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ENTROPIES FOR RADIALLY SYMMETRIC HIGHER-ORDER NONLINEAR DIFFUSION EQUATIONS

MARIO BUKAL, ANSGAR JÜNGEL, AND DANIEL MATTHES

Abstract. A previously developed algebraic approach to proving entropy production inequalities is extended to deal with radially symmetric solutions for a class of higher-order diffusion equations in multiple space dimensions. In application of the method, novel a priori estimates are derived for the thin-film equation, the fourth-order Derrida-Lebowitz-Speer-Spohn equation, and a sixth-order quantum diffusion equation.

1. Introduction

In the last two decades, there has been a growing interest in the mathematical analysis of fourth and higher-order nonlinear diffusion equations. Such equations arise, for instance, in lubrication theory and as models for the electron transport in semi-conductors; below, we will briefly review several specific examples and their origins in physics. Rigorous results about the existence of solutions and their qualitative behavior are typically much harder to obtain than in the context of the well-studied second-order diffusion equations. One of the principal difficulties is the non-applicability of comparison principles. To substitute for this loss, one has to rely on suitable a priori estimates.

In [9], the last two authors have proposed a systematic approach to the derivation of a priori estimates for certain classes of nonlinear evolution equations of even order. This procedure allows one to determine Lyapunov functionals, which we call *entropies* in the following, and to derive integral bounds from their dissipation, called *entropy production inequalities*. The developed method has been successfully applied to several equations in one space dimension. The main idea is to translate the procedure of integration by parts — which is the core element in most derivations of a priori estimates — into an algebraic problem about the positivity of polynomials. Roughly speaking, to each evolution equation, a polynomial in the spatial derivatives of the solution is associated, and integration by parts allows one to modify the coefficients of this polynomial. If a suitable change of coefficients can be found that makes the resulting polynomial nonnegative, then this corresponds (formally) to a proof of an a priori estimate on the solutions. The key point is that such

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polynomial decision problems are well-known in real algebraic geometry, and there exist powerful methods to solve them.

The approach of [9] can, in principle, be generalized in a straightforward way to multidimensional higher-order equations by taking all partial derivatives as polynomial variables. However, this leads, even in simple situations, to huge polynomial expressions, and the corresponding algebraic problem is too complex to be solved directly, even with the aid of computer algebra systems. The method has been successfully adapted to deal with certain multidimensional equations of second order [13, 17] and fourth order [10, 18], but the systematic extension of the scheme to the general multidimensional case is still under development. In this paper, we propose a further adaption that works generally for *radially symmetric* solutions to homogeneous higher-order nonlinear equations. Furthermore, we prove its practicability by applying our scheme to the equations listed below.

Before describing our main results, we briefly review the example equations. The first is the fourth-order thin-film equation

$$\partial_t U + \operatorname{div}(U^\beta \nabla\Delta U) = 0, \tag{1}$$

which models the flow of a thin liquid along a solid surface with film height $U(t;x) \geq 0$ (for $\beta = 2$ or $\beta = 3$) or the thin neck of a Hele-Shaw flow in the lubrication approximation (for $\beta = 1$). For details, we refer to the reviews [2, 19]. The one-dimensional family of equations has been first analyzed by Bernis and Friedman [1]; for the multidimensional case, we refer to the work of Dal Passo et al. [5] and references therein.

The other examples we are dealing with arise as approximations of the quantum diffusion model by Degond et al. In [6], an equation for the dynamics of the electron density in a plasma has been derived. Although essentially non-local in its nature, the partial pseudo-differential equation can be developed asymptotically in terms of the reduced Planck constant $\hbar$, and this provides a family of approximative (genuine) partial differential equations.

The equation for the electron density $U(t;x) \geq 0$ obtained at order $\hbar^2$ is (after neglecting electric fields)

$$\partial_t U + \operatorname{div}\left(U\nabla\Big(\frac{\Delta\sqrt{U}}{\sqrt{U}}\Big)\right) = 0. \tag{2}$$

Interestingly, this equation – in one space dimension – also describes the fluctuations of the interface between the regions of positive and negative particle spins in the Toom model. It has been derived by Derrida et al. in [7]; we shall therefore refer to (2) as Derrida-Lebowitz-Speer-Spohn (DLSS) equation in the following. It has been first analyzed in [4] for local positive smooth solutions and then in [12] for global nonnegative weak solutions. The existence of weak solutions to the multidimensional equation has been proven recently in [8, 10].

When the non-local quantum diffusion model is expanded to order $\hbar^4$, the main part of the differential operator is of sixth order, and the corresponding equation reads as

$$\partial_t U - \operatorname{div}\left(U\nabla\Big(\frac{1}{2}\|\nabla^2\log U\|^2 + \frac{1}{U}\nabla^2 : (U\nabla^2\log U)\Big)\right) = 0, \tag{3}$$

where $\nabla^2 \log U$ denotes the Hessian of $\log U$, $\|\cdot\|$ is the Euclidean matrix norm, and the double point ":" means summation over both matrix indices. The one-dimensional version of (3) has been derived in [9]; see the appendix for the derivation in the general case. The one-dimensional equations with periodic boundary conditions have been analyzed in [11].

The objective of this paper is to prove, for radially symmetric smooth positive solutions $U(t)$ to (1), (2), or (3) satisfying no-flux and Neumann-type boundary conditions (see below for the precise conditions), estimates of the type

$$\frac{dE_\alpha}{dt}[U(t)] + cQ_\alpha[U(t)] \leq 0, \tag{4}$$

on a specific range of parameters α, where

$$\begin{aligned} E_\alpha[U] &= \frac{1}{\alpha(\alpha-1)}\int_\Omega U^\alpha \mathrm{d}x, \quad \alpha \neq 0, 1, \\ E_0[U] &= \int_\Omega \big(U - \log U\big)\mathrm{d}x, \\ E_1[U] &= \int_\Omega \Big(U\big(\log U - 1\big) + 1\Big)\mathrm{d}x. \end{aligned} \tag{5}$$

Above, $\Omega = B^d = \{|x| < 1\}$ is the unit ball in $\mathbb{R}^d$, $c \geq 0$ is a constant independent of the solution U, and Q_α is a nonnegative functional containing higher-order derivatives of U. We call E_α an *entropy* if (4) holds with some suitable choice of Q_α and $c \geq 0$ for arbitrary solutions $U(t)$ of the evolution equation under consideration. The estimate (4) is referred to as an *entropy production inequality*, and Q_α is the corresponding *entropy production*. Inequalities like (4) provide a priori bounds for the evolution; they are a necessary first step in proofs for existence of solutions; and they allow to describe the equilibration behavior of the solutions.

Entropy production inequalities for the evolution equations reviewed above have been extensively studied in the literature. Concerning the thin-film equation, with no-flux and homogeneous Neumann bondary conditions, it has been shown in [3, 5] that E_α is an entropy if $3/2 \leq \alpha+\beta \leq 3$. The same result holds for periodic boundary conditions [9]. This bound turns out to be sharp, at least in the one-dimensional case [15]. Moreover, the entropy production Q_α in (4) can be made explicit: a valid choice is $Q_\alpha[U] = \int_\Omega |(U^{(\alpha+\beta)/2})_{xx}|^2 \mathrm{d}x$ with a suitable $c > 0$ if $3/2 < \alpha + \beta < 3$, see [9].

Let U be a smooth solution to the DLSS equation (2) with periodic boundary conditions. Then (4) holds with

$$c = \frac{2p(\alpha)}{\alpha^2(p(\alpha) - p(0))}, \tag{6}$$

where $p(\alpha) = -\alpha^2 + 2\alpha(d+1)/(d+2) - (d-1)^2/(d+2)^2$, and $Q_\alpha[U] = \int_\Omega (\Delta U^{\alpha/2})^2 \mathrm{d}x$ for all $0 < \alpha < 2(d+1)/(d+2)$ [10]. In the one-dimensional case, this estimate holds true for a larger range of values for α, with $c = 2/\alpha^2$ for $0 < \alpha < 4/3$ and $c = 8(3-2\alpha)/\alpha^3$ for $4/3 < \alpha < 3/2$.

Entropy estimates for the sixth-order quantum diffusion model (3) with periodic boundary conditions are available only in one space dimension. In fact, it has been shown in [11] that E_1 is an entropy and (4) holds for some $c > 0$ and with $Q_1[U] = \int_\Omega ((\sqrt{U})^2_{xxx} + (\sqrt[6]{U})^6_x)\mathrm{d}x$.

To our knowledge, no entropy production inequalities (4) are available for the DLSS equation with no-flux and Neumann boundary conditions and for the sixth-order equation with $\alpha \neq 1$. In this paper, we will prove such results for radially symmetric solutions.

The advantage of considering radially symmetric solutions $U(t;x) = u(t;|x|)$ — in comparison to solutions of the full multidimensional problem — is that the reduced function $u(t;r)$ satisfies an evolution equation with only one spatial variable $r > 0$. Still, the proof of entropy production inequalities (4) is substantially more difficult than in the genuinely one-dimensional situation considered before [9]. The reason is that the variable r appears explicitly in the evolution equation. On the algebraic level, this adds one polynomial variable.

In the following we summarize our main results. Below, $\Omega = B^d \subset \mathbb{R}^d$ denotes the d-dimensional unit ball, and ν is the exterior unit normal vector to $\partial\Omega$.

Theorem 1 (Thin-film equation)**.** *Let U be a radially symmetric smooth and positive solution to the thin-film equation with homogeneous Neumann and no-flux boundary conditions:*

$$\partial_t U + \mathrm{div}(U^\beta \nabla\Delta U) = 0 \quad \textit{in } \Omega, \textit{ for } t > 0,$$
$$\nabla U \cdot \nu = U^\beta \nabla\Delta U \cdot \nu = 0 \quad \textit{on } \partial\Omega, \textit{ for } t > 0.$$

Then the functionals E_α, defined in (5)*, are entropies provided that $3/2 \leq \alpha + \beta \leq 3$. In this case, the entropy production inequality* (4) *holds with*

$$c = \frac{16}{(\alpha+\beta)^4}(3-\alpha-\beta)(2(\alpha+\beta)-3) \quad \textit{and} \quad Q_\alpha[U] = \int_\Omega \big(\Delta U^{(\alpha+\beta)/2}\big)^2 \mathrm{d}x.$$

The fact that E_α is a Lyapunov functional for $3/2 \leq \alpha \leq 3$ is well known. The production estimate with the explicit constant c is new. The dependence of c on $\alpha + \beta$ is illustrated in Figure 1.

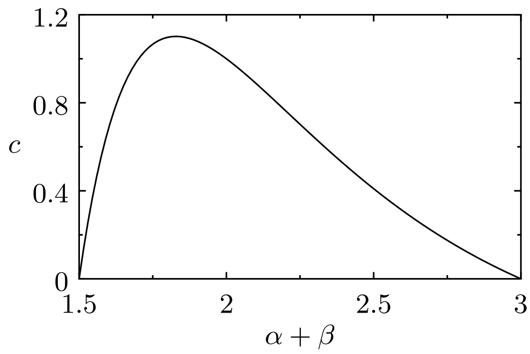

FIGURE 1. Thin-film equation: Values of c as a function of $\alpha + \beta$.

Theorem 2 (DLSS equation)**.** *Let U be a radially symmetric smooth positive solution to the DLSS equation with homogeneous Neumann and no-flux boundary conditions:*

$$\partial_t U + \operatorname{div}\left(U\nabla\Big(\frac{\Delta\sqrt{U}}{\sqrt{U}}\Big)\right) = 0 \quad \text{in } \Omega,\ \text{for } t>0,$$

$$\nabla U\cdot\nu = U\nabla\Big(\frac{\Delta\sqrt{U}}{\sqrt{U}}\Big)\cdot\nu = 0 \quad \text{on } \partial\Omega,\ \text{for } t>0.$$

Then the functionals E_α, defined in (5)*, are entropies if*

$$\begin{aligned} d=1,2,3,\ \text{or } 4,\ \text{and}\quad & \frac{(\sqrt{d}-1)^2}{d+2}\le\alpha\le\frac{3}{2},\\ d=5,6,\ \text{or } 7,\ \text{and}\quad & \frac{(\sqrt{d}-1)^2}{d+2}\le\alpha\le\frac{(\sqrt{d}+1)^2}{d+2},\\ d\ge 8\ \text{and}\quad & \frac{d-4}{2(d-2)}\le\alpha\le\frac{(\sqrt{d}+1)^2}{d+2}, \end{aligned}$$

and the entropy production inequality (4) *holds with $Q_\alpha[U]=\int_\Omega(\Delta U^{\alpha/2})^2\mathrm{d}x$ and*

$$d=1,2,3:\quad c=\begin{cases}\dfrac{2p(\alpha)}{\alpha^2(p(\alpha)-p(0))} & \text{for } \dfrac{(\sqrt{d}-1)^2}{d+2}<\alpha\le\dfrac{5d+7}{3d+6},\\ \dfrac{8(3-2\alpha)}{\alpha^3} & \text{for } \dfrac{5d+7}{3d+6}<\alpha<\dfrac{3}{2},\end{cases}$$

$$d=4,5,6,7:\quad c=\frac{2p(\alpha)}{\alpha^2(p(\alpha)-p(0))} \qquad \text{for } \frac{(\sqrt{d}-1)^2}{d+2}<\alpha<\frac{(\sqrt{d}+1)^2}{d+2},$$

$$d\ge 8:\quad c=\begin{cases}\dfrac{16(d-2)\alpha-8(d-4)}{d^2\alpha^3} & \text{for } \dfrac{d-4}{2(d-2)}<\alpha\le\dfrac{d^2-5d-8}{d^2-2d-8},\\ \dfrac{2p(\alpha)}{\alpha^2(p(\alpha)-p(0))} & \text{for } \dfrac{d^2-5d-8}{d^2-2d-8}<\alpha<\dfrac{(\sqrt{d}+1)^2}{d+2},\end{cases}$$

where $p(\alpha)=-\alpha^2+2\alpha(d+1)/(d+2)-(d-1)^2/(d+2)^2$.

