')^2 \le \sum_{E \in \mathcal{E}^\Gamma_\ell} \mathrm{apx}_\ell(E)^2 - \frac{1}{2} \sum_{E \in \mathcal{E}^\Gamma_\ell \cap \mathcal{M}_\ell} \mathrm{apx}_\ell(E)^2. \tag{51}$$

Combining the estimates (46)–(51) and using the Dörfler marking (33), we easily obtain

$$\mu^2_{\ell+1} \le (1+\delta)(1-\theta/4)\, \mu^2_\ell + (1+\delta^{-1}) C \, |||U_{\ell+1} - U_\ell|||^2.$$

Finally, we may choose $\delta > 0$ sufficiently small to guarantee $q := (1+\delta)(1-\theta/4) < 1$ to end up with (45). □

Remark. *In the conforming case $\mathcal{A}_\ell \subseteq \mathcal{A}_{\ell+1}$, it is easily seen that the sequence U_ℓ of discrete solutions tends to some limit U_∞ which is the unique Galerkin solution with respect to the closure of $\bigcup_{\ell=0}^\infty \mathcal{A}_\ell$, see* [31, Lemma 3.3.8]. *Therefore,* $\lim_\ell |||U_{\ell+1} - U_\ell||| = 0$, *and elementary calculus predicts that* (45) *already implies* $\lim_\ell \mu_\ell = 0$, *cf. e.g.* [2, Proposition 1.2]. *Then, Theorem 6 would conclude that* $\lim_\ell \|u - U_\ell\|_{H^1(\Omega)} = 0$, *i.e.* $u = U_\infty$. *— Now that $\mathcal{A}_\ell \not\subseteq \mathcal{A}_{\ell+1}$, we did neither succeed to prove that the a priori limit U_∞ exists nor the much weaker claim that* $\lim_\ell |||U_{\ell+1} - U_\ell||| = 0$. □

6.2. A priori convergence of discrete Dirichlet data. In the adaptive algorithm, the mesh $\mathcal{T}_{\ell+1}$ is obtained by local refinement of elements in $\mathcal{T}_\ell$. Consequently, the discrete spaces $\mathcal{S}^1(\mathcal{T}_\ell)$ are nested, i.e.,

$$\mathcal{S}^1(\mathcal{T}_\ell) \subseteq \mathcal{S}^1(\mathcal{T}_{\ell+1}) \quad \text{for all } \ell \in \mathbb{N} \tag{52}$$

and the analogous inclusion also holds for the spaces $\mathcal{S}^1(\mathcal{T}_\ell|_\Gamma) \subseteq \mathcal{S}^1(\mathcal{T}_{\ell+1}|_\Gamma)$ on the boundary. The following lemma is part of the proof of [1, Proposition 10].

Lemma 12. *The nodal interpolants $g_\ell \in \mathcal{S}^1(\mathcal{T}_\ell|_\Gamma)$ of some Dirichlet data $g \in H^1(\Gamma)$ converge to some a priori limit in $H^1(\Gamma)$, i.e. there holds*

$$\|g_\infty - g_\ell\|_{H^1(\Gamma)} \xrightarrow{\ell\to\infty} 0 \tag{53}$$

for a certain element $g_\infty \in H^1(\Gamma)$. □

In the following, we will only use the convergence of g_ℓ to g_∞ in $H^{1/2}(\Gamma)$ as well as the uniform boundedness $\sup\limits_{\ell\in\mathbb{N}} \|g_\ell\|_{H^{1/2}(\Gamma)} < \infty$ which is an immediate consequence.

Lemma 13. *The sequence $u_\ell \in \mathcal{A}_\ell^\star$ of solutions of the continuous auxiliary problem* (28) *converges to some a priori limit $u_\infty \in H^1(\Omega)$, i.e.*

$$\|u_\infty - u_\ell\|_{H^1(\Omega)} \xrightarrow{\ell\to\infty} 0 \tag{54}$$

for some function $u_\infty \in H^1(\Omega)$. In particular, there holds

$$\mathcal{J}(u_\infty) = \lim_{\ell\to\infty} \mathcal{J}(u_\ell). \tag{55}$$

Proof. Since convergence in $H^{1/2}(\Gamma)$ implies convergence in $L^2(\Gamma)$, the Weyl theorem applies and proves that a subsequence (g_{ℓ_k}) converges to g_∞ pointwise almost everywhere on Γ. Therefore, the limit function $g_\infty \in H^{1/2}(\Gamma)$ satisfies $g_\infty \geq \chi$ almost everywhere on Γ. In particular, the obstacle problem (8) with $g = g_\infty$ has also a unique solution $u_\infty \in H^1(\Gamma)$. Applying the stability result from Proposition 10, we obtain convergence of u_ℓ to u_∞ in $H^1(\Omega)$ as $\ell \to \infty$. □

6.3. Discrete energy estimates.

We start with the observation that the energy functional $\mathcal{J}(\cdot)$ is coercive.

Lemma 14. *For each sequence $w_\ell \in H^1(\Omega)$ with changing boundary data $w_\ell|_\Gamma = g_\ell$, bounded energy implies uniform boundedness in $H^1(\Omega)$, i.e.*

$$\sup_{\ell\in\mathbb{N}} \mathcal{J}(w_\ell) < \infty \quad \Longrightarrow \quad \sup_{\ell\in\mathbb{N}} \|w_\ell\|_{H^1(\Omega)} < \infty. \tag{56}$$

Proof. Note that triangle inequalities and the Friedrichs inequality yield

$$\begin{aligned}\|w_\ell\|_{L^2(\Omega)} &\leq \|w_\ell - \mathcal{L}g_\ell\|_{L^2(\Omega)} + \|\mathcal{L}g_\ell\|_{L^2(\Omega)} \\ &\lesssim \|\nabla(w_\ell - \mathcal{L}g_\ell)\|_{L^2(\Omega)} + \|\mathcal{L}g_\ell\|_{L^2(\Omega)} \\ &\lesssim \|\nabla w_\ell\|_{L^2(\Omega)} + \|\mathcal{L}g_\ell\|_{H^1(\Omega)} \\ &\lesssim \|\nabla w_\ell\|_{L^2(\Omega)} + M\end{aligned}$$

with $M = \sup_\ell \|g_\ell\|_{H^{1/2}(\Gamma)} < \infty$, where we have finally used continuity of $\mathcal{L}$ and a priori convergence of g_ℓ, cf. Lemma 12. Consequently, we obtain

$$\|w_\ell\|_{H^1(\Omega)} \lesssim \|\nabla w_\ell\|_{L^2(\Omega)} + M$$

and therefore

$$\mathcal{J}(w_\ell) = \frac{1}{2}\|\nabla w_\ell\|_{L^2(\Omega)}^2 - (f, w_\ell) \gtrsim (\|w_\ell\|_{H^1(\Omega)} - M)^2 - \|f\|_{L^2(\Omega)}\|w_\ell\|_{H^1(\Omega)}.$$

The implication (56) is an immediate consequence. □

Lemma 15. *The sequence of discrete solutions $U_\ell \in \mathcal{A}_\ell$ from Algorithm 7 satisfies*

$$\sup_{\ell\in\mathbb{N}} |\mathcal{J}(U_\ell)| < \infty \quad \textit{as well as} \quad \sup_{\ell\in\mathbb{N}} \|U_\ell\|_{H^1(\Omega)} < \infty. \tag{57}$$

Proof. According to Lemma 14 with $w_\ell = U_\ell$, it is sufficient to prove boundedness of the energy. To that end, we define

$$V_\ell := \max_\ell\{P_\ell \mathcal{L} g_\ell - \chi, 0\} + \chi \in \mathcal{A}_\ell,$$

where $\max_\ell$ denotes the nodewise max-function, i.e. $\max_\ell\{v, w\} \in \mathcal{S}^1(\mathcal{T}_\ell)$ is defined by

$$\max_\ell\{v, w\}(z) := \max\{v(z), w(z)\} \quad \text{for all } z \in \mathcal{N}_\ell.$$

We consider the function $W_\ell := \max_\ell\{P_\ell \mathcal{L} g_\ell - \chi, 0\}$. Note that, by definition, $|W_\ell(z)| \le |(P_\ell \mathcal{L} g_\ell - \chi)(z)|$ for all $z \in \mathcal{N}_\ell$. According to a standard scaling argument and $P_\ell \mathcal{L} g_\ell - \chi \in \mathcal{S}^1(\mathcal{T}_\ell)$, we infer, for all $T \in \mathcal{T}_\ell$,

$$\|W_\ell\|_{L^2(T)} \lesssim \|P_\ell \mathcal{L} g_\ell - \chi\|_{L^2(T)} \quad \text{as well as} \quad \|\nabla W_\ell\|_{L^2(T)} \lesssim \|\nabla(P_\ell \mathcal{L} g_\ell - \chi)\|_{L^2(T)}$$

due to the nodewise estimate. From this, we obtain a constant $C > 0$ with

$$\|V_\ell\|_{H^1(\Omega)} \lesssim \|P_\ell \mathcal{L} g_\ell - \chi\|_{H^1(\Omega)} + \|\chi\|_{H^1(\Omega)} \lesssim \|g_\ell\|_{H^{1/2}(\Gamma)} + \|\chi\|_{H^1(\Omega)} \le C < \infty$$

since $\sup_\ell \|g_\ell\|_{H^{1/2}(\Gamma)} < \infty$ and the operator norm of P_ℓ depends only on $\sigma(\mathcal{T}_\ell)$. Therefore,

$$\mathcal{J}(U_\ell) \le \mathcal{J}(V_\ell) = \frac{1}{2} \|\nabla V_\ell\|^2_{L^2(\Omega)} - (f, V_\ell) \lesssim C\,(C + \|f\|_{L^2(\Omega)}) < \infty$$

yields $\sup_{\ell\in\mathbb{N}} \mathcal{J}(U_\ell) < \infty$ and concludes the proof. □

The next lemma is a key ingredient of the upcoming proofs. It states that one may change the boundary data of a discrete function and control the influence within Ω.

Lemma 16. *Let $W_{\ell+1} \in \mathcal{S}^1(\mathcal{T}_{\ell+1})$ with $W_{\ell+1}|_\Gamma = g_{\ell+1}$. Define $W^\ell_{\ell+1} \in \mathcal{S}^1(\mathcal{T}_{\ell+1})$ by*

$$W^\ell_{\ell+1}(z) = \begin{cases} W_{\ell+1}(z) & \textit{for } z \in \mathcal{N}_{\ell+1}\backslash\Gamma, \\ g_\ell(z) & \textit{for } z \in \mathcal{N}_{\ell+1} \cap \Gamma. \end{cases} \tag{58}$$

Then, there holds

$$\|W_{\ell+1} - W^\ell_{\ell+1}\|_{H^1(\Omega)} \le C_5\, \|h_\ell^{1/2}(g_{\ell+1} - g_\ell)'\|_{L^2(\Gamma)} \le C_6\, \|g_{\ell+1} - g_\ell\|_{H^{1/2}(\Gamma)}, \tag{59}$$

where $C_5, C_6 > 0$ depend only on $\mathcal{T}_0$ and on Ω.

Proof. By definition, we have $W^\ell_{\ell+1} \in \mathcal{S}^1(\mathcal{T}_{\ell+1})$ with $W^\ell_{\ell+1}|_\Gamma = g_\ell$. Moreover, there holds $W^\ell_{\ell+1}|_T = W_{\ell+1}|_T$ for all $T \in \mathcal{T}_{\ell+1} \setminus \mathcal{T}^\Gamma_{\ell+1}$. Let $T = \mathrm{conv}\{z_1, z_2, z_3\} \in \mathcal{T}^\Gamma_{\ell+1}$ and $\Phi_T : T_{\mathrm{ref}} \to T$ be the affine diffeomorphism with reference element $T_{\mathrm{ref}} = \mathrm{conv}\{(0,0), (0,1), (1,0)\}$.

The transformation formula and norm equivalence on $\mathcal{P}^1(T_{\mathrm{ref}})$ prove

$$\begin{aligned} \|W_{\ell+1} - W^\ell_{\ell+1}\|_{L^2(T)} &= |T|^{1/2}\, \|(W_{\ell+1} - W^\ell_{\ell+1}) \circ \Phi_T\|_{L^2(T_{\mathrm{ref}})} \\ &\lesssim |T|^{1/2} \sum_{j=1}^3 |(W_{\ell+1} - W^\ell_{\ell+1}) \circ \Phi_T(z_j^{\mathrm{ref}})| \\ &\le |\Omega|^{1/2} \sum_{j=1}^3 |(W_{\ell+1} - W^\ell_{\ell+1})(z_j)| \end{aligned}$$

as well as

$$\begin{aligned}\|\nabla(W_{\ell+1}-W_{\ell+1}^{\ell})\|_{L^2(T)} &\lesssim \sigma(\mathcal{T}_\ell)\,\|\nabla\big((W_{\ell+1}-W_{\ell+1}^{\ell})\circ\Phi_T\big)\|_{L^2(T_{\mathrm{ref}})}\\ &\lesssim \sigma(\mathcal{T}_\ell)\sum_{j=1}^{3}|(W_{\ell+1}-W_{\ell+1}^{\ell})(z_j)|.\end{aligned}$$

Now, we consider the nodes $z\in\{z_1,z_2,z_3\}$ of the triangle T: For $z\in\mathcal{N}_\ell\cap\Gamma$ holds $g_\ell(z)=g_{\ell+1}(z)$ by definition of the nodal interpolants. Therefore,

$$(W_{\ell+1}-W_{\ell+1}^{\ell})(z)=0\quad\text{if } z\in(\mathcal{N}_{\ell+1}\cap\Omega)\cup(\mathcal{N}_\ell\cap\Gamma).$$

Thus, it only remains to consider nodes $z\in(\mathcal{N}_{\ell+1}\backslash\mathcal{N}_\ell)\cap\Gamma$. In this case, z is the midpoint of a refined edge $\widehat{E}_z$ of the unique father $\widehat{T}\in\mathcal{T}_\ell$ with $T\subseteq\widehat{T}$, and there even holds $\widehat{E}_z\subset\Gamma$. Let $\widehat{z}\in\mathcal{N}_\ell$ be an arbitrary endpoint of $\widehat{E}_z$. Then,

$$(W_{\ell+1}-W_{\ell+1}^{\ell})(\widehat{z})=(g_{\ell+1}-g_\ell)(\widehat{z})=0,$$

and the fundamental theorem of calculus, now applied for the arclength derivative, yields

$$|(W_{\ell+1}-W_{\ell+1}^{\ell})(z)|\le\Big|\int_{\widehat{z}}^{z}(g_{\ell+1}-g_\ell)'\,d\Gamma\Big|\le|z-\widehat{z}|^{1/2}\,\|(g_\ell-g_{\ell+1})'\|_{L^2(\widehat{E}_z)}.$$

Combining our observations, we have now shown

$$\|W_{\ell+1}-W_{\ell+1}^{\ell}\|_{H^1(T)}\lesssim\sum_{\substack{\widehat{E}\in\mathcal{E}_\ell^\Gamma\\ \widehat{E}\subset\partial\widehat{T}}}h_{\widehat{E}}^{1/2}\|(g_\ell-g_{\ell+1})'\|_{L^2(\widehat{E})}\simeq\Big(\sum_{\substack{\widehat{E}\in\mathcal{E}_\ell^\Gamma\\ \widehat{E}\subset\partial\widehat{T}}}h_{\widehat{E}}\|(g_\ell-g_{\ell+1})'\|_{L^2(\widehat{E})}^2\Big)^{1/2}.$$

Finally, note that each edge $\widehat{E}\in\mathcal{E}_\ell^\Gamma$ belongs only to one element $\widehat{T}\in\mathcal{T}_\ell$. Thus, $\widehat{E}$ can at most be selected by all (but at most four) sons $T\in\mathcal{T}_{\ell+1}$ of $\widehat{T}$, cf. Figure 1. This observation yields

$$\|W_{\ell+1}-W_{\ell+1}^{\ell}\|_{H^1(\Omega)}^2=\sum_{T\in\mathcal{T}_{\ell+1}^\Gamma}\|W_{\ell+1}-W_{\ell+1}^{\ell}\|_{H^1(T)}^2\lesssim\sum_{\widehat{E}\in\mathcal{E}_\ell^\Gamma}h_{\widehat{E}}^{1/2}\|(g_\ell-g_{\ell+1})'\|_{L^2(\widehat{E})}^2.$$