The dependence of c on α is illustrated in Figure 2 for various dimensions d. The values for c for $d=4,5,6,7$ are the same as those derived in [10]. We are able to improve the results from [10] in the radially symmetric case for space dimensions $d=2,3$ and $d\ge 8$, see Figure 3. Moreover, the range of parameters α leading to entropies is larger than in [10].

It is known from [9] that the bounds $0\le\alpha\le 3/2$ are optimal if $d=1$. We prove in Section 5 that in the two-dimensional case, no entropies exist for $\alpha\le 0$. The lower bound $\alpha=(d-4)/(2d-4)$ for $d\ge 8$ is optimal.

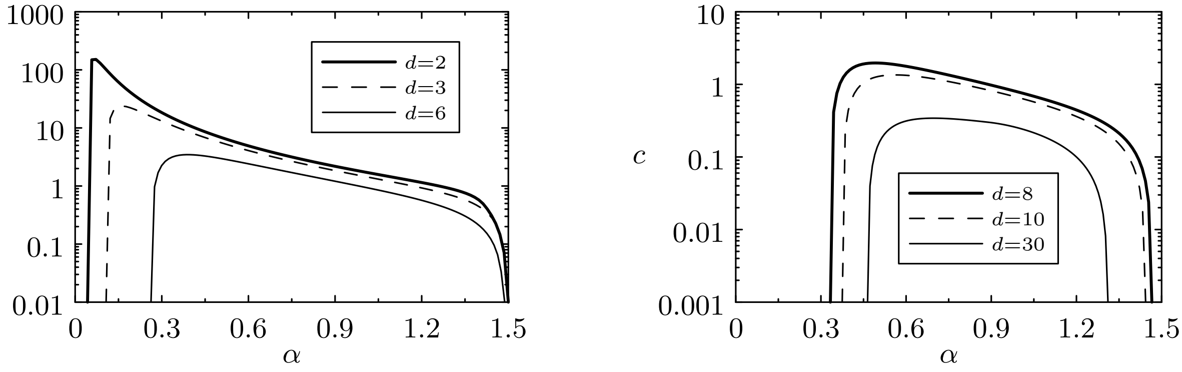

FIGURE 2. DLSS equation: Values of c as a function of d and α.

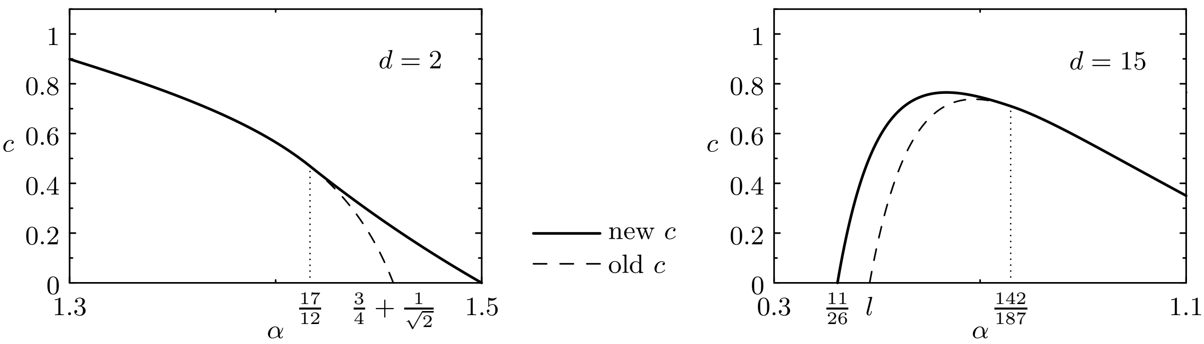

FIGURE 3. DLSS equation: Values of c as a function of α. The solid line represents the values from Theorem 2, the dashed line those from [10]. Here, $l = 2(8-\sqrt{15})/17$.

Theorem 3 (Sixth-order quantum diffusion equation)**.** *Let U be a radially symmetric smooth and positive solution to the sixth-order quantum diffusion equation:*

$$\partial_t U - \operatorname{div}\left(U\nabla\Big(\frac{1}{2}\|\nabla^2\log U\|^2 + \frac{1}{U}\nabla^2 : (U\nabla^2\log U)\Big)\right) = 0 \quad \text{in } \Omega, \text{ for } t>0,$$

$$\nabla U\cdot\nu = U\nabla\Big(\frac{\Delta\sqrt{U}}{\sqrt{U}}\Big)\cdot\nu = U\nabla\left(\frac{1}{2}\|\nabla^2\log U\|^2 + \frac{1}{U}\nabla^2 : (U\nabla^2\log U)\right)\cdot\nu = 0 \quad \text{on } \partial\Omega.$$

Then the functionals E_α, defined in (5)*, are entropies if*

$$\begin{aligned}
d = 1 \text{ and } &\quad 0.1927\ldots \le \alpha \le 1.1572\ldots,\\
d = 2 \text{ and } &\quad 0.2827\ldots \le \alpha \le 1.0982\ldots,\\
d = 3 \text{ and } &\quad 0.3470\ldots \le \alpha \le 1.0517\ldots,\\
d = 4 \text{ and } &\quad 0.3968\ldots \le \alpha \le 1.0123\ldots,\\
d = 5 \text{ and } &\quad 0.4380\ldots \le \alpha \le 0.9775\ldots
\end{aligned}$$

Moreover, in dimensions $d = 1, \dots, 4$ and for $\alpha = 1$, the entropy production inequality (4) *holds for some $c > 0$ if one chooses*

$$Q_1[U] = \int_\Omega U \Big| \nabla \Big(\frac{\Delta\sqrt{U}}{\sqrt{U}} \Big) \Big|^2 \mathrm{d}x. \tag{7}$$

The bounds for α are roots of certain polynomials and can be determined only numerically, see Figure 4. The Lyapunov property of E_α for $\alpha \neq 1$ and $d > 1$ as well as the entropy production inequality are new. Interestingly, it seems that the logarithmic functional E_1 is no longer a Lyapunov functional for the sixth-order equation in higher space dimensions. We remark that in dimension $d = 2$, the results from Section 5 show that there are no entropies if $\alpha > 4/3$.

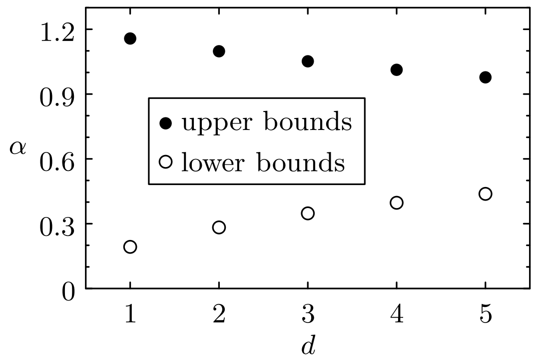

FIGURE 4. Sixth-order quantum diffusion equation: Upper and lower bounds for α depending on the dimension d.

The paper is organized as follows. The algebraic formalism is developed in Section 2. Section 3 is devoted to the proof of two auxiliary results about quadratic polynomials. The proofs for Theorems 1 to 3 are given in Section 4. In Section 5, a sufficient condition is provided under which E_α is *not* an entropy. Finally, the derivation of the quantum diffusion equation (3) from the Wigner-BGK model is outlined in the appendix.

2. Decision problem and shift polynomials

2.1. **Formulation as a decision problem.** To start with, we need to introduce some notation. First, observe that $U : \overline{\Omega} \to \mathbb{R}_+$ is a smooth and positive radially symmetric function if and only if there exists some

$$u \in \mathcal{U} := \big\{ u \in C^\infty([0,1]; \mathbb{R}_+) \;\big|\; \partial_r^m\big|_{r=0} u(r) = 0 \text{ for all odd } m \in \mathbb{N} \big\}$$

such that $U(x) = u(r)$ with $r = |x|$ for all $x \in \Omega = B^d$. We shall refer to u as the (radial) reduction of U, and to U as the (radially symmetric) extension of u.

Throughout this article, η and $\xi_1, \xi_2, \dots$ denote real variables. For $k \in \mathbb{N}$, let Σ_k be the linear span of all monomials $\eta^s \xi_1^{p_1} \cdots \xi_k^{p_k}$ satisfying $s + 1 \cdot p_1 + \cdots + k \cdot p_k = k$. Alternatively, one can define Σ_k as the set of polynomials P in $(\eta, \xi_1, \dots, \xi_k)$ with the homogeneity property

$$P(\lambda\eta, \lambda\xi_1, \lambda^2\xi_2, \dots, \lambda^k\xi_k) = \lambda^k P(\eta, \xi_1, \xi_2, \dots, \xi_k) \tag{8}$$

for all $\lambda \in \mathbb{R}$. To any $P \in \Sigma_k$, we associate a non-linear differential operator D_P of order k by

$$\mathrm{D}_P[u, r] = P\left(\frac{1}{r}, \frac{\partial_r u}{u}(r), \dots, \frac{\partial_r^k u}{u}(r)\right)$$

acting on functions $u \in \mathcal{U}$.

The key point behind this formalism is that the reductions $u(t; r)$ of radially symmetric solutions $U(t; x)$ to the evolution equations under consideration satisfy equations of the form

$$\partial_t u + r^{-(d-1)}\partial_r\big(r^{d-1}u^{\beta+1}\,\mathrm{D}_P[u, r]\big) = 0, \quad t > 0, \tag{9}$$

where $\beta \in \mathbb{R}$ is a parameter, $P \in \Sigma_{K-1}$ and K is order of the equation.

Example 4. Recall the representation of the gradient, divergence and Laplacian in radial coordinates: If $W(x) = w(r)$ is a radially symmetric function on $\Omega = B^d$, and $\mathbf{e}_r = x/r$ is the unit vector in radial direction, then

$$\nabla_x W(x) = w_r(r)\mathbf{e}_r, \quad \mathrm{div}_x(W(x)\mathbf{e}_r) = w_r(r) + \frac{d-1}{r}w(r) = r^{-(d-1)}\partial_r\big(r^{d-1}w(r)\big),$$

and, in combination,

$$\Delta_x W(x) = w_{rr}(r) + \frac{d-1}{r}w_r(r) =: \Delta_r w(r).$$

For our examples, this leads to the following:

- A radially symmetric solution $U(t; x) = u(t; r)$ to the thin-film equation (1) satisfies:

$$\partial_t U = \mathrm{div}_x(U^\beta \nabla_x \Delta_x U) = \mathrm{div}_x\left[u^{\beta+1}\left(\frac{u_{rrr}}{u} + \frac{d-1}{r}\Big(\frac{u_{rr}}{u} - \frac{u_r}{ru}\Big)\right)\mathbf{e}_r\right].$$

  This equation is of the form (9), with

$$P(\eta, \xi) = \xi_3 + (d-1)\big(\eta\xi_2 - \eta^2\xi_1\big).$$

- A radially symmetric solution to the DLSS equation (2) satisfies:

$$\begin{aligned}\partial_t U = \mathrm{div}_x\left[U\nabla_x\left(\frac{\Delta_x\sqrt{U}}{\sqrt{U}}\right)\right] &= \mathrm{div}_x\left[u\,\partial_r\left(\frac{(\sqrt{u})_{rr}}{\sqrt{u}} + \frac{d-1}{r}\frac{(\sqrt{u})_r}{\sqrt{u}}\right)\mathbf{e}_r\right] \\ &= \mathrm{div}_x\left[u\left(\frac{1}{2}\frac{u_{rrr}}{u} - \frac{u_{rr}}{u}\frac{u_r}{u} + \frac{1}{2}\frac{u_r^3}{u^3} + \frac{d-1}{2r}\left(\frac{u_{rr}}{u} - \frac{u_r^2}{u^2} - \frac{u_r}{ru}\right)\right)\mathbf{e}_r\right].\end{aligned}$$

  Also this equation is of the form (9), with $\beta = 0$ and

$$P(\eta, \xi) = \frac{1}{2}\big(\xi_3 - 2\xi_2\xi_1 + \xi_1^3 + (d-1)(\eta\xi_2 - \eta\xi_1^2 - \eta^2\xi_1)\big).$$

- If $U(t;x) = u(t;r)$ is a radially symmetric solution to the sixth order equation (3), then a tedious but straightforward computations show that $\partial_t U = \mathrm{div}_x(u\, G(u)\mathbf{e}_r)$, where

$$\begin{aligned} G(u) = & -6\frac{u_r^5}{u^5} + 18\frac{u_r^3}{u^3}\frac{u_{rr}}{u} - 11\frac{u_r}{u}\frac{u_{rr}^2}{u^2} - 8\frac{u_r^2}{u}\frac{u_{rrr}}{u} + 3\frac{u_r}{u}\frac{u_{rrrr}}{u} + 5\frac{u_{rr}}{u}\frac{u_{rrr}}{u} - \frac{u_{rrrrr}}{u} \\ & - (d-1)\frac{1}{r}\bigg(-6\frac{u_r^4}{u^4} + (2d-7)\frac{1}{r}\frac{u_r^3}{u^3} + 14\frac{u_r^2}{u^2}\frac{u_{rr}}{u} + (3d-8)\frac{1}{r^2}\frac{u_r^2}{u^2} - 4\frac{u_{rr}^2}{u^2} \\ & - 3(d-4)\frac{1}{r}\frac{u_r}{u}\frac{u_{rr}}{u} - 6\frac{u_r}{u}\frac{u_{rrr}}{u} + 3(d-3)\frac{1}{r^3}\frac{u_r}{u} - 3(d-3)\frac{1}{r^2}\frac{u_{rr}}{u} \\ & + (d-5)\frac{1}{r}\frac{u_{rrr}}{u} + 2\frac{u_{rrrr}}{u}\bigg). \end{aligned}$$

  In principle, one can easily deduce the correct choice of P from here.