With the local mesh-width function $h_\ell\in L^\infty(\Gamma)$ defined by $h_\ell|_{\widehat{E}}=h_{\widehat{E}}$, this estimate reads

$$\|W_{\ell+1}-W_{\ell+1}^{\ell}\|_{H^1(\Omega)}\lesssim\|h_\ell^{1/2}(g_\ell-g_{\ell+1})'\|_{L^2(\Gamma)}\lesssim\|g_\ell-g_{\ell+1}\|_{H^{1/2}(\Gamma)},$$

where we have used a local inverse estimate from [14, Proposition 3.1] in the last step. We stress that the constant depends only on the local mesh-ratio of $\mathcal{T}_{\ell+1}|_\Gamma$, whence on $\sigma(\mathcal{T}_{\ell+1})$. This concludes the proof. $\square$

Next, we consider a discrete auxiliary problem: *Find $U_{\ell+1,\ell}\in\mathcal{A}_{\ell+1,\ell}$ such that*

$$\mathcal{J}(U_{\ell+1,\ell})=\min_{V_{\ell+1}\in\mathcal{A}_{\ell+1,\ell}}\mathcal{J}(V_{\ell+1}).\tag{60}$$

where the admissible set reads

$$\mathcal{A}_{\ell+1,\ell}:=\big\{V_{\ell+1}\in\mathcal{S}^1(\mathcal{T}_{\ell+1})\,:\,V_{\ell+1}\ge\chi\text{ in }\Omega,\,V_{\ell+1}=g_\ell\text{ on }\Gamma\big\}.\tag{61}$$

Applying Proposition 5 to the data $(\mathcal{A}_{\ell+1,\ell},g_\ell)$ instead of $(\mathcal{A}_\ell,g_\ell)$, we see that the auxiliary problem (60) admits a unique solution $U_{\ell+1,\ell}\in\mathcal{A}_{\ell+1,\ell}$.

Lemma 17. *The sequence of discrete solutions $U_{\ell+1,\ell} \in \mathcal{A}_{\ell+1,\ell}$ satisfies*

$$\sup_{\ell\in\mathbb{N}} |\mathcal{J}(U_{\ell+1,\ell})| < \infty \quad \textit{as well as} \quad \sup_{\ell\in\mathbb{N}} \|U_{\ell+1,\ell}\|_{H^1(\Omega)} < \infty. \tag{62}$$

Moreover, with the minimizer $U_{\ell+1} \in \mathcal{A}_{\ell+1}$, there holds

$$\lim_{\ell\to\infty} |\mathcal{J}(U_{\ell+1,\ell}) - \mathcal{J}(U_{\ell+1})| = 0 \quad \textit{as well as} \quad \lim_{\ell\to\infty} |\mathcal{J}(U_{\ell+1,\ell}) - \mathcal{J}(U^{\ell}_{\ell+1})| = 0. \tag{63}$$

Proof. According to Lemma 14 with $w_\ell = U_{\ell+1,\ell}$, it is sufficient to prove boundedness of the energy to verify (62): With

$$M_1 := \sup_{\ell\in\mathbb{N}} \|g_{\ell+1} - g_\ell\|_{H^{1/2}(\Gamma)} < \infty \quad \text{and} \quad M_2 := \sup_{\ell\in\mathbb{N}} \|U_\ell\|_{H^1(\Omega)} < \infty$$

and $U^{\ell}_{\ell+1} \in \mathcal{A}_{\ell+1,\ell}$, we obtain from Lemma 16

$$\begin{aligned}
\mathcal{J}(U_{\ell+1,\ell}) &\le \mathcal{J}(U^{\ell}_{\ell+1}) = \frac{1}{2}\, |||U^{\ell}_{\ell+1}|||^2 - (f, U^{\ell}_{\ell+1}) \\
&\le \frac{1}{2}\left(|||U_{\ell+1}||| + |||U^{\ell}_{\ell+1} - U_{\ell+1}|||\right)^2 - (f, U_{\ell+1}) + \|f\|_{L^2(\Omega)}\|U^{\ell}_{\ell+1} - U_{\ell+1}\|_{L^2(\Omega)} \\
&= \mathcal{J}(U_{\ell+1}) + \frac{1}{2}|||U^{\ell}_{\ell+1} - U_{\ell+1}|||^2 + |||U_{\ell+1}|||\, |||U^{\ell}_{\ell+1} - U_{\ell+1}||| + \|f\|_{L^2(\Omega)}\|U^{\ell}_{\ell+1} - U_{\ell+1}\|_{L^2(\Omega)} \\
&\le \mathcal{J}(U_{\ell+1}) + \frac{1}{2}\, C_6^2\, \|g_\ell - g_{\ell+1}\|^2_{H^{1/2}(\Gamma)} + C_6\,(M_2 + \|f\|_{L^2(\Omega)})\, \|g_{\ell+1} - g_\ell\|_{H^{1/2}(\Gamma)},
\end{aligned} \tag{64}$$

whence

$$\sup_{\ell\in\mathbb{N}} \mathcal{J}(U_{\ell+1,\ell}) \le \sup_{\ell\in\mathbb{N}} \mathcal{J}(U_{\ell+1}) + \frac{1}{2}\, C_6^2\, M_1^2 + C_6\,(M_2 + \|f\|_{L^2(\Omega)})\, M_1 < \infty.$$

This concludes the proof of (62). Moreover, Estimate (64) reveals

$$\mathcal{J}(U_{\ell+1,\ell}) - \mathcal{J}(U_{\ell+1}) \lesssim \|g_\ell - g_{\ell+1}\|^2_{H^{1/2}(\Gamma)} + \|g_{\ell+1} - g_\ell\|_{H^{1/2}(\Gamma)}.$$

Similar calculations with the ansatz $\mathcal{J}(U_{\ell+1}) \le \mathcal{J}(U^{\ell+1}_{\ell+1,\ell})$ show

$$\mathcal{J}(U_{\ell+1}) - \mathcal{J}(U_{\ell+1,\ell}) \lesssim \|g_\ell - g_{\ell+1}\|^2_{H^{1/2}(\Gamma)} + \|g_{\ell+1} - g_\ell\|_{H^{1/2}(\Gamma)}.$$

The combination of the last two inequalities yields

$$|\mathcal{J}(U_{\ell+1}) - \mathcal{J}(U_{\ell+1,\ell})| \lesssim \|g_\ell - g_{\ell+1}\|^2_{H^{1/2}(\Gamma)} + \|g_{\ell+1} - g_\ell\|_{H^{1/2}(\Gamma)} \xrightarrow{\ell\to\infty} 0,$$

which provides the first limit in (63). Finally, we recall $\mathcal{J}(U_{\ell+1,\ell}) \le \mathcal{J}(U^{\ell}_{\ell+1})$. Then, the above calculation also yields

$$\begin{aligned}
|\mathcal{J}(U^{\ell}_{\ell+1}) - \mathcal{J}(U_{\ell+1,\ell})| &= \mathcal{J}(U^{\ell}_{\ell+1}) - \mathcal{J}(U_{\ell+1,\ell}) \\
&= \left[\mathcal{J}(U^{\ell}_{\ell+1}) - \mathcal{J}(U_{\ell+1})\right] + \left[\mathcal{J}(U_{\ell+1}) - \mathcal{J}(U_{\ell+1,\ell})\right] \\
&\lesssim \|g_\ell - g_{\ell+1}\|^2_{H^{1/2}(\Gamma)} + \|g_{\ell+1} - g_\ell\|_{H^{1/2}(\Gamma)} \xrightarrow{\ell\to\infty} 0.
\end{aligned}$$

This concludes the proof. □

We are now ready to prove that the sequence $U_\ell \in \mathcal{A}_\ell$ of discrete solutions generated by Algorithm 7 indeed converges towards the exact solution $u \in \mathcal{A}$.