Equation (9) is supplemented by initial conditions at $t = 0$,

$$u(0,r) = u_0(r). \tag{10}$$

For the fourth order equations ($K = 4$), homogeneous Neumann and no-flux boundary conditions are assumed,

$$u_r(r) = 0, \quad r^{d-1}\,\mathrm{D}_P[u,r] = 0 \qquad \text{at } r = 0 \text{ and } r = 1. \tag{11}$$

An additional boundary conditions will be specified for the sixth order equation (3), when $K = 6$.

Notice that the Neumann condition at $r = 0$ is already implied by $u \in \mathcal{U}$. On the other hand, the no-flux condition at $r = 0$ is in general *not* trivially satisfied since $\mathrm{D}_P[u,r]$ might contain terms with negative powers of r. More precisely, the condition is that

$$\lim_{r\downarrow 0}\big(r^{d-1}\,\mathrm{D}_P[u,r]\big) = 0.$$

In terms of the radially symmetric extension $U(t;x) = u(t;r)$, the homogeneous Neumann and no-flux boundary conditions (11) for an equation of the form $\partial_t U = \mathrm{div}_x(U^{\beta+1}G(U))$ correspond to

$$\nu\cdot\nabla U(t;x) = 0, \quad \nu\cdot U^{\beta+1}G(U) = 0 \qquad \text{for all } x\in\partial\Omega \text{ and } t>0,$$

with $\nu = \mathbf{e}_r$ denoting the outer normal vector at the boundary of the unit sphere.

For radially symmetric solutions $U(t;x) = u(t;r)$, the entropy functionals in (5) become

$$E_\alpha[U(t)] = \frac{\omega_d}{\alpha(\alpha-1)}\int_0^1 u(t;r)^\alpha r^{d-1}\mathrm{d}r,$$

where ω_d is the surface of the unit sphere in $\mathbb{R}^d$. For the time derivative along (9), one calculates

$$
\begin{aligned}
\frac{d}{dt}E_\alpha[U(t)] &= \frac{\omega_d}{\alpha-1}\int_0^1 u(t;r)^{\alpha-1}\partial_t u(t;r)r^{d-1}dr \\
&= -\omega_d\int_0^1 u^{\alpha+\beta}\left(-\frac{\partial_r u}{u}\right)P\left(\frac{1}{r},\frac{\partial_r u}{u},\ldots,\frac{\partial_r^{K-1}u}{u}\right)r^{d-1}\mathrm{d}r, \qquad (12)
\end{aligned}
$$

where the no-flux boundary conditions in (11) have been taken into account. The integrand in (12) is again of polynomial structure: defining $S_0\in\Sigma_K$ by

$$S_0(\eta,\xi) = -\xi_1 P(\eta,\xi_1,\ldots,\xi_{K-1}),$$

one can write

$$\frac{d}{dt}E_\alpha[U(t)] = -\omega_d I_0[u(t)] \quad \text{with} \quad I_0[u(t)] := \int_0^1 u(t;r)^{\alpha+\beta}\,\mathrm{D}_{S_0}[u(t),r]r^{d-1}\mathrm{d}r. \qquad (13)$$

Following [9], we call S_0 the *canonical symbol* that characterizes the dissipation of E_α by (9).

Recall that the primary goal is to identify — for a given equation of the form (9) — those entropies E_α which are monotone in time along all smooth radially symmetric solutions. Thus, we wish to determine values $\alpha\in\mathbb{R}$ such that the corresponding functional I_0 in (13) is nonnegative on $\mathcal{U}$. To prove nonnegativity, we apply integration by parts to the integral expression for I_0 in a systematic way that we explain now.

Let $\gamma\in\mathbb{R}$ and a polynomial $R\in\Sigma_{K-1}$ be given. Introduce the *divergence* $T=\delta_\gamma R$ as the unique element $T\in\Sigma_K$ which satisfies

$$\partial_r\big(r^{d-1}u(r)^\gamma\,\mathrm{D}_R[u,r]\big) = r^{d-1}u(r)^\gamma\,\mathrm{D}_T[u,r]$$

for all $u\in\mathcal{U}$. Formally, $\delta_\gamma:\Sigma_{K-1}\to\Sigma_K$ is a linear map that acts on monomials $R(\eta,\xi)=\eta^s\xi_1^{p_1}\cdots\xi_{K-1}^{p_{K-1}}$ as follows,

(14)
$$\delta_\gamma R(\eta,\xi) = \Big[(d-1-s)\eta + (\gamma-p_1-\cdots-p_{K-1})\xi_1 + p_1\frac{\xi_2}{\xi_1}+\cdots+p_{K-1}\frac{\xi_K}{\xi_{K-1}}\Big]R(\eta,\xi).$$

For $S=S_0+T$ with $T=\delta_\gamma R$, where $\gamma=\alpha+\beta$ and $R\in\Sigma_{K-1}$, it follows by the fundamental theorem of calculus that

$$
\begin{aligned}
I[u] := \int_0^1 u(r)^{\alpha+\beta}\,\mathrm{D}_S[u,r]r^{d-1}\mathrm{d}r &= \int_0^1 u(r)^{\alpha+\beta}\big(\mathrm{D}_{S_0}[u,r]+\mathrm{D}_T[u,r]\big)r^{d-1}\mathrm{d}r \\
&= I_0[u] + \Big[u(r)^{\alpha+\beta}\,\mathrm{D}_R[u,r]r^{d-1}\Big]_{r=0}^{r=1}.
\end{aligned}
$$

Assuming that u satisfies boundary conditions which imply in particular that

$$r^{d-1}\,\mathrm{D}_R[u,r]=0 \quad \text{at } r=1 \text{ and for } r\downarrow 0, \qquad (15)$$

then $I[u]=I_0[u]$, i.e., the replacement $S_0\mapsto S=S_0+T$ modifies the integrand but does *not* change the value of the integral. Hence, if there exists an $R\in\Sigma_{K-1}$ for which

$S = S_0 + \delta_{\alpha+\beta} R$ is a nonnegative polynomial, then it follows that $I_0[u] = I[u]$ is nonnegative for all $u \in \mathcal{U}$ that satisfy (15). Consequently, if the boundary conditions (11) for (9) imply (15), then $E_\alpha[U(t)]$ is monotone in time for all smooth radially symmetric solutions.

In practice, it is more convenient to work directly with the polynomials $T = \delta_\gamma R \in \Sigma_K$ rather than with their pre-images $R \in \Sigma_{K-1}$. Let R_1 to R_m be a collection of linearly independent polynomials in Σ_{K-1} for which (15) holds; we refer to Section 2.2 below for details on how to select appropriate R's. Denote by $T_1 = \delta_\gamma R_1$ to $T_m = \delta_\gamma R_m$ their respective divergences, which can be explicitly calculated using the rule (14) above. In analogy to [9], we call them *shift polynomials*. In conclusion of our discussion, the following is now obvious.

Lemma 5. *If the algebraic decision problem*

$$\exists c_1, \dots, c_m \in \mathbb{R} : \forall (\eta, \xi) \in \mathbb{R}^{K+1} : \big(S_0 + c_1 T_1 + \cdots c_m T_m\big)(\eta, \xi) \geq 0 \tag{16}$$

can be solved affirmatively, then E_α is a Lyapunov functional for (9)*.*

We remark that it suffices to prove (16) for all $\xi \in \mathbb{R}^K$ and *positive* $\eta \in \mathbb{R}$ only, since $\eta = 1/r > 0$. However, since both S_0 and the T_j satisfy the homogeneity property (8) with an *even* K, their values at (η, ξ) and $(-\eta, -\xi)$ agree; thus, (16) is true under the restriction $\eta > 0$ if and only if it is true without this restriction. We prefer to work directly with (16).

2.2. **Determination of the shift polynomials.** Let $u \in \mathcal{U}$ satisfy homogeneous Neumann boundary conditions,

$$u_r(0) = u_r(1) = 0.$$

We wish to find all polynomials $R \in \Sigma_{K-1}$ for which (15) holds. To this end, observe that

$$\big(r^{d-1}\,\mathrm{D}_R[u,r]\big)\big|_{r=1} = R\Big(1, 0, \frac{u_{rr}(1)}{u(1)}, \dots\Big),$$

and further that $R(1, 0, \xi_2, \xi_3, \dots) = 0$ for arbitrary ξ_2, $\xi_3 \dots$ if and only if R can be factored in the form $R(\eta, \xi_1, \xi_2, \dots) = \xi_1 Q(\eta, \xi_1, \xi_2, \dots)$ with some $Q \in \Sigma_{K-2}$. Among polynomials R of this type, it remains to single out those for which also

$$\lim_{r\downarrow 0} \big(r^{d-1}\,\mathrm{D}_R[u,r]\big) = 0. \tag{17}$$

The result depends sensitively on the order K of the operator.

First, we discuss the case $K = 4$ of the DLSS and the thin film equation. Then $\Sigma_{K-2} = \Sigma_2$ is spanned by ξ_2, ξ_1^2 and $\eta\xi_1$, so we need to investigate (17) for $R_1 = \xi_1\xi_2$, $R_2 = \xi_1^3$ and $R_3 = \eta\xi_1^2$, respectively. Since R_1 and R_2 are independent of η, both satisfy (17). Further,

$$\lim_{r\downarrow 0}\Big[r^{d-1} R_3\Big(\frac{1}{r}, \frac{u_r(r)}{u(r)}\Big)\Big] = \lim_{r\downarrow 0}\big(r^{d-2} u_r(r)^2\big) = 2\lim_{r\downarrow 0}\big(r^{d-1} u_r(r) u_{rr}(r)\big) = 0$$

by l'Hospital's rule, and since $u_r(0) = 0$ and $d \geq 1$. According to (14), the corresponding shift polynomials are

$$T_1(\eta,\xi) = \delta_{\alpha+\beta} R_1(\eta,\xi) = (\alpha+\beta-2)\xi_1^2\xi_2 + \xi_1\xi_3 + \xi_2^2 + (d-1)\eta\xi_1\xi_2, \tag{18}$$

$$T_2(\eta,\xi) = \delta_{\alpha+\beta} R_2(\eta,\xi) = (\alpha+\beta-3)\xi_1^4 + 3\xi_1^2\xi_2 + (d-1)\eta\xi_1^3, \tag{19}$$

$$T_3(\eta,\xi) = \delta_{\alpha+\beta} R_3(\eta,\xi) = (\alpha+\beta-2)\eta\xi_1^3 + (d-2)\eta^2\xi_1^2 + 2\eta\xi_1\xi_2. \tag{20}$$

We turn to the sixth-order equation (3). There are 12 monomials of the form $R_i(\xi,\eta) = \xi_1 Q(\xi,\eta)$ with $Q \in \Sigma_{K-2} = \Sigma_4$, listed in Table 1. We investigate the limits (17) correspond-

#	s	p_1	p_2	p_3	p_4	p_5
1	0	5	0	0	0	0
2	0	3	1	0	0	0
3	0	1	2	0	0	0
4	0	2	0	1	0	0
5	0	1	0	0	1	0
6	1	4	0	0	0	0
7	1	2	1	0	0	0
8	1	1	0	1	0	0
9	2	3	0	0	0	0
10	2	1	1	0	0	0
11	3	2	0	0	0	0
12	4	1	0	0	0	0

TABLE 1. Exponents of the monomials $\eta^s\xi_1^{p_1}\cdots\xi_5^{p_5}$ satisfying $s+p_1+2p_2+\cdots+5\cdot p_5 = 5$ and $p_1 \geq 1$.

ing to these R_i. For $R_8 = \eta\xi_1\xi_3$, $R_9 = \eta^2\xi_1^3$, $R_{10} = \eta^2\xi_1\xi_2$, $R_{11} = \eta^3\xi_1^2$, and $R_{12} = \eta^4\xi_1$, respectively, one obtains by l'Hospital's rule (using that $u_r(0) = u_{rrr}(0) = 0$ for all $u \in \mathcal{U}$) that

$$\lim_{r\downarrow 0}\left(r^{d-1}\,\mathrm{D}_{R_8}[u,r]\right) = \lim_{r\downarrow 0}\left(\frac{u_r(r)}{r}\frac{r^{d-1}u_{rrr}(r)}{u(r)^2}\right) = \frac{u_{rr}(0)u_{rrrr}(0)}{u(0)^2}\lim_{r\downarrow 0} r^d = 0,$$

$$\lim_{r\downarrow 0}\left(r^{d-1}\,\mathrm{D}_{R_9}[u,r]\right) = \lim_{r\downarrow 0}\left(\frac{u_r(r)^3}{r^3}\frac{r^d}{u(r)^3}\right) = \left(\frac{u_{rr}(0)}{u(0)}\right)^3\lim_{r\downarrow 0} r^d = 0,$$

$$\lim_{r\downarrow 0}\left(r^{d-1}\,\mathrm{D}_{R_{10}}[u,r]\right) = \lim_{r\downarrow 0}\left(\frac{u_r(r)}{r}\frac{r^{d-2}u_{rr}(r)}{u(r)^2}\right) = \left(\frac{u_{rr}(0)}{u(0)}\right)^2\lim_{r\downarrow 0} r^{d-2},$$

$$\lim_{r\downarrow 0}\left(r^{d-1}\,\mathrm{D}_{R_{11}}[u,r]\right) = \lim_{r\downarrow 0}\left(\frac{u_r(r)^2}{r^2}\frac{r^{d-2}}{u(r)^2}\right) = \left(\frac{u_{rr}(0)}{u(0)}\right)^2\lim_{r\downarrow 0} r^{d-2},$$

$$\lim_{r\downarrow 0}\left(r^{d-1}\,\mathrm{D}_{R_{12}}[u,r]\right) = \lim_{r\downarrow 0}\left(\frac{u_r(r)}{r}\frac{r^{d-4}}{u(r)}\right) = \frac{u_{rr}(0)}{u(0)}\lim_{r\downarrow 0} r^{d-4}.$$

The limits corresponding to R_{10}, R_{11} and R_{12} do not vanish in general in dimensions $d = 1$ or $d = 2$; we thus cannot use these mononomials directly for the derivation of shift polynomials; however, we shall employ a suitable linear combination of them below. Omitting the analogous calculation, we remark that (17) is also satisfied for $R_6 = \eta\xi_1^4$ and $R_7 = \eta\xi_1^2\xi_2$ in $d \geq 1$. For all the remaining monomials R_1 to R_5, property (17) holds trivially since these R_i are independent of η.

Since equation (3) is of sixth order, additional boundary conditions can be imposed. We choose

$$\nabla\left(\frac{\Delta\sqrt{U}}{\sqrt{U}}\right)\cdot\nu = 0 \quad \text{on } \partial\Omega.$$

In terms of the reduction u, this means that we assume

$$\left(\frac{u_{rrr}(r)}{u(r)} + (d-1)\frac{u_{rr}(r)}{ru(r)}\right) = 0 \quad \text{at } r = 1. \tag{21}$$

There are polynomials $R \in \Sigma_5$ for which $r^{d-1}\,\mathrm{D}_R[u,r]$ vanishes for $r \downarrow 0$ and at $r = 1$ because of (21), and *not* on grounds of the homogeneous Neumann conditions alone. In analogy to the case of Neumann boundary data, these polynomials can be written in the form $R(\eta,\xi) = (\xi_3 + (d-1)\xi_2\eta)Q(\eta,\xi)$ with an appropriate $Q \in \Sigma_2$. There is no need to consider $Q = \xi_1^2$, since then R contains ξ_1 as a factor, and this has already been investigated above. It is easily seen that the choice $R = (\xi_3 + (d-1)\eta\xi_2)\eta^2$ does not satisfy (15). On the other hand, $R_* = (\xi_3 + (d-1)\eta\xi_2)\xi_2$ gives

$$\begin{aligned}\lim_{r\downarrow 0}\left(r^{d-1}\,\mathrm{D}_{R_*}[u,r]\right) &= \lim_{r\downarrow 0}\frac{r^{d-1}u_{rrr}(r)}{u(r)} + \left(\frac{u_{rr}(0)}{u(0)}\right)^2\lim_{r\downarrow 0} r^{d-2} \\ &= \frac{u_{rrrr}(0)}{u(0)}\lim_{r\downarrow 0} r^d + \left(\frac{u_{rr}(0)}{u(0)}\right)^2\lim_{r\downarrow 0} r^{d-2}.\end{aligned}$$

While the first term vanishes in all dimensions, the second diverges for $d = 1$ or is finite but generally nonzero for $d = 2$. However, it can be annihilated by a suitable linear combination of R_{10} and R_{11}. Indeed, replacing R_{10} by

$$R'_{10}(\eta,\xi) := (d-1)\eta^2\xi_1\xi_2 - 2(d-1)\eta^3\xi_1^2 + (\xi_3 + (d-1)\eta\xi_2)\xi_2,$$

it is now easily verfied that R'_{10} has the property (15).

In summary, we shall use the following expressions for the definition of the shift polynomials:

$$\begin{aligned}&R_1 = \xi_1^5, \quad R_2 = \xi_1^3\xi_2, \quad R_3 = \xi_1\xi_2^2, \quad R_4 = \xi_1^2\xi_3, \quad R_5 = \xi_1\xi_4, \\ &R_6 = \eta\xi_1^4, \quad R_7 = \eta\xi_1^2\xi_2, \quad R_8 = \eta\xi_1\xi_3, \quad R_9 = \eta^2\xi_1^3, \\ &R'_{10} = \xi_2\xi_3 + (d-1)(\eta^2\xi_1\xi_2 - 2\eta^3\xi_1^2 + \eta\xi_2^2).\end{aligned}$$

The corresponding shift polynomials read as follows:

(22) $T_1(\eta,\xi) = (\alpha+\beta-5)\xi_1^6 + 5\xi_1^4\xi_2 + (d-1)\eta\xi_1^5,$

(23) $T_2(\eta,\xi) = (\alpha+\beta-4)\xi_1^4\xi_2 + 3\xi_1^2\xi_2^2 + \xi_1^3\xi_3 + (d-1)\eta\xi_1^3\xi_2,$

(24) $T_3(\eta,\xi) = (\alpha+\beta-3)\xi_1^2\xi_2^2 + \xi_2^3 + 2\xi_1\xi_2\xi_3 + (d-1)\eta\xi_1\xi_2^2,$

(25) $T_4(\eta,\xi) = (\alpha+\beta-3)\xi_1^3\xi_3 + 2\xi_1\xi_2\xi_3 + \xi_1^2\xi_4 + (d-1)\eta\xi_1^2\xi_3,$

(26) $T_5(\eta,\xi) = (\alpha+\beta-2)\xi_1^2\xi_4 + \xi_1\xi_5 + \xi_2\xi_4 + (d-1)\eta\xi_1\xi_4,$

(27) $T_6(\eta,\xi) = (\alpha+\beta-4)\eta\xi_1^5 + 4\eta\xi_1^3\xi_2 + (d-2)\eta^2\xi_1^4,$

(28) $T_7(\eta,\xi) = (\alpha+\beta-3)\eta\xi_1^3\xi_2 + 2\eta\xi_1\xi_2^2 + \eta\xi_1^2\xi_3 + (d-2)\eta^2\xi_1^2\xi_2,$

(29) $T_8(\eta,\xi) = (\alpha+\beta-2)\eta\xi_1^2\xi_3 + \eta\xi_2\xi_3 + \eta\xi_1\xi_4 + (d-2)\eta^2\xi_1\xi_3,$

(30) $T_9(\eta,\xi) = (\alpha+\beta-3)\eta^2\xi_1^4 + 3\eta^2\xi_1^2\xi_2 + (d-3)\eta^3\xi_1^3,$

(31)
$$\begin{aligned} T_{10}(\eta,\xi) = {} & \xi_2\xi_4 + \xi_3^2 + (\alpha+\beta-2)\xi_1\xi_2\xi_3 + 3(d-1)\eta\xi_2\xi_3 + (d-1)(\alpha+\beta-2)\eta\xi_1\xi_2^2 \\ & + (d-1)\eta^2\xi_1\xi_3 + (d-1)^2\eta^2\xi_2^2 + (d-1)(\alpha+\beta-2)\eta^2\xi_1^2\xi_2 \\ & + (d-1)(d-7)\eta^3\xi_1\xi_2 - 2(d-1)(\alpha+\beta-2)\eta^3\xi_1^3 - 2(d-1)(d-4)\eta^4\xi_1^2. \end{aligned}$$

Remark 6. The no-flux condition in (11) is identical to (15) with $R = P \in \Sigma_{K-1}$. Thus, this polynomial may define another shift polynomial. However, P contains the monomial ξ_{K-1} and its divergence contains the indefinite term ξ_K, which cannot be annihilated in the linear combination (16). As a consequence, we omit this shift polynomial in our computations.

3. Two auxiliary lemmas

In this section, we solve two easy quantifier elimination problems.

Lemma 7. *Let the real polynomial*

$$P(\eta,\xi_1,\xi_2) = a_1\xi_1^4 + a_2\xi_1^2\xi_2 + a_3\xi_2^2 + a_4\eta\xi_1^3 + a_5\eta^2\xi_1^2 + a_6\eta\xi_1\xi_2$$

be given. Then the quantified formula

$$\forall(\eta,\xi_1,\xi_2)\in\mathbb{R}^3:\ P(\eta,\xi_1,\xi_2)\ge 0 \tag{32}$$

is equivalent to the quantifier free formula

$$\begin{aligned} & \textit{either } a_3>0 \textit{ and} \\ & \quad \big[(4a_3a_5-a_6^2>0 \textit{ and } 4a_1a_3a_5-a_3a_4^2-a_2^2a_5-a_1a_6^2+a_2a_4a_6\ge 0) \textit{ or} \\ & \quad (4a_3a_5-a_6^2=2a_4a_3-a_2a_6=0 \textit{ and } 4a_3a_1-a_2^2\ge 0)\big] \\ & \textit{or } a_3=0 \textit{ and } a_2=a_6=0 \textit{ and} \\ & \quad \big[(a_5>0 \textit{ and } 4a_5a_1-a_4^2\ge 0) \textit{ or } (a_4=a_5=0 \textit{ and } a_1\ge 0)\big]. \end{aligned} \tag{33}$$

Proof. The polynomial P is nonnegative on the hyperplane $\xi_1 = 0$ if and only if $a_3 \geq 0$. For $\xi_1 \neq 0$, formula (32) is equivalent to the statement that the quadratic polynomial

$$p(x_1, x_2) = a_1 + a_2 x_2 + a_3 x_2^2 + a_4 x_1 + a_5 x_1^2 + a_6 x_1 x_2$$

is nonnegative for all real values $x_1 = \eta/\xi_1$ and $x_2 = \xi_2/\xi_1^2$. For fixed $x_1^* \in \mathbb{R}$, the quadratic polynomial in x_2,

$$p(x_1^*, x_2) = (a_1 + a_4 x_1^* + a_5 (x_1^*)^2) + (a_2 + a_6 x_1^*) x_2 + a_3 x_2^2,$$

is nonnegative if and only if

$$\begin{aligned} &\text{either } a_3 > 0 \text{ and } q_1(x_1^*) := 4a_3(a_1 + a_4 x_1^* + a_5(x_1^*)^2) - (a_2 + a_6 x_1^*)^2 \geq 0 \\ &\text{or } a_3 = 0 \text{ and } q_2(x_1^*) := a_2 + a_6 x_1^* = 0 \text{ and } q_3(x_1^*) := a_1 + a_4 x_1^* + a_5 (x_1^*)^2 \geq 0. \end{aligned} \tag{34}$$