6.4. Proof of Theorem 8. With the help of Lemma 16 and Lemma 4 applied twice for $U_\ell \in \mathcal{A}_{\ell+1,\ell}$, and $U_{\ell+1}^\ell \in \mathcal{A}_{\ell+1,\ell}$, we obtain

$$\begin{aligned}\frac{1}{2}\,|||U_{\ell+1}-U_\ell|||^2 &\le |||U_{\ell+1}-U_{\ell+1,\ell}|||^2 + |||U_{\ell+1,\ell}-U_\ell|||^2\\ &\le 2\,|||U_{\ell+1}-U_{\ell+1}^\ell|||^2 + 2\,|||U_{\ell+1}^\ell - U_{\ell+1,\ell}|||^2 + 2\big[\mathcal{J}(U_\ell)-\mathcal{J}(U_{\ell+1,\ell})\big]\\ &\le 2\,C_5^2\|h_\ell^{1/2}(g_{\ell+1}-g_\ell)'\|_{L^2(\Gamma)}^2 + 4\big[\mathcal{J}(U_{\ell+1}^\ell)-\mathcal{J}(U_{\ell+1,\ell})\big] + 2\big[\mathcal{J}(U_\ell)-\mathcal{J}(U_{\ell+1,\ell})\big].\end{aligned}$$

Recall that $\Delta_{\ell+1} = \big[\mathcal{J}(U_{\ell+1})-\mathcal{J}(u_{\ell+1})\big] + \gamma\mu_{\ell+1}^2 + \lambda\,\mathrm{apx}_{\ell+1}^2$. We now use the last estimate to see

$$\begin{aligned}\big[\mathcal{J}(U_{\ell+1})-\mathcal{J}(u_{\ell+1})\big] &= \big[\mathcal{J}(U_\ell)-\mathcal{J}(u_\ell)\big] + \big[\mathcal{J}(u_\ell)-\mathcal{J}(u_{\ell+1})\big] + \big[\mathcal{J}(U_{\ell+1})-\mathcal{J}(U_{\ell+1,\ell})\big]\\ &\qquad + \big[\mathcal{J}(U_{\ell+1,\ell})-\mathcal{J}(U_\ell)\big]\\ &\le \big[\mathcal{J}(U_\ell)-\mathcal{J}(u_\ell)\big] + \big[\mathcal{J}(u_\ell)-\mathcal{J}(u_{\ell+1})\big] + \big[\mathcal{J}(U_{\ell+1})-\mathcal{J}(U_{\ell+1,\ell})\big]\\ &\qquad + C_5^2\|h_\ell^{1/2}(g_{\ell+1}-g_\ell)'\|_{L^2(\Gamma)}^2 - \frac{1}{4}\,|||U_{\ell+1}-U_\ell|||^2\\ &\qquad + 2\big[\mathcal{J}(U_{\ell+1}^\ell)-\mathcal{J}(U_{\ell+1,\ell})\big].\end{aligned}$$

Note that according to Lemma 13, Lemma 17, and Lemma 12, there holds

$$\alpha_\ell := \big|\mathcal{J}(u_\ell)-\mathcal{J}(u_{\ell+1})\big| + \big|\mathcal{J}(U_{\ell+1})-\mathcal{J}(U_{\ell+1,\ell})\big| + 2\,\big|\mathcal{J}(U_{\ell+1}^\ell)-\mathcal{J}(U_{\ell+1,\ell})\big| \xrightarrow{\ell\to\infty} 0.$$

Next, we recall that on the 1D manifold Γ, the derivative g_ℓ' of the nodal interpolant is the elementwise best approximation of the derivative g' by piecewise constants, i.e.

$$\|h_\ell^{1/2}(g-g_{\ell+1})'\|_{L^2(\Gamma)}^2 + \|h_\ell^{1/2}(g_{\ell+1}-g_\ell)'\|_{L^2(\Gamma)}^2 = \|h_\ell^{1/2}(g-g_\ell)'\|_{L^2(\Gamma)}^2 \tag{65}$$

according to the elementwise Pythagoras theorem. So far and with $\lambda = C_5^2$, we thus have derived

$$\begin{aligned}\Delta_{\ell+1} &\le \big[\mathcal{J}(U_\ell)-\mathcal{J}(u_\ell)\big] + \gamma\mu_{\ell+1}^2 + \lambda\mathrm{apx}_{\ell+1}^2 + \lambda\,\|h_\ell^{1/2}(g_{\ell+1}-g_\ell)'\|_{L^2(\Gamma)}^2 - \frac{1}{4}\,|||U_{\ell+1}-U_\ell|||^2 + \alpha_\ell,\\ &\le \big[\mathcal{J}(U_\ell)-\mathcal{J}(u_\ell)\big] + \gamma\mu_{\ell+1}^2 + \lambda\mathrm{apx}_\ell^2 - \frac{1}{4}\,|||U_{\ell+1}-U_\ell|||^2 + \alpha_\ell,\end{aligned}$$

where we have used the Pythagoras theorem in the second step.

Due to the equivalence (36) of ϱ_ℓ and μ_ℓ, the Dörfler marking provides

$$\frac{\theta}{4}\mu_\ell^2 \le \theta\varrho_\ell^2 \le \sum_{E\in\mathcal{E}_\ell^\Omega\cap\mathcal{M}_\ell}\big(\eta_\ell(E)^2 + \mathrm{osc}_\ell(E)^2\big) + \sum_{T\in\mathcal{T}_\ell^\Gamma\cap\mathcal{M}_\ell}\mathrm{osc}_\ell(T)^2 + \sum_{E\in\mathcal{E}_\ell^\Gamma\cap\mathcal{M}_\ell}\mathrm{apx}_\ell(E)^2,$$

i.e. $\mathcal{M}_\ell$ satisfies the Dörfler marking (44) with modified parameter $\widetilde{\theta} = \theta/4 \in (0,1)$. Consequently, the estimator reduction of Proposition 11 applies and provides constants $0 < q < 1$ and $C_4 > 0$ with

$$\begin{aligned}\Delta_{\ell+1} &\le \big[\mathcal{J}(U_\ell)-\mathcal{J}(u_\ell)\big] + \gamma q\mu_\ell^2 + \lambda\mathrm{apx}_\ell^2 + \Big(\gamma C_4 - \frac{1}{4}\Big)\,|||U_{\ell+1}-U_\ell|||^2 + \alpha_\ell\\ &\le \big[\mathcal{J}(U_\ell)-\mathcal{J}(u_\ell)\big] + \gamma q\mu_\ell^2 + \lambda\mathrm{apx}_\ell^2 + \alpha_\ell\end{aligned}$$

provided the constant $0 < \gamma < 1$ is chosen sufficiently small.

Next, from Theorem 6, we infer $C_3^{-2}\big[\mathcal{J}(U_\ell)-\mathcal{J}(u_\ell)\big] \le \varrho_\ell^2 \le \mu_\ell^2$.

We plug this estimate into the last one and use the fact that $\mathrm{apx}_\ell^2 \le \mu_\ell^2$ to see

$$\Delta_{\ell+1} \le (1-\gamma\varepsilon C_3^{-2})\left[\mathcal{J}(U_\ell) - \mathcal{J}(u_\ell)\right] + \gamma(q+2\,\varepsilon)\,\mu_\ell^2 + (1-\varepsilon\gamma)\lambda\mathrm{apx}_\ell^2 + \alpha_\ell \le \kappa\,\Delta_\ell + \alpha_\ell,$$

with $\kappa := \max\{1-\gamma\varepsilon C_3^{-2}\,, q+2\,\varepsilon, 1-\gamma\varepsilon\}$. For $0 < 2\,\varepsilon < 1-q$, we obtain $0 < \kappa < 1$ and conclude the proof of the contraction property (38).

The application of Lemma 4 for $\mathcal{A}_\ell \subseteq \mathcal{A}_\ell^\star$ yields $\mathcal{J}(u_\ell) \le \mathcal{J}(U_\ell)$ and hence $\Delta_\ell \ge 0$. Together with $\alpha_\ell \ge 0$ and $\alpha_\ell \to 0$, elementary calculus yields $\Delta_\ell \to 0$ as $\ell \to \infty$, cf. [2, Proposition 1.2]. From $0 \le \mu_\ell \lesssim \Delta_\ell$, we obtain $\lim_\ell \mu_\ell = 0 = \lim_\ell \varrho_\ell$, and the weak reliability of ϱ_ℓ stated in Theorem 6 concludes the proof of (39). □

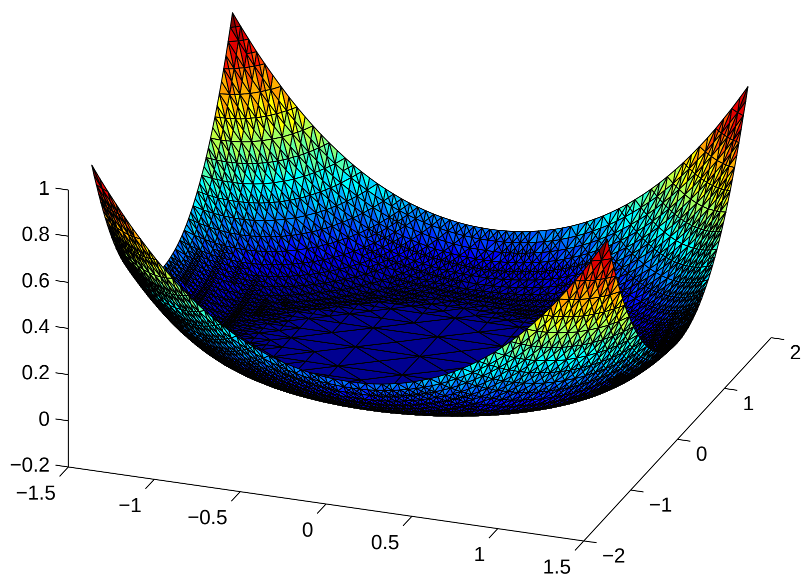

FIGURE 2. Galerkin solution U_6 on adaptively generated mesh $\mathcal{T}_6$ with $N = 4.159$ elements for $\theta = 0.8$.

7. NUMERICAL EXPERIMENTS

We consider numerical examples that basically also have been treated in [9]. The mesh in each step is adaptively generated by Algorithm 7. For the solution at each level, we use the primal-dual active set strategy from [22]. The numerical results are quite similar to those in [9]. We stress, however, that our approach includes the adaptive resolution of the Dirichlet data and, contrary to [9], the upcoming examples are thus covered by theory.

7.1. Example 1. We consider the obstacle problem with constant obstacle $\chi \equiv 0$ on the square $\Omega := (-1.5, 1.5)^2$ and a constant force $f \equiv -2$. The Dirichlet boundary data $g \in H^1(\Gamma)$ are given by the trace of the exact solution

$$u := \begin{cases} \frac{r^2}{2} - \ln(r) - \frac{1}{2}, & r \ge 1 \\ 0, & \text{else,} \end{cases}$$

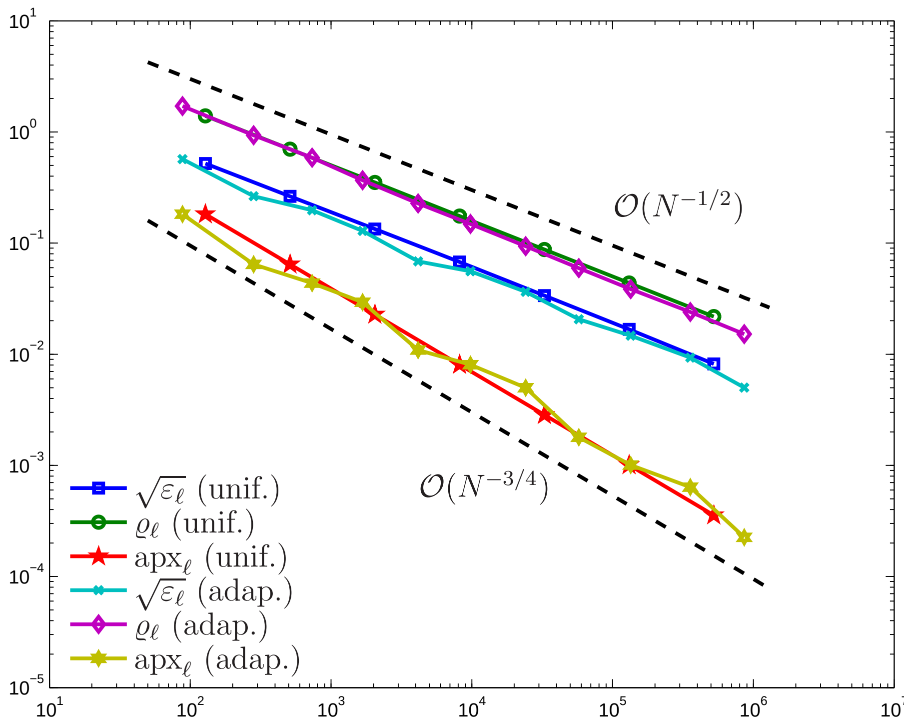

FIGURE 3. Numerical results for uniform and adaptive mesh-refinement with $\theta = 0.8$, where $\sqrt{\varepsilon_\ell}, \varrho_\ell$, and apx_ℓ are plotted over the number of elements $N = \#\mathcal{T}_\ell$.

where $r = |x|$ and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$. The solution is visualized in Figure 2. We compare uniform and adaptive mesh-refinement, where the adaptivity parameter θ varies between 0.4 and 0.8. The convergence history for uniform and adaptive refinement with $\theta = 0.8$ is plotted in Figure 3, where the error is given in the energy functional, i.e.

$$\varepsilon_\ell = \big|\mathcal{J}(U_\ell) - \mathcal{J}(u)\big| \tag{66}$$

and the Dirichlet data oscillations apx_ℓ are defined by (26). Note that due to Theorem 8, the adaptive algorithm drives the error and thus also the energy ε_ℓ to zero, whence it makes sense to plot these physically relevant terms. All quantities are plotted over the number of elements $N = \#\mathcal{T}_\ell$ of the given triangulation. Due to high regularity of the exact solution, there are no significant benefits of adaptive refinement. We observe, however, that error and error estimator, as well as Dirichlet oscillations show optimal convergence behaviour $\mathcal{O}(N^{-1/2})$ and $\mathcal{O}(N^{-3/4})$ respectively. Also, the curves of ϱ_ℓ and $\sqrt{\varepsilon_\ell}$ are parallel, which experimentally confirms classical reliability and efficiency of the underlying estimator ϱ_ℓ in the sense of $\|u - U_\ell\|_{H^1(\Omega)} \simeq \varrho_\ell$.

Figure 4 compares different values of θ and we can see that each choice of $\theta \in \{0.4, 0.6, 0.8\}$ leads to optimal convergence, i.e. the curves basically coincide.

Finally, Figure 5 shows the adaptively generated meshes after 5 and 11 iterations. As expected, refinement basically takes place in the inactive zone, i.e. elements where the discrete solution U_ℓ does not touch the obstacle.