Therefore, $p(x_1, x_2)$ is nonnegative if and only if $q_1(x_1) \geq 0$ or if $q_2(x_1) = 0$ and $q_3(x_1) \geq 0$ for all $x_1 \in \mathbb{R}$. The polynomial

$$q_1(x_1) = 4a_3 a_1 - a_2^2 + 2(2a_3 a_4 - a_2 a_6) x_1 + (4a_3 a_5 - a_6^2) x_1^2$$

is nonnegative if and only if

$$\begin{aligned} &\text{either } 4a_3a_5 - a_6^2 > 0 \text{ and } (4a_3a_5 - a_6^2)(4a_3a_1 - a_2^2) - (2a_3a_4 - a_2a_6)^2 \geq 0 \\ &\text{or } 4a_3a_5 - a_6^2 = 2a_4a_3 - a_2a_6 = 0 \text{ and } 4a_3a_1 - a_2^2 \geq 0. \end{aligned}$$

The polynomial q_2 vanishes on $\mathbb{R}$ if and only if $a_2 = a_6 = 0$, and $q_3(x_1) = a_1 + a_4 x_1 + a_5 x_1^2$ is nonnegative if and only if

$$\begin{aligned} &\text{either } a_5 > 0 \text{ and } 4a_5 a_1 - a_4^2 \geq 0 \\ &\text{or } a_4 = a_5 = 0 \text{ and } a_1 \geq 0. \end{aligned}$$

Inserting these statements into (34) yields (33). □

Lemma 8. *Let the real polynomial $P(x) = b_0 + b_1 x + b_2 x^2$ with $b_2 \geq 0$ and real numbers $z_1 < z_2$ be given. Then the quantified formula*

$$\exists x \in (z_1, z_2): \ P(x) \leq 0 \tag{35}$$

is equivalent to the quantifier free expression

$$\begin{aligned} &\textit{either } b_2 > 0 \textit{ and } \big[b_0 + b_1 z_1 + b_2 z_1^2 < 0 \textit{ or } (4b_0b_2 - b_1^2 \leq 0 \textit{ and } 2b_2 z_1 + b_1 < 0)\big] \\ &\quad \textit{and } \big[b_0 + b_1 z_2 + b_2 z_2^2 < 0 \textit{ or } (4b_0 b_2 - b_1^2 \leq 0 \textit{ and } 2b_2 z_2 + b_1 > 0)\big] \\ &\textit{or } b_2 = 0 \textit{ and } \big[(b_1 > 0 \textit{ and } b_0 + b_1 z_1 < 0) \textit{ or } (b_1 < 0 \textit{ and } b_0 + b_1 z_2 < 0) \\ &\quad \textit{or } (b_1 = 0 \textit{ and } b_0 \leq 0)\big]. \end{aligned} \tag{36}$$

Proof. First assume that $b_2 > 0$. Then the quadratic polynomial P is nonpositive in some interval if and only if $4b_0 b_2 - b_1^2 \leq 0$ and exactly for those x which lie in between the two real roots $x_\pm = (\pm\sqrt{b_1^2 - 4b_0 b_2} - b_1)/2b_2$. The statement (35) is then equivalent to $z_1 < x_+$ and $z_2 > x_-$, which can be rephrased as the first two lines of (36). Indeed, if $z_1 + b_1/2b_2 < 0$ then $z_1 < x_+$ is always satisfied, and if $z_1 + b_1/2b_2 \geq 0$ then $z_1 < x_+$ is equivalent to $b_0 + b_1 z_1 + b_2 z_1^2 < 0$. Notice that this inequality is satisfied only if $4b_0 b_2 - b_1^2 \leq 0$.

If $b_2 = 0$, then P is linear. If additionally $b_1 = 0$, (35) is equivalent to $b_0 \le 0$. Therefore, let $b_1 \ne 0$. Then P vanishes at $x_0 = -b_0/b_1$, and (35) is equivalent to $z_1 < x_0$ (if $b_1 > 0$) or $z_2 > x_0$ (if $b_1 < 0$). This leads to the last two lines of (36). □

4. Proofs of the theorems

4.1. **Proof of Theorem 1.** By Example 4 and (13), the canonical symbol of (1) reads as follows:

$$S_0(\eta,\xi) = -\xi_1\xi_3 - (d-1)\eta\xi_1\xi_2 + (d-1)\eta^2\xi_1^2.$$

We have to solve the decision problem

$$\exists c_1, c_2, c_3 \in \mathbb{R} : \forall(\eta,\xi_1,\xi_2,\xi_3) \in \mathbb{R}^4 : S(\eta,\xi) = (S_0 + c_1T_1 + c_2T_2 + c_3T_3)(\eta,\xi) \ge 0, \tag{37}$$

where the shift polynomials T_1, T_2, and T_3 are given by (18)-(20).

This problem can be simplified. Indeed, the variable ξ_3 appears in S only in the term $\xi_1\xi_3$, and its coefficient $-1 + c_1$ has to vanish; otherwise, $S(\eta,\xi)$ becomes negative for $\xi_1 \equiv 1$ and $\xi_3 \to +\infty$. Thus, $c_1 = 1$. Hence, we have to solve the decision problem: Find c_2, $c_3 \in \mathbb{R}$ such that for all $(\eta,\xi) = (\eta,\xi_1,\xi_2) \in \mathbb{R}^3$,

$$\begin{aligned} S(\eta,\xi) &= (S_0 + T_1 + c_2T_2 + c_3T_3)(\eta,\xi) \\ &= a_1\xi_1^4 + a_2\xi_1^2\xi_2 + a_3\xi_2^2 + a_4\eta\xi_1^3 + a_5\eta^2\xi_1^2 + a_6\eta\xi_1\xi_2 \ge 0, \end{aligned}$$

where, setting $\gamma = \alpha + \beta$,

$$\begin{aligned} &a_1 = (\gamma-3)c_2, && a_2 = \gamma - 2 + 3c_2, && a_3 = 1, \\ &a_4 = (\gamma-2)c_3 + (d-1)c_2, && a_5 = (d-2)c_3 + d - 1, && a_6 = 2c_3. \end{aligned}$$

By Lemma 7, this decision problem is equivalent to either

$$0 < 4a_3a_5 - a_6^2 = -4(c_3+1)(c_3-d+1) =: -4C, \tag{38}$$

$$\begin{aligned} 0 \le q(c_2,c_3) &:= 4a_1a_3a_5 - a_3a_4^2 - a_2^2a_5 - a_1a_6^2 + a_2a_4a_6 \\ &= \big(9C - (d-3c_3-1)^2\big)c_2^2 + 2C\gamma c_2 + (\gamma-2)^2C \end{aligned} \tag{39}$$

or

$$0 = 4a_3a_5 - a_6^2 = -4(c_3+1)(c_3-d+1), \tag{40}$$

$$0 = 2a_3a_4 - a_2a_6 = 2c_2(d-3c_3-1), \tag{41}$$

$$\begin{aligned} 0 \le 4a_1a_3 - a_2^2 &= 4(\gamma-3)c_2 - (3c_2+\gamma-2)^2 \\ &= -9\left(c_2 + \frac{\gamma}{9}\right)^2 + \frac{8}{9}(3-\gamma)\left(\gamma - \frac{3}{2}\right). \end{aligned} \tag{42}$$

First, we solve (40)-(42). Equation (41) yields $c_2 = 0$ or $c_3 = (d-1)/3$. Because of (40), the latter case is only possible if $d = 1$. Let $c_2 = 0$. Then (42) is fulfilled if and only if $\gamma = 2$. On the other hand, if $c_3 = (d-1)/3$ (and hence, $d = 1$), the largest range for γ fulfilling (42) is obtained by choosing the maximizing value $c_2 = -\gamma/9$. With this choice, (42) is fulfilled if and only if $3/2 \le \gamma \le 3$. This shows that (40)-(42) holds for some c_2, $c_3 \in \mathbb{R}$ if and only if $d = 1$ and $3/2 \le \gamma \le 3$ or if $d > 1$ and $\gamma = 2$.

Next, we solve (38)-(39). The first inequality implies that $-1 < c_3 < d-1$. For any fixed c_3, the polynomial $q(c_2, c_3)$ is quadratic in c_2 with a strictly negative leading coefficient (since $C < 0$ by (38)). Thus, there exists $c_2 \in \mathbb{R}$ such that $q(c_2, c_3) \le 0$ if and only if the discriminant of $q(\cdot, c_3)$ is nonnegative:

$$0 \le \big(2C\gamma\big)^2 - 4\big(9C - (d - 3c_3 - 1)^2\big)(\gamma - 2)^2 C = 4C\Delta(c_3),$$

where

$$\Delta(c_3) = \gamma^2 c_3^2 + 3(\gamma-2)^2(d - 4 - \gamma^2 d)c_3 + (\gamma-2)^2(d-1)(d+8) + \gamma^2 - \gamma^2 d.$$

Since $C < 0$, the discriminant is nonnegative if and only if the quadratic polynomial $\Delta(c_3)$ is nonpositive for some $-1 < c_3 < d-1$. By Lemma 8, this is the case if either $d = 1$ and $3/2 < \gamma < 3$ or $d > 1$ and $3/2 \le \gamma \le 3$. Thus, there exist $c_2 \in \mathbb{R}$, $c_3 \in (-1, d-1)$ such that (38)-(39) holds if and only if $3/2 \le \gamma \le 3$. This shows that E_α are entropies for all $3/2 \le \alpha + \beta \le 3$.

We wish to quantify the constant $c > 0$ in the entropy production inequality (4) for the choice

$$Q_\alpha[U] = \int_\Omega \big(\Delta U^{\gamma/2}\big)^2 \mathrm{d}x = \omega_d \int_0^1 u^\gamma \, \mathrm{D}_W[u, r] r^{d-1} \mathrm{d}r.$$

The symbol W that characterizes Q_α is

$$\begin{aligned} W(\eta, \xi) = {}& \left(\frac{\gamma}{2}\right)^2 \left(\frac{\gamma}{2} - 1\right)^2 \xi_1^4 + 2\left(\frac{\gamma}{2}\right)^2 \left(\frac{\gamma}{2} - 1\right) \xi_1^2 \xi_2 + \left(\frac{\gamma}{2}\right)^2 \xi_2^2 \\ & + 2(d-1)\left(\frac{\gamma}{2}\right)^2 \left(\frac{\gamma}{2} - 1\right) \eta \xi_1^3 + (d-1)^2 \left(\frac{\gamma}{2}\right)^2 \eta^2 \xi_1^2 + 2(d-1)\left(\frac{\gamma}{2}\right)^2 \eta \xi_1 \xi_2. \end{aligned}$$

We wish to find the largest $c > 0$ for which there exist c_2, $c_3 \in \mathbb{R}$ such that for all $(\eta, \xi) = (\eta, \xi_1, \xi_2) \in \mathbb{R}^3$ it holds

$$S_c(\eta, \xi) = (S - cW)(\eta, \xi) = a_1 \xi_1^4 + a_2 \xi_1^2 \xi_2 + a_3 \xi_2^2 + a_4 \eta \xi_1^3 + a_5 \eta^2 \xi_1^2 + a_6 \eta \xi_1 \xi_2 \ge 0,$$

where

$$\begin{aligned} a_1 &= (\gamma - 3)c_2 - c\left(\frac{\gamma}{2}\right)^2 \left(\frac{\gamma}{2} - 1\right)^2, \\ a_2 &= \gamma - 2 + 3c_2 - 2c\left(\frac{\gamma}{2}\right)^2 \left(\frac{\gamma}{2} - 1\right), \\ a_3 &= 1 - c\left(\frac{\gamma}{2}\right)^2, \\ a_4 &= (\gamma - 2)c_3 + (d-1)c_2 - 2c(d-1)\left(\frac{\gamma}{2}\right)^2 \left(\frac{\gamma}{2} - 1\right), \\ a_5 &= (d-2)c_3 + d - 1 - c(d-1)^2 \left(\frac{\gamma}{2}\right)^2, \\ a_6 &= 2c_3 - 2c(d-1)\left(\frac{\gamma}{2}\right)^2. \end{aligned}$$

We consider the cases $a_3 > 0$ and $a_3 = 0$ separately. First, let $a_3 = 0$, which is equivalent to $c = 4/\gamma^2$. By Lemma 7, we find that $a_2 = a_6 = 0$, which gives $c_2 = 0$ and $c_3 = d - 1$. Furthermore, we obtain $a_5 = 0$. Hence, by the same lemma, $a_4 = 0$ and $a_1 = -(\gamma/2-1)^2 \ge$

0, implying that $\gamma = 2$. Next, let $a_3 > 0$. By Lemma 7, the nonnegativity of S_c for certain values c, c_2, and c_3 is equivalent to either

$$0 < 4a_3a_5 - a_6^2 = -(c_3 - d + 1)(4c_3 - \gamma^2 dc + 4) =: -E, \tag{43}$$

$$\begin{aligned} 0 \le q(c_2, c_3, c) &:= 4a_1a_3a_5 - a_3a_4^2 - a_2^2a_5 - a_1a_6^2 + a_2a_4a_6 \\ &= \frac{1}{4 - \gamma^2 c}\big(9E - (2d - 2 - 6c_3 + \gamma^2(d-1))^2 c\big)c_2^2 + \frac{E}{2}\gamma c_2 + \frac{E}{4}(\gamma - 2)^2 \end{aligned} \tag{44}$$

or

$$0 = 4a_3a_5 - a_6^2 = -(c_3 - d + 1)(4c_3 - \gamma^2 cd + 4), \tag{45}$$

$$0 = 2a_3a_4 - a_2a_6 = c_2\big(2d - 2 - 6c_3 + \gamma^2 c(d-1)\big), \tag{46}$$

$$\begin{aligned} 0 \le 4a_1a_3 - a_2^2 &= -9c_2^2 + \frac{\gamma}{2}(\gamma^2 c - 4) + \frac{1}{2}(\gamma - 2)^2(\gamma^2 c - 4) \\ &= -9\left(c_2 - \frac{\gamma}{36}(\gamma^2 c - 4)\right)^2 + \frac{1}{144}(\gamma^2 c - 4)\big(\gamma^4 c + 32\gamma^2 + 144(1-\gamma)\big). \end{aligned} \tag{47}$$

First, we solve (45)-(47). We obtain a maximal value for c by choosing $c_2 = \gamma(\gamma^2 c - 4)/36$. Since $a_3 = 1 - \gamma^2 c/4 > 0$ by assumption, we have $c_2 < 0$. With this choice of c_2, condition (47) implies that $c \le 16(2\gamma - 3)(3 - \gamma)/\gamma^4$. Furthermore, by (46), $c_3 = (d-1)(\gamma^2 c + 2)/6$. Condition (45) can be satisfied only if $d = 1$.