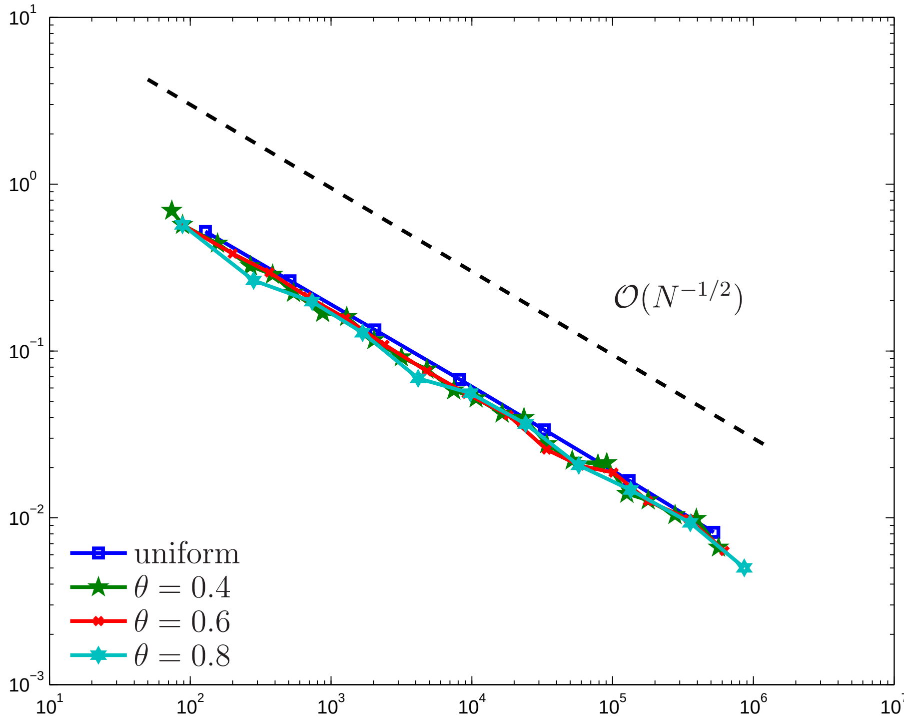

FIGURE 4. Numerical results for $\sqrt{\varepsilon_\ell}$ for uniform and adaptive mesh-refinement with $\theta \in \{0.4, 0.6, 0.8\}$, plotted over the number of elements $N = \#\mathcal{T}_\ell$.

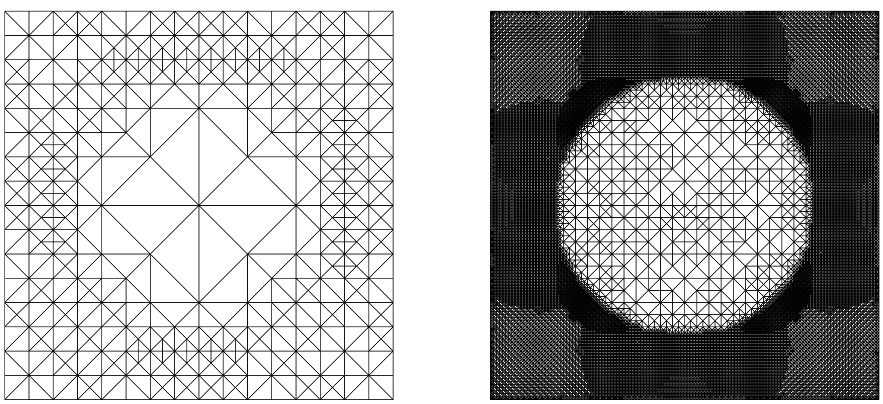

FIGURE 5. Adaptively generated meshes $\mathcal{T}_5$ (*left*) and $\mathcal{T}_{11}$ (*right*) with $N = 678$ and $N = 34.024$ elements, respectively, for $\theta = 0.6$.

7.2. Example 2. This example is a mixture of the two examples treated in [9]. Again we consider a constant zero-obstacle, i.e. $\chi \equiv 0$. As computational domain, we choose the L-shape $\Omega := (-1.5, 1.5)^2 \backslash [0, 1.5) \times (-1.5, 0]$. The right hand side is given in polar coordinates by

$$f(r,\varphi) := \begin{cases} -2, & r \geq 0.75 \\ -r^{2/3}\sin(2\varphi/3)\big(d/dr(\gamma_1)(r)/r + d^2/dr^2\gamma_1(r)\big) & \\ \quad -\frac{4}{3}\,r^{-1/3}d/dr(\gamma_1)(r)\sin(2\varphi/3) - \gamma_2(r), & r < 0.75, \end{cases}$$

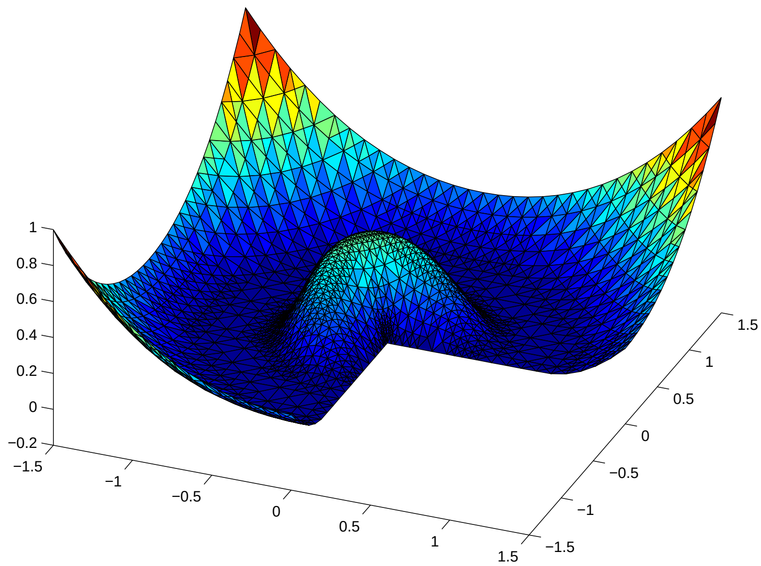

FIGURE 6. Galerkin solution U_9 on adaptively generated mesh $\mathcal{T}_9$ with $N = 4.616$ elements for $\theta = 0.6$.

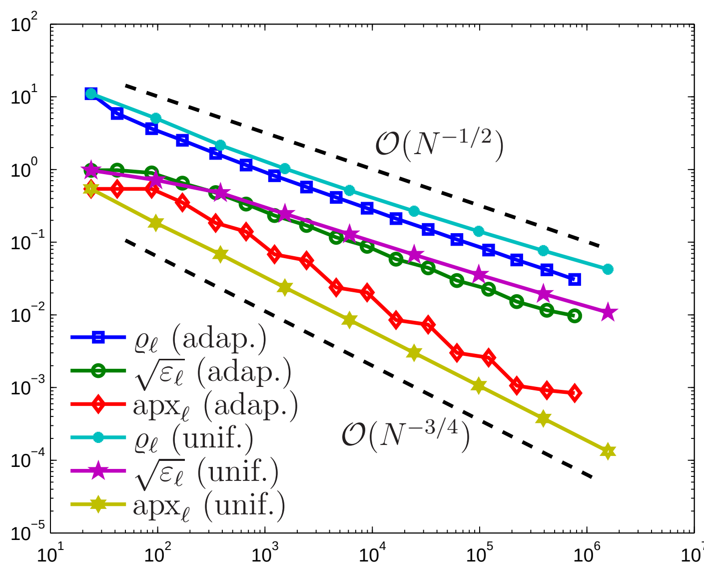

FIGURE 7. Numerical results for uniform and adaptive mesh-refinement with $\theta = 0.6$, where $\sqrt{\varepsilon_\ell}, \varrho_\ell$, and apx_ℓ are plotted over the number of elements $N = \#\mathcal{T}_\ell$.

where d/dr denotes the radial derivative. Moreover, $\bar{r} := 2(r - 1/4)$ and

$$\gamma_1(r) := \begin{cases} 1, & \bar{r} < 0, \\ -6\bar{r}^5 + 15\bar{r}^4 - 10\bar{r}^3 + 1, & 0 \leq \bar{r} < 1, \\ 0, & \bar{r} \geq 1, \end{cases}$$

$$\gamma_2(r) := \begin{cases} 0, & r \leq 5/4, \\ 1, & \text{else.} \end{cases}$$

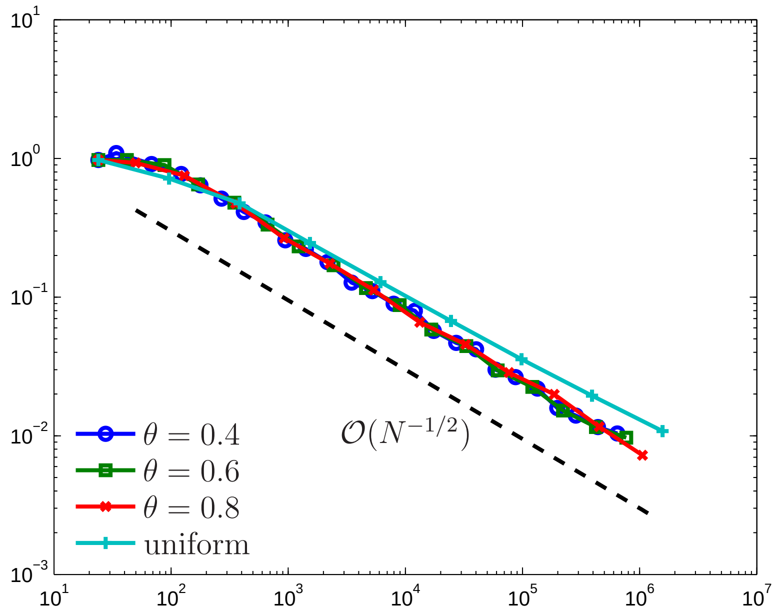

FIGURE 8. Numerical results for $\sqrt{\varepsilon_\ell}$ for uniform and adaptive mesh-refinement with $\theta \in \{0.4, 0.6, 0.8\}$, plotted over the number of elements $N = \#\mathcal{T}_\ell$.