Next, we consider (43)-(44). The polynomial $q(\cdot, c_3, c)$ is quadratic in c_2 with a negative leading coefficient (since $a_3 > 0$). Hence, there exists $c_2 \in \mathbb{R}$ such that $q(c_2, c_3, c)$ is nonnegative if and only if its discriminant $D(c_3, c) = E\Delta_0(c_3, c)/4$ is nonnegative, where $E < 0$ (by (43)) and

$$\begin{aligned} \Delta_0(c_3, c) &= 4\gamma^2 c_3^2 + \big(8\gamma^2 + 12(\gamma - 2)^2(d - 4) - 4\gamma^2 d - \gamma^4 cd\big)c_3 \\ &\quad + 4(\gamma - 2)^2(d-1)(d+8) - 4\gamma^2 d + 4\gamma^2 - 4\gamma^2 c(\gamma - 2)^2(d-1)^2 - \gamma^4 cd + \gamma^4 cd^2 \end{aligned}$$

is a quadratic polynomial in c_3. Applying Lemma 8, we find that

$$\begin{aligned} &\text{if } d = 1 \text{ and } \gamma \in \left(\frac{3}{2}, 3\right): && c < \frac{16}{\gamma^4}(2\gamma - 3)(3 - \gamma); \\ &\text{if } d > 1 \text{ and } \gamma \in \left(\frac{3}{2}, 3\right)\setminus\{2\}: && c \le \frac{16}{\gamma^4}(2\gamma - 3)(3 - \gamma). \end{aligned}$$

The case $a_3 = 0$ provides the choice $\gamma = 2$ with $c = 16/\gamma^4 = 1$. This proves the theorem.

4.2. **Proof of Theorem 2.** By Example 4, the canonical symbol S_0 for entropy dissipation along the DLSS equation (2) is given by

$$S_0(\eta, \xi) = -\frac{1}{2}\xi_1\xi_3 + \xi_2\xi_1^2 - \frac{1}{2}\xi_1^4 - \frac{1}{2}(d-1)\eta\xi_1(\xi_2 - \xi_1^2 - \eta\xi_1).$$

Again, we have to solve the decision problem (37). The same argument as in the previous subsection shows that $c_1 = 1$. Thus, we wish to find c_2, $c_3 \in \mathbb{R}$ such that for all $(\eta, \xi) = (\eta, \xi_1, \xi_2) \in \mathbb{R}^3$,

$$2S(\eta, \xi) = a_1\xi_1^4 + a_2\xi_1^2\xi_2 + a_3\xi_2^2 + a_4\eta\xi_1^3 + a_5\eta^2\xi_1^2 + a_6\eta\xi_1\xi_2 \ge 0,$$

where

$$a_1 = (\alpha-3)c_2 - 1, \qquad a_2 = \alpha + 3c_2, \qquad a_3 = 1,$$
$$a_4 = (\alpha-2)c_3 + (d-1)(c_2+1), \qquad a_5 = (d-2)c_3 + d - 1, \qquad a_6 = 2c_3.$$

According to Lemma 7, the above decision problem is equivalent to either

$$0 < 4a_3a_5 - a_6^2 = -4(c_3+1)(c_3-d+1) =: -4C, \tag{48}$$

$$\begin{aligned} 0 \le q(c_2,c_3) &:= 4a_1a_3a_5 - a_3a_4^2 - a_2^2a_5 - a_1a_6^2 + a_2a_4a_6 \\ &= \big(9C - (d-3c_3-1)^2\big)c_2^2 - 2\big(d^2+4d+(d-7)c_3 - 5 - \alpha C\big)c_2 \\ &\quad + \alpha^2 C - d^2 - 2d + 4c_3 + 3 \end{aligned} \tag{49}$$

or

$$0 = 4a_3a_5 - a_6^2 = -4(c_3+1)(c_3-d+1), \tag{50}$$

$$0 = 2a_3a_4 - a_2a_6 = -2(c_2+2c_3+3c_2c_3+1) + 2(c_2+1)d, \tag{51}$$

$$0 \le 4a_1a_3 - a_2^2 = -4 - \alpha^2 - 12c_2 - 2\alpha c_2 - 9c_2^2. \tag{52}$$

First, we solve (50)-(52). Condition (50) implies that either $c_3 = -1$ or $c_3 = d-1$. In the former case, (51) gives $c_2 = -(d+1)/(d+2)$. Then (52) is equivalent to

$$\alpha^2 - \frac{2(d+1)}{d+2}\alpha + \frac{(d-1)^2}{2(d+2)^2} \le 0,$$

which is satisfied if and only if

$$\frac{(\sqrt{d}-1)^2}{d+2} \le \alpha \le \frac{(\sqrt{d}+1)^2}{d+2}. \tag{53}$$

In the latter case $c_3 = d-1$, (51) is satisfied if $d=1$ or if $d>1$ and $c_2 = -1/2$. If $d=1$, we choose the maximizing value $c_2 = -(\alpha+6)/9$ for (52). Then, this inequality is satisfied if and only if $0 \le \alpha \le 3/2$. On the other hand, if $d>1$, (52) can be written as $\alpha^2 - \alpha + 1/4 \le 0$, which is satisfied if and only if $\alpha = 1/2$. We have shown that the decision problem is solvable if $d=1$ and $0 \le \alpha \le 3/2$ or if $d>1$ and (53) hold.

Next, we solve (48)-(49). The discriminant $D(c_3)$ of the quadratic polynomial $q(\cdot, c_3)$ factorizes, $D(c_3) = 4C\Delta(c_3)$, where

$$\begin{aligned} \Delta(c_3) &= \alpha^2c_3^2 + 2(\alpha^2(d-5) - \alpha(d-7))c_3 + (d^2+6d-7)\alpha^2 \\ &\quad - 2\alpha(d^2+4d-5) + (d-1)^2. \end{aligned}$$

Notice that by (48), $C<0$. An application of Lemma 8 shows that $D(c_3)$ is nonnegative if $d=1$ and $0<\alpha<3/2$, or $d \in \{2,3\}$ and $(\sqrt{d}-1)^2/(d+2) < \alpha \le 3/2$, or $d \in \{4,5,6,7\}$ and $(\sqrt{d}-1)^2/(d+2) < \alpha < (\sqrt{d}+1)^2/(d+2)$, or $d \ge 8$ and $(d-4)/(2d-4) \le \alpha < (\sqrt{d}+1)^2/(d+2)$. This proves that $dE_\alpha/dt \le 0$ if these conditions are satisfied.

The estimates for the entropy production term $\omega_d \int (\Delta_r u^{\alpha/2})^2 r^{d-1} \mathrm{d}r$ are obtained by similar arguments as in the previous subsection. We leave the details to the reader.

4.3. **Proof of Theorem 3.** The canonical symbol associated to the sixth-order equation (3) can be read off from the representation of its radially symmetric solutions as given in Example 4. One finds

$$\begin{aligned} S_0(\eta,\xi) = {} & 6\xi_1^6 - 18\xi_1^4\xi_2 + 11\xi_1^2\xi_2^2 + 8\xi_1^3\xi_3 - 3\xi_1^2\xi_4 - 5\xi_1\xi_2\xi_3 + \xi_1\xi_5 \\ & + (d-1)\big[-6\eta\xi_1^5 + (2d-7)\eta^2\xi_1^4 + 14\eta\xi_1^3\xi_2 + (3d-8)\eta^3\xi_1^3 - 4\eta\xi_1\xi_2^2 \\ & - 3(d-4)\eta^2\xi_1^2\xi_2 - 6\eta\xi_1^2\xi_3 + 3(d-3)\eta^4\xi_1^2 - 3(d-3)\eta^3\xi_1\xi_2 + (d-5)\eta^2\xi_1\xi_3 \\ & + 2\eta\xi_1\xi_4\big]. \end{aligned}$$

We have to solve the decision problem

$$\exists c_1,\ldots,c_{10}\in\mathbb{R} : \forall(\eta,\xi) : S(\eta,\xi) = (S_0 + c_1T_1 + \cdots + c_{10}T_{10})(\eta,\xi) \geq 0,$$

where the shift polynomials T_i are given by (22)-(31) with $\beta = 0$. Again, we can simplify this problem by eliminating the terms whose sign cannot be controlled. We choose $c_3 = 0$ to eliminate ξ_2^3, $c_5 = -1$ to eliminate $\xi_1\xi_5$, $c_8 = -(d-1)$ to eliminate $\eta\xi_1\xi_4$, $c_4 = \alpha - 2$ to eliminate $\xi_1^2\xi_4$, and $c_{10} = 1$ to eliminate the product $\xi_2\xi_4$ introduced by T_5. With these choices,

$$\begin{aligned} S(\eta,\xi) = {} & \big(c_1T_1 + c_2T_2 + 0\cdot T_3 + (\alpha-2)T_4 + (-1)\cdot T_5 + c_6T_6 + c_7T_7 - (d-1)T_8 \\ & + c_9T_9 + 1\cdot T_{10}\big)(\eta,\xi) \\ = {} & \big((\alpha-5)c_1 + 6\big)\xi_1^6 + \big(5c_1 + (\alpha-4)c_2 - 18\big)\xi_1^4\xi_2 + (3c_2 + 11)\xi_1^2\xi_2^2 \\ & + \big(c_2 + (\alpha+1)(\alpha-3) + 8\big)\xi_1^3\xi_3 + (3\alpha-5)\xi_1\xi_2\xi_3 \\ & + \big((\alpha-4)c_6 + (d-1)(c_1-6)\big)\eta\xi_1^5 \\ & + \big((\alpha-3)c_9 + (d-2)c_6 + (d-1)(2d-7)\big)\eta^2\xi_1^4 \\ & + \big((\alpha-3)c_7 + 4c_6 + (d-1)(c_2+14)\big)\eta\xi_1^3\xi_2 \\ & + \big((d-3)c_9 - 2(\alpha-2)(d-1) + (d-1)(3d-8)\big)\eta^3\xi_1^3 \\ & + \big(2c_7 + (\alpha-6)(d-1)\big)\eta\xi_1\xi_2^2 \\ & + \big((\alpha-2)(d-1) + 3c_9 + (d-2)c_7 - 3(d-1)(d-4)\big)\eta^2\xi_1^2\xi_2 \\ & + \big(c_7 - 3(d-1)\big)\eta\xi_1^2\xi_3 + (d-1)^2\eta^4\xi_1^2 - 2(d-1)^2\eta^3\xi_1\xi_2 \\ & - 2(d-1)\eta^2\xi_1\xi_3 + 2(d-1)\eta\xi_2\xi_3 + (d-1)^2\eta^2\xi_2^2 + \xi_3^2. \end{aligned}$$

The corresponding decision problem contains the four variables η, $\xi_1,\ldots,\xi_3$ and the five coefficients c_1, c_2, c_6, c_7 and c_9. For further simplification, we make a change of variables. Let

$$\zeta_1 = \frac{\eta}{\xi_1}, \quad \zeta_2 = \frac{\xi_2}{\xi_1^2} - \frac{\eta}{\xi_1}, \quad \zeta_3 = \frac{\xi_3}{\xi_1^3} - 3\frac{\eta}{\xi_1}\left(\frac{\xi_2}{\xi_1^2} - \frac{\eta}{\xi_1}\right). \tag{54}$$