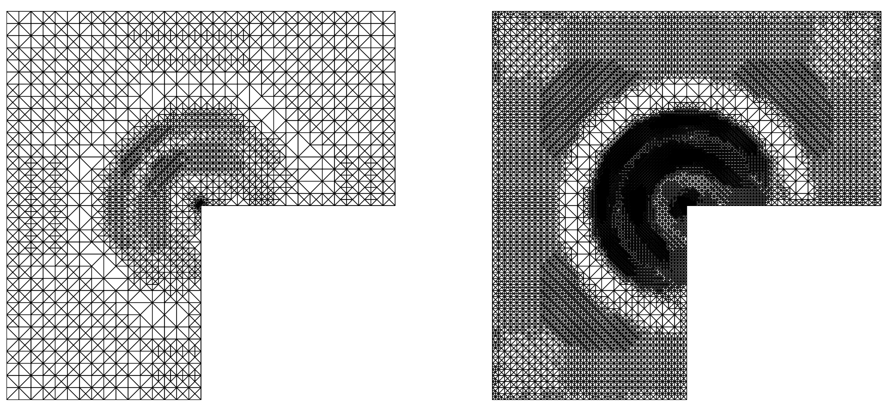

FIGURE 9. Adaptively generated meshes $\mathcal{T}_9$ (*left*) and $\mathcal{T}_{12}$ (*right*) with $N = 4.616$ and $N = 33.076$ elements, respectively, for $\theta = 0.6$.

The Dirichlet data g are given by the trace of the exact solution

$$u(r,\varphi) = \begin{cases} \frac{r^2}{2} - \ln(r) - \frac{1}{2}, & r \geq 1 \\ 0, & 0.75 \leq r < 1 \\ r^{2/3}\gamma_1(r)\sin(2\varphi/3), & r < 0.75, \end{cases}$$

which is visualized in Figure 6.

Again, we compare uniform and adaptive mesh-refinement for different adaptivity parameters θ between 0.4 and 0.8. Figure 7 shows the convergence history for $\theta = 0.6$ plotted over the number of elements $N = \#\mathcal{T}_\ell$. As before, adaptive refinement leads to the optimal convergence rates $\mathcal{O}(N^{-1/2})$ and $\mathcal{O}(N^{-3/4})$ for $\sqrt{\varepsilon_\ell}$ and apx_ℓ, respectively. Due to the corner singularity of the exact solution at 0, we observe that uniform mesh-refinement leads to a suboptimal convergence behaviour. Even though the suboptimality of uniform refinement can easily be seen, we observe that the curves of adaptive and uniform refinement drift apart only after a few iterations. This is of course due to the fact that the active zone in this example is quite small. In the beginning, the domain is refined nearly everywhere, see Figure 9.

Figure 8 compares the error for uniform and adaptive mesh refinement, where the adaptivity parameter θ varies between 0.4 and 0.8. Again, we observe that each adaptive strategy leads to optimal convergence rates, whereas the convergence rate for uniform refinement is suboptimal.

In Figure 9, the adaptively generated meshes after 6 and 11 iterations are visualized. As before, refinement is basically restricted to the inactive zone.

8. Concluding Remarks

In this paper, we introduced an adaptive, lowest-order FE scheme for an elliptic 2D obstacle problem with inhomogeneous Dirichlet boundary conditions, which is proven to converge in the sense of $\lim_\ell U_\ell = u$ in $H^1(\Omega)$, where U_ℓ denotes the Galerkin approximation of the exact solution u in the ℓ-th step of the adaptive algorithm. Our analysis extends recent results of [9, 32] for affine obstacle problems with homogeneous Dirichlet data $u|_\Gamma = g = 0$. Although this extension might look only minor, it came as a surprise to us that the convergence analysis is much more involved due to the non-conformity $\mathcal{A}_\ell \not\subseteq \mathcal{A}_{\ell+1}$ of the discrete ansatz sets. In particular, we did not succeed to prove a strict contraction property

$$\Delta_{\ell+1} \leq \kappa\,\Delta_\ell$$

for the quasi-error quantity as in [32], but only a contraction

$$\Delta_{\ell+1} \leq \kappa\,\Delta_\ell + \alpha_\ell$$

up to a zero sequence $\alpha_\ell \geq 0$. We stress, however, that in case of homogeneous Dirichlet data $g = 0$ our algorithm and result coincides with that of [32], where $\alpha_\ell = 0$.

Analytically, one could transform the obstacle problem with non-homogeneous Dirichlet data into an obstacle problem with homogeneous Dirichlet data. Numerically one difficulty for this approach is that a particular extension $\mathcal{L}g \in H^1(\Omega)$ has to be provided. Moreover, this would lead to a non-affine obstacle $\widetilde{\chi} = \chi - \mathcal{L}g$, see Lemma 3, and the convergence of obstacle problems with general non-affine obstacles is widely open, see e.g. [10], where the obstacle problem reads: *Find* $u \in \widetilde{\mathcal{A}} = \{v \in H^1_0(\Omega) \,:\, v \geq \chi \text{ a.e. in } \Omega\}$ *with*

$$\mathcal{J}(u) = \min_{v\in\widetilde{\mathcal{A}}} \mathcal{J}(v).$$

However, the case of non-affine obstacles $\chi \in H^2(\Omega)$, even with inhomogeneous Dirichlet data, is implicitly treated by this work, since it can be reduced to an affine obstacle problem with inhomogeneous boundary conditions. This can be seen as follows: For $\chi \in H^2(\Omega)$, the set of admissible functions reads

$$\begin{aligned}\mathcal{A}_1 :&= \{v \in H^1(\Omega) \,:\, v \geq \chi \text{ a.e. in } \Omega, v|_\Gamma = g\}\\ &= \{v \in H^1(\Omega) \,:\, v \geq 0 - (-\chi) \text{ a.e. in } \Omega, v|_\Gamma = g\}.\end{aligned}$$

The obstacle problem then reads: *Find* $u \in \mathcal{A}_1$ *with* $\mathcal{J}(u) = \min_{v\in\mathcal{A}_1} \mathcal{J}(v)$, which is equivalent to the variational inequality

$$\langle\!\langle u\,,\,v-u\rangle\!\rangle \geq (f, v-u),$$

for all $v \in \mathcal{A}_1$. It is easily seen that this is again equivalent to the variational formulation

$$\langle\!\langle \widetilde{u}\,,\,\widetilde{v}-\widetilde{u}\rangle\!\rangle \geq (f, \widetilde{v}-\widetilde{u}) - \langle\!\langle \chi\,,\,\widetilde{v}-\widetilde{u}\rangle\!\rangle = (f+\Delta\chi, \widetilde{v}-\widetilde{u}),$$

for all $\widetilde{v} \in \mathcal{A} := \{v \in H^1(\Omega) \,:\, v \geq 0 \text{ a.e. in } \Omega, v|_\Gamma = g - \chi|_\Gamma\}$ with $\widetilde{u} = u - \chi$. Here, we have used integration by parts in the second inequality. Note that the boundary terms disappear due to $\widetilde{v} - \widetilde{u} \in H^1_0(\Omega)$. In particular, this variational form can be treated by means of this work.

For our analysis, we used nodal interpolation g_ℓ of the Dirichlet data g and an oscillation-type quantity to control the influence of the approximation and the resolution of the Dirichlet data. Although observed experimentally, we did not succeed to prove Lipschitz continuous dependence of u on the Dirichlet data g and thus reliability $\|u - U_\ell\|_{H^1(\Omega)} \lesssim \varrho_\ell$ of the proposed error estimator. Instead, Theorem 6 only predicts weak reliability in the sense that $\lim_\ell \varrho_\ell = 0$ implies $\lim_\ell U_\ell = u$.

We stress that our approach applies also to the linear case with $\mathcal{A} = H^1(\Omega)$ and $\mathcal{A}_\ell = \mathcal{S}^1(\mathcal{T}_\ell)$ respectively, i.e. we solve $-\Delta u = f$ subject to $u|_\Gamma = g$ by an adaptive P1-finite element method. Then, the proposed adaptive algorithm even leads to quasi-optimal convergence results in the sense of [35, 15], see [19].

In 3D, $H^1(\Gamma)$ is not embedded into $C(\Gamma)$, and nodal interpolation must thus not be used. Instead, one could use the $L^2(\Gamma)$-projection to discretize g. Since the latter is $H^1(\Gamma)$-stable if $\mathcal{T}_\ell$ is refined by newest vertex bisection, one can prove that a priori convergence of g_ℓ (Lemma 12) and the oscillation estimate $\|g - g_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}\nabla(g - g_\ell)\|_{L^2(\Gamma)}$ still hold true, cf. [23]. Since we loose the orthogonality relation (65) in the proof of Theorem 8, we now have to choose $\lambda = 0$ and extend α_ℓ by the additional term

$$\|h_\ell^{1/2}\nabla(g_{\ell+1} - g_\ell)\|_{L^2(\Gamma)} \simeq \|g_{\ell+1} - g_\ell\|_{H^{1/2}(\Gamma)}.$$

Then, Theorem 8 holds accordingly for the 3D case or even in 2D if the $L^2(\Gamma)$-projection instead of nodal interpolation is used to discretize g.

References

[1] M. Aurada, M. Feischl, D. Praetorius: *Convergence of some adaptive FEM-BEM coupling*, ASC Report **06/2010**, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien 2010.

[2] M. Aurada, S. Ferraz-Leite, D. Praetorius: *Estimator reduction and convergence of adaptive FEM and BEM*, ASC Report **27/2009**, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien 2009.

[3] M. Aurada, P. Goldenits, D. Praetorius: *Convergence of data perturbed adaptive boundary element methods*, ASC Report **40/2009**, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien 2009.

[4] M. Ainsworth, T. Oden: *A posteriori error estimation in finite element analysis*, Wiley-Interscience, New-York, 2000.

[5] M. Ainsworth, C. Lee, T. Oden: *Local a posterori error estimators for variational inequalities*, Numer. Meth. Part. D. E. **9** (1993), 23–33.

[6] D. Braess: *Finite Elemente*, Springer, Berlin-Heidelberg ^{4}2007.

[7] D. Braess: *A posteriori error estimators for obstacle problems — another look*, Numer. Math. **101** (2005), 415–421.

[8] S. Bartels, C. Carstensen: *Averaging techniques yield reliable a posteriori finite element error control for obstacle problems*, Numer. Math. **99** (2004), 225–249.
[9] D. Braess, C. Carstensen, R. Hoppe: *Convergence analysis of a conforming adaptive finite element method for an obstacle problem*, Numer. Math. **107** (2007), 455–471.
[10] D. Braess, C. Carstensen, R. Hoppe: *Error reduction in adaptive finite element approximations of elliptic obstacle problems*, J. Comput. Math. **27** (2009), 148–169.
[11] R. Becker, S. Mao: *Convergence and quasi-optimal complexity of a simple adaptive finite element method*, ESAIM, M2AN **43** (2009), 1203–1219.
[12] C. Carstensen, R. Hoppe: *Error reduction and convergence for an adaptive mixed finite element method*, Math. Comp. **75** (2006), 1033–1042.
[13] C. Carstensen, R. Hoppe: *Convergence analysis of an adaptive nonconforming finite element method*, Numer. Math. **103** (2006), 251–266.
[14] C. Carstensen, D. Praetorius: *Averaging techniques for the a posteriori BEM error control for a hypersingular integral equation in two dimensions*, SIAM J. Sci. Comput. **29** (2007), 782–810.
[15] J. Cascon, C. Kreuzer, R. Nochetto, K, Siebert: *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal. **46** (2008), 2524–2550.
[16] Z. Chen, R. Nochetto: *Residual type a posteriori error estimates for elliptic obstacle problems*, Numer. Math. **84** (2000), 527–548.
[17] V. Dominguez, F. Sayas: *Stability of discrete liftings*, C. R. Acad. Sci. Paris, Ser. I **337** (2003), 805–808.
[18] W. Dörfler: *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. **33** (1996), 1106–1124.
[19] M. Feischl, M. Page, D. Praetorius: *Convergence and quasi-optimality of adaptive FEM for the Poisson problem with inhomogeneous Dirichlet data*, work in progress.
[20] A. Friedmann: *Variational principles and free-boundary problems*, Wiley, New York 1982.
[21] R. Glowinski: *Numerical methods for nonlinear variational problems*, Springer, Berlin-Heidelberg 2008.
[22] K. Ito, K. Kunisch: *Lagrange multiplier approach to variational problems and applications*, SIAM, Advances in Design and Control, Philadelphia 2008.
[23] M. Karkulik, D. Praetorius: *Convergence of adaptive 3D-BEM for weakly singular integral equations based on isotropic mesh refinement*, work in progress.
[24] D. Kinderlehrer, G. Stampacchia: *An introduction to variational inequalities*, Academic Press, New York, 1980.
[25] W. Liu, N. Yan: *A posteriori error estimates for a class of variational inequalities*, J. Sci. Comput. **15** (2000), 361–393.
[26] W. McLean: *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge 2000.
[27] U. Mosco: *Convergence of convex sets and of solutions of variational inequalities*, Advances in Math. **3** (1969), 510–585.
[28] P. Morin, R. Nochetto, K. Siebert: *Data oscillation and convergence of adaptive FEM*, SIAM. J. Numer. Anal. **18** (2000), 466–488.
[29] P. Morin, K. Siebert, A. Veeser: *A basic convergence result for conforming adaptive finite elements*, Math. Models Methods Appl. Sci. **18** (2008), 707–737.
[30] R. Nochetto, K. Siebert, A. Veeser: *Fully localized a posteriori error estimators and barrier sets for contact problems*, SIAM J. Numer. Anal. **42** (2005), 2118–2135.
[31] M. Page: *Schätzerreduktion und Konvergenz adaptiver FEM für Hindernisprobleme*, Master thesis (in German), Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien 2010.
[32] M. Page, D. Praetorius: *Convergence of adaptive FEM for some elliptic obstacle problem*, ASC Report **05/2010**, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien 2010.
[33] J. Rodrigues: *Obstacle problems in mathematical physics*, Elsevier Science Publishers B.V., Amsterdam 1987.

[34] L. R. Scott, S. Zhang: *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), 483–493.

[35] R. Stevenson: *Optimality of standard adaptive finite element method*, Found. Comput. Math. (2007), 245–269.

[36] R. Verfürth: *A review of a posteriori error estimation and adaptive mesh-refinement techniques*, Wiley-Teubner, New York 1996.

[37] A. Veeser: *Efficient and reliable a posteriori error estimators for elliptic obstacle problems*, SIAM J. Numer. Anal. **39** (2001), 146–167.

[38] A. Veeser: *Convergent adaptive finite elements for the nonlinear Laplacian*, Numer. Math. **92** (2002), 743–770.

Acknowledgement. The authors M.F. and D.P. are partially funded by the Austria Science Fund (FWF) under grant P21732 *Adaptive Boundary Element Methode*, which is thankfully acknowledged. M.P. acknowledges partial support through the project *Micromagnetic Simulations and Computational Design of Future Devices*, funded by the Viennese Science and Technology Fund (WWTF) under grant MA09-029.

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