These definitions are motivated by the observation that for any radially symmetric function $U(x) = u(r)$, the tensors $\nabla_x U$, $\nabla_x^2 U$ and $\nabla_x^3 U$ of the first, second and third total derivatives take the form

$$\nabla_x U(x) = u\xi_1 \mathbf{e}_r,$$
$$\nabla_x^2 U(x) = u\xi_1^2\big(\zeta_2 \mathbf{e}_r \otimes \mathbf{e}_r + \zeta_1 \mathbf{1}\big),$$
$$\nabla_x^3 U(x) = u\xi_1^3\big(\zeta_3 \mathbf{e}_r \otimes \mathbf{e}_r \otimes \mathbf{e}_r + \zeta_1\zeta_2 \mathbf{e}_r \otimes_s \mathbf{1}\big),$$

where $(\mathbf{e}_r \otimes_s \mathbf{1})_{ijk} = \delta_{ij}x_k + \delta_{jk}x_i + \delta_{ik}x_j$. It turns out that S can be expressed in terms of $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ only. Furthermore, choosing $c_7 = -c_9 = (\alpha + 1/2)(d-1)$ (see Remark 9 below), some higher-order terms cancel, and we end up with

$$\begin{aligned} S_1(\zeta) = \xi_1^6 S(\eta,\xi) &= \zeta_3^2 + 2(d+2)\zeta_1\zeta_2\zeta_3 + (3\alpha-5)\zeta_2\zeta_3 + \frac{1}{2}\big(2\alpha(d+2) - 5(d+1)\big)\zeta_1\zeta_3 \\ &+ \big(\alpha^2 - 2\alpha + 5 + c_2\big)\zeta_3 + (d+2)^2\zeta_1^2\zeta_2^2 + \frac{1}{2}(d+2)\big(2\alpha(d+2) - 5(d+1)\big)\zeta_1^2\zeta_2 \\ &+ \big((d+2)(c_2+c_6) + 2d^2 + 5d + 4\big)\zeta_1^2 + (3\alpha-5)(d+2)\zeta_1\zeta_2^2 + (3c_2+11)\zeta_2^2 \\ &+ \frac{1}{2}\big(2\alpha^2(d+2) - \alpha(5d+7) + 2(d+8)c_2 + 8c_6 + 25d + 49\big)\zeta_1\zeta_2 \\ &+ \big((\alpha-4)(c_2+c_6) + (d+4)c_1 - 6(d+2)\big)\zeta_1 + \big((\alpha-4)c_2 + 5c_1 - 18\big)\zeta_2 \\ &+ (\alpha-5)c_1 + 6. \end{aligned} \tag{55}$$

For any fixed ζ_1 and ζ_2, the polynomial $S_1(\zeta)$ is quadratic in ζ_3, with leading coefficient equal to one. This quadratic polynomial is nonnegative if and only if its discriminant

$$\begin{aligned} D(\zeta_1,\zeta_2) &= \big(\partial_{\zeta_3} S_1(\zeta_1,\zeta_2,0)\big)^2 - 4S_1(\zeta_1,\zeta_2,0) \\ &= -\frac{1}{4}\big[20(d^2+3d+2)\alpha - 4\alpha^2(d+2)^2 + 16(d+2)(c_2+c_6) + 7d^2 + 30d + 39\big]\zeta_1^2 \\ &\quad + (9\alpha^2 - 30\alpha - 19 - 12c_2)\zeta_2^2 \\ &\quad + \big[6(d+2)\alpha^2 - (23d+37)\alpha - 24c_2 - 16c_6 - 5d - 33\big]\zeta_1\zeta_2 \\ &\quad + \big[2(d+2)\alpha^3 - (9d+13)\alpha^2 + 2\alpha(d(c_2+10) - 2c_6 + 15) - 4c_1(d+4) \\ &\quad + c_2(11-5d) + 16c_6 - d + 23\big]\zeta_1 + 2\big[3\alpha^3 - 11\alpha^2 + (c_2+25)\alpha - 10c_1 + 3c_2 \\ &\quad + 11\big]\zeta_2 + \alpha^4 - 4\alpha^3 + 2\alpha^2(c_2+7) - 4\alpha(c_1+c_2+5) + c_2(c_2+10) + 20c_1 + 1 \end{aligned}$$

is nonpositive. Thus, the nonnegativity of S_0 for some coefficients c_i is reduced to the following decision problem:

$$\exists c_1, c_2, c_6 \in \mathbb{R} : \forall \zeta_1, \zeta_2 \in \mathbb{R} : -D(\zeta_1,\zeta_2) \geq 0.$$

The discriminant $D(\zeta_1,\zeta_2)$ is again of quadratic type, now in terms of ζ_1 and ζ_2. Thus Lemma 7 is applicable and yields several conditions on c_1, c_2 and c_6 for the nonpositivity of D. This nonlinear system of equations and inequalities is solved by the computer algebra system `Mathematica`. As a result, we obtain, for given dimension $d \geq 1$, conditions on the admissible values of α. More precisely, α has to be in between the numbers $\alpha_0(d)$ and $\alpha_1(d)$, and $\alpha_i(d)$ are the positive roots of certain higher-order polynomials which are explicit. Their roots, however, can be calculated only numerically and are given in the statement of the theorem.

The entropy production

$$\omega_d \int_0^1 u \left(\frac{\Delta_r \sqrt{u}}{\sqrt{u}} \right)_r^2 r^{d-1} \mathrm{d}r$$

is represented by the symbol

$$\begin{aligned} W(\eta, \xi) &= \frac{1}{4} \big(\xi_3 - 2\xi_2\xi_1 + \xi_1^3 + (d-1)(\eta\xi_2 - \eta\xi_1^2 - \eta^2\xi_1) \big)^2 \\ &= \frac{\xi_1^6}{4} \big(\zeta_3 + (d+2)\zeta_1\zeta_2 - 2\zeta_2 - (d+1)\zeta_1 + 1 \big)^2 =: \xi_1^6 W_1(\zeta). \end{aligned}$$

Setting $\alpha = 1$ in (55), we obtain the decision problem

$$\exists c_1, c_2, c_6 \in \mathbb{R}, c > 0 : \forall \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3 : S_1(\zeta) - cW_1(\zeta) \geq 0.$$

Our solution strategy is the same as before. We observe that $S_1 - cW_1$ is a quadratic polynomial in ζ_3, and we calculate the respective discriminant. The latter turns out to be quadratic in the remaining variables ζ_1 and ζ_2. Omitting the details, we remark that the reduced decision problem for the discriminant is again solvable with the aid of Lemma 7 and `Mathematica`. This results in numerical values for $c > 0$ such that (4) holds.

Remark 9. The choice of the coefficients c_7 and c_9 in the above proof can be explained as follows. First, choosing $c_9 = -c_7$ cancels the coefficient of the indefinite term ζ_1^3 (obtained after writing S_1 in terms of $(\zeta_1, \zeta_2, \zeta_3)$). Second, by the choice $c_7 = (\alpha + 1/2)(d-1)$, the coefficient of the term $\zeta_1\zeta_2^2$ in the discriminant $D(\zeta_1, \zeta_2)$ cancels, such that the remaining polynomial becomes quadratic in ζ_1 and ζ_2. This choice also helps to determine a sum-of-squares representation of S_1 by employing the numerical tool Yalmip [16].

5. Absence of entropies

Similarly as in [9, 15], it is possible to prove that certain functionals E_α *cannot* be entropies. Below, we generalize Theorem 19 in [9] to the multidimensional, radially symmetric situation. Specifically, let $\gamma \in \mathbb{R}$ and $S \in \Sigma_K$ be given, and define

$$\mathbb{I}(u) = \int_0^1 u(r)^\gamma \, \mathrm{D}_S[u, r] r^{d-1} \mathrm{d}r.$$

Further, define the components of a vector $\bar{\xi} \in \mathbb{R}^K$ by

$$\bar{\xi}_1 = \sigma, \quad \bar{\xi}_2 = \sigma(\sigma - 1), \ldots, \quad \bar{\xi}_K = \sigma(\sigma - 1) \cdots (\sigma - K + 1),$$

where $\sigma = (K - d)/\gamma$. By inserting $(\eta, \xi) = (1, \bar{\xi})$ into formula (14), one easily verifies that all shift polynomials T_k vanish at this particular point. Therefore, the values of any two characteristic symbols S and S' coincide at $(1, \bar{\xi})$. Hence, if the given S is negative at $(1, \bar{\xi})$, so is *any* affine combination $S + c_1T_1 + \cdots + c_mT_m$. In this case, $\mathbb{I}(u)$ cannot be written as an integral over a pointwise nonnegative expression by the method developed before. This statement can be strengthened as follows.

Theorem 10. *Assume that $S(1,\bar\xi) < 0$. Then there exists a family of functions $u_\varepsilon \in \mathcal{U}$ with $u_\varepsilon(r) = 1$ for $r \in [2/3, 1]$ satisfying $\lim_{\varepsilon\downarrow 0} \mathbb{I}(u_\varepsilon) = -\infty$.*

The set $\mathcal{U}$ is defined on page 7. We remark that, since the functions u_ε are equal to a positive constant for $r > 2/3$, they satisfy any homogeneous boundary condition that involves derivatives at $r = 1$.

Proof. We shall use a more straight-forward construction of the functions u_ε than those employed in [9, 15]. For this, let a cut-off function $\phi \in C^\infty(\mathbb{R})$ with $0 \le \phi \le 1$ be given that satisfies

$$\phi(r) = 1 \quad \text{for } r \le 1/3 \text{ and} \quad \phi(r) = 0 \quad \text{for } r \ge 2/3.$$

Choose $\varepsilon \in (0, 1/2)$ arbitrary and define u_ε by

$$u_\varepsilon(r) = \phi(r/\varepsilon)\,\varepsilon^\sigma + [1 - \phi(r/\varepsilon)]\phi(r)\,r^\sigma + 1 - \phi(r).$$

Clearly, u_ε is positive and of class C^∞. Moreover, notice that $u_\varepsilon(r) = 1$ for $2/3 \le r \le 1$ as stated in the theorem. We need to evaluate the integral

$$\mathbb{I}(u_\varepsilon) = \int_0^1 u_\varepsilon(r)^\gamma\,\mathrm{D}_P[u_\varepsilon, r] r^{d-1}\mathrm{d}r.$$

This is done by splitting the domain $[0, 1]$ into three intervals. To start with, let $r \in [0, 2\varepsilon/3]$. Then $u_\varepsilon(r) = \varepsilon^\sigma\psi(r/\varepsilon)$, where $\psi(\rho) = \phi(\rho) + [1 - \phi(\rho)]\rho^\sigma$, and consequently

$$\frac{\partial_r^k u_\varepsilon(r)}{u_\varepsilon(r)} = \varepsilon^{-k}\frac{\partial_\rho^k\psi(\rho)}{\psi(\rho)},$$

with $\rho = r/\varepsilon$. The homogeneity (8) of $S \in \Sigma_K$ now implies

$$\mathrm{D}_S[u_\varepsilon, r] = \varepsilon^{-K}\,\mathrm{D}_S[\psi, \rho].$$

Substitution of $r = \varepsilon\rho$ under the integral leads to

$$I_1 := \int_0^{2\varepsilon/3} u_\varepsilon(r)^\gamma\,\mathrm{D}_P[u_\varepsilon, r] r^{d-1}\mathrm{d}r = \varepsilon^{\sigma\gamma-K+d}\int_0^{2/3}\psi(\rho)^\gamma\,\mathrm{D}_P[\psi, \rho]\rho^{d-1}\mathrm{d}\rho.$$

Since ψ is positive and smooth, and all of its derivatives vanish at $\rho = 0$, the last integral is well-defined and finite. In fact, the value of I_1 is independent of ε, since $\sigma\gamma = K - d$ by definition of σ.

Next, let $r \in [2\varepsilon/3, 1/3]$ and notice that $u_\varepsilon(r) = r^\sigma$. It follows that

$$\partial_r^k u_\varepsilon(r) = \sigma(\sigma - 1)\cdots(\sigma - k + 1)r^{\sigma-k} = r^{-k}\bar\xi_k u_\varepsilon(r).$$

Using the homogeneity (8) once again, we obtain $\mathrm{D}_S[u_\varepsilon, r] = r^{-K}S(1, \bar\xi_1, \ldots, \bar\xi_K)$, and thus

$$\begin{aligned} I_2 : &= \int_{2\varepsilon/3}^{1/3} u_\varepsilon(r)^\gamma\,\mathrm{D}_S[u_\varepsilon, r] r^{d-1}\mathrm{d}r = S(1, \bar\xi_1, \ldots, \bar\xi_K)\int_{2\varepsilon/3}^{1/3} r^{\gamma\sigma+d-K}\frac{\mathrm{d}r}{r} \\ &= S(1, \bar\xi_1, \ldots, \bar\xi_K)\ln[1/(2\varepsilon)]. \end{aligned}$$

Finally, for $r \in [1/3, 1]$, the function $u_\varepsilon(r)$ is smooth and positive, and does not depend on $\varepsilon > 0$. In other words,

$$I_3 := \int_{1/3}^1 u_\varepsilon(r)^\gamma \,\mathrm{D}_S[u_\varepsilon, r] r^{d-1} \mathrm{d}r$$

is a finite, ε-independent value. In summary, there is some constant $C > 0$ for which

$$\mathbb{I}(u_\varepsilon) = I_1 + I_2 + I_3 = C + S(1, \bar{\xi}_1, \ldots, \bar{\xi}_K) \ln[1/(2\varepsilon)].$$

This sum converges to $-\infty$ as $\varepsilon \downarrow 0$ since $S(1, \bar{\xi}_1, \ldots, \bar{\xi}_K) < 0$ by assumption. □

As a corollary, we obtain that E_α cannot be an entropy for the evolution equation (9) if the associated canonical symbol S_0 has the property that

$$S_0\big(1, \sigma, \sigma(\sigma - 1), \ldots, \sigma(\sigma - 1) \cdots (\sigma - K + 1)\big) < 0$$

for $\sigma = (K - d)/(\alpha + \beta)$. Indeed, we may use the corresponding function u_ε constructed in the proof of Theorem 10 above as an initial condition u_0 in (10). The functions u_ε are positive and smooth, and they satisfy the boundary conditions since u_ε is constant close to the boundary. By classical parabolic theory, there exists a corresponding solution $u_\varepsilon(t)$, at least locally in time, i.e. for $t \in [0, \tau]$, and this solution and its spatial derivatives depend continuously on $t \in [0, \tau]$. Hence,

$$E_\alpha[u_\varepsilon(\tau)] - E_\alpha[u_\varepsilon] = -\omega_d \int_0^\tau \int_0^1 u_\varepsilon(t; r)^\gamma \,\mathrm{D}_{S_0}[u_\varepsilon(t), r] r^{d-1} \mathrm{d}r \,\mathrm{d}t.$$

Choosing ε and τ sufficiently small, the double integral on the right-hand side is negative, and one concludes that $E_\alpha[u_\varepsilon(\tau)] > E_\alpha[u_\varepsilon]$.

We apply this result to the fourth- and sixth-order equations introduced in the introduction. It turns out that for the thin-film equation (1), we have $S_0(1, \bar{\xi}) < 0$ if and only if $\alpha + \beta \notin [3/2, 3]$ for $d = 1$, $\alpha + \beta \in (-\infty, 1)$ for $d = 2$, $\alpha + \beta \in (-1, 1/2)$ for $d = 3$, and $\alpha + \beta \in (-(d - 4)/2, (d - 4/(d + 2))$ for $d > 4$. (Our method does not give any statement for $d = 4$.) In one space dimension, we achieve the optimal bounds for $\alpha + \beta$, being in the interval $[3/2, 3]$ (as in [9]). However, we obtain much less information for $d > 1$.

For the DLSS equation (2), $S_0(1, \bar{\xi}) < 0$ holds if and only if $\alpha \notin [0, 3/2]$ for $d = 1$, $\alpha \in (-\infty, 0)$ for $d = 2$, $\alpha \in (-1/2, 0)$ for $d = 3$, and $\alpha \in (0, (d - 4)/(2d - 4))$ for $d \geq 4$. We recover the optimal range in the one-dimensional case. Moreover, we see that the lower bound for $d \geq 8$ is optimal, at least for nonnegative values for α.

Finally, for the sixth-order equation (3), we have $S_0(1, \bar{\xi}) < 0$ if and only if $\alpha \in (5/4, 10/3)$ for $d = 1$, $\alpha \in (4/3, \infty)$ for $d = 2$, $\alpha \notin [-3(1 - \sqrt{33})/8, -3(1 + \sqrt{33})/8]$ for $d = 3$, and $\alpha \in (-\infty, -1)$ for $d = 4$. For higher space dimensions, $S_0(1, \bar{\xi}) \geq 0$ holds for all $\alpha \in \mathbb{R}$, and we do not obtain any information. In the two-dimensional case, there are no entropies for $\alpha > 4/3$, which is not far from the upper bound $\alpha = 1.0982\ldots$ obtained in Theorem 3.

Appendix

In this appendix we give a sketch of the derivation of the sixth-order equation (3). This equation is formally derived from an $O(\varepsilon^6)$ approximation of the generalized quantum drift-diffusion model of Degond et al. [6], where ε is the scaled Planck constant. Without electric field, this model is given by

$$\partial_t U = \operatorname{div}(U\nabla A), \tag{56}$$

where the particle density $U(t;x)$ and the function $A(t;x)$ are related through the integral

$$U(t;x) = \frac{1}{(2\pi\varepsilon)^d}\int_{\mathbb{R}^d} \operatorname{Exp}\left(A(t;x) - \frac{|p|^2}{2}\right) \mathrm{d}p, \quad x\in\mathbb{R}^d,\ t>0.$$

Here, the so-called quantum exponential Exp is defined by $\operatorname{Exp}(a) = W(\exp(W^{-1}(a)))$, where $a(t;x,p)$ is a function in the phase-space, W is the Wigner transform, W^{-1} its inverse and exp is the operator exponential. For precise definitions and the derivation of the quantum drift-diffusion model we refer to [6].

The crucial step in the $O(\varepsilon^6)$ derivation of (56) is to determine an $O(\varepsilon^6)$ approximation of $\operatorname{Exp}(a)$ with $a(x,p) = A(t;x) - |p|^2/2$. To this end, we follow the strategy proposed in [6]. Define $F(z) = \operatorname{Exp}(za)$ and expand $F(z)$ formally as a series in ε, i.e. $F(z) = \sum_{k=0}^\infty \varepsilon^k F_k(z)$. The functions $F_k(z)$ can be computed by pseudo-differential calculus. For odd indices k, we have $F_k(z) = 0$, and for even indices we have to solve the following differential equation:

$$\frac{d}{dz}F_k(z) = a\circ_0 F_k(z) + a\circ_2 F_{k-2}(z) + \ldots + a\circ_k F_0(z), \quad z>0,$$

with the initial condition $F_k(0) = \delta_{k0}$. The multiplication $\circ_n$ is defined for any two smooth functions ω_1 and ω_2 by (see also (5.19) in [6])

$$\omega_1 \circ_n \omega_2 = \sum_{|\alpha|+|\beta|=n} \left(\frac{i}{2}\right)^n \frac{(-1)^{|\beta|}}{\alpha!\beta!} \partial_x^\alpha \partial_p^\beta \omega_1 \partial_x^\beta \partial_p^\alpha \omega_2, \tag{57}$$

where $\alpha,\beta\in\mathbb{N}^d$ are multi-indices. Let ∇^k denote the k-tensor of partial derivatives of order k.

Lemma. *It holds*

$$\omega_1 \circ_n \omega_2 = \frac{i^n}{2^n n!}\left(\sum_{k=0}^n (-1)^k \binom{n}{k} \nabla_x^{n-k}\otimes\nabla_p^k\omega_1 : \nabla_p^{n-k}\otimes\nabla_x^k\omega_2\right), \tag{58}$$

where "$\otimes$" denotes the tensor product and ":" the component-wise inner product.

Proof. Let $k = |\beta| = \beta_1 + \cdots + \beta_d$ for $\beta\in\mathbb{N}^d$. The tensor ∇_p^k consists of the partial derivatives ∂_p^β. According to the Schwartz rule, each partial derivative ∂_p^β appears in ∇_p^k on exactly $k!/\beta!$ positions, where $\beta! = \beta_1!\ldots\beta_d!$. Analogusly, for $|\alpha| = n-k$, ∂_x^α appears in ∇_x^{n-k} on $(n-k)!/\alpha!$ positions. Thus, the expression $\partial_x^\alpha\partial_p^\beta$ appears in $\nabla_x^{n-k}\otimes\nabla_p^k$ on $(n-k)!k!/(\alpha!\beta!)$ positions, and the same number of appearance holds for the expression $\partial_x^\alpha\partial_p^\beta\partial_x^\beta\partial_p^\alpha$ in $\nabla_x^{n-k}\otimes\nabla_p^k : \nabla_p^{n-k}\otimes\nabla_x^k$. Using these combinatorial observations, formula (58) follows immediately. □

The functions $F_0(z)$ and $F_2(z)$ have already been calculated in [6]:

$$\begin{aligned} F_0(z)(x,p) &= e^{za(x,p)}, \\ F_2(z)(x,p) &= \frac{1}{8}e^{za(x,p)}\Big(z^2\Delta_x A + \frac{z^3}{3}|\nabla_x A|^2 - \frac{z^3}{3}\nabla_x^2 A : p\otimes p\Big). \end{aligned}$$

Thus, it remains to solve

$$\begin{aligned} \frac{d}{dz}F_4(z) &= a\circ_0 F_4(z) + a\circ_2 F_2(z) + a\circ_4 F_0(z) = a\cdot F_4(z) \\ &\quad + \frac{e^{za}}{192}\Big[z^5|\nabla A|^4 + 5z^4|\nabla A|^2\Delta A - 2z^5|\nabla A|^2(\nabla^2 A : p\otimes p) \\ &\quad - 4z^4(\nabla^2 A : \nabla^2 Ap\otimes p) + z^5(\nabla^2 A : p\otimes p)^2 + 2z^3\|\nabla^2 A\|^2 \\ &\quad - 5z^4\Delta A(\nabla^2 A : p\otimes p) + 6z^3(\Delta A)^2 + 3z^2\Delta^2 A + z^3\Delta|\nabla A|^2 \\ &\quad - z^3\Delta(\nabla^2 A : p\otimes p) + 6z^3\nabla A\cdot\nabla\Delta A + 2z^4\nabla A\cdot\nabla|\nabla A|^2 \\ &\quad - 2z^4\nabla A\cdot\nabla(\nabla^2 A : p\otimes p)\Big] + \frac{\mathbf{e}^{za}}{384}\Big[z^4(\nabla^4 A : p\otimes p\otimes p\otimes p) \\ &\quad - z^3(\nabla^4 A : (p\otimes p\otimes \mathbb{I})) - z^3(\nabla^4 A : p\otimes\nabla_p(p\otimes p)) \\ &\quad - z^3(\nabla^4 A : \nabla_p(p\otimes p\otimes p)) + z^2(\nabla^4 A : \nabla_p(p\otimes\mathbb{I})) \\ &\quad + z^2(\nabla^4 A : \nabla_p^2(p\otimes p))\Big], \end{aligned}$$

with $F_4(0) = 0$. In the above computations, we have exhaustively used the above lemma. By the variation-of-constants formula, we obtain

$$\begin{aligned} F_4(1) = \frac{e^a}{384}\Big[&\frac{1}{3}|\nabla A|^4 + 2|\nabla A|^2\Delta A - \frac{2}{3}|\nabla A|^2(\nabla^2 A : p\otimes p) \\ &- \frac{8}{5}(\nabla^2 A : \nabla^2 Ap\otimes p) + \frac{1}{3}(\nabla^2 A : p\otimes p)^2 + \|\nabla^2 A\|^2 \\ &- 2\Delta A(\nabla^2 A : p\otimes p) + (\Delta A)^2 + 2\Delta^2 A + \frac{1}{2}\Delta|\nabla A|^2 \\ &- \frac{1}{2}\Delta(\nabla^2 A : p\otimes p) + 3\nabla A\cdot\nabla\Delta A + \frac{4}{5}\nabla A\cdot\nabla|\nabla A|^2 \\ &- \frac{4}{5}\nabla A\cdot\nabla(\nabla^2 A : p\otimes p) + \frac{1}{5}(\nabla^4 A : p\otimes p\otimes p\otimes p) \\ &- \frac{1}{4}\big((\nabla^4 A : (p\otimes p\otimes\mathbb{I})) + (\nabla^4 A : p\otimes\nabla_p(p\otimes p)) \\ &+ (\nabla^4 A : \nabla_p(p\otimes p\otimes p))\big) + \frac{1}{3}\big((\nabla^4 A : \nabla_p(p\otimes\mathbb{I})) \\ &+ (\nabla^4 A : \nabla_p^2(p\otimes p))\big)\Big]. \end{aligned}$$

This gives us the $O(\varepsilon^6)$ expansion of the quantum exponential.

It remains to represent the density u as a function of A. We integrate F_0, F_2, and F_4 with respect to $p \in \mathbb{R}^d$ and employ the formulas

$$\frac{1}{(2\pi\varepsilon)^d}\int_{\mathbb{R}^d} p_i p_j e^{A-|p|^2/2}\mathrm{d}p = \frac{e^A}{(\sqrt{2\pi}\varepsilon)^d}\delta_{ij},$$

$$\frac{1}{(2\pi\varepsilon)^d}\int_{\mathbb{R}^d} p_r p_s p_i p_j e^{A-|p|^2/2}\mathrm{d}p = \frac{e^A}{(\sqrt{2\pi}\varepsilon)^d}(\delta_{rs}\delta_{ij} + \delta_{ri}\delta_{sj} + \delta_{rj}\delta_{sj}),$$

where δ_{ij} denotes the Kronecker symbol. This gives

$$\begin{aligned} U &= \frac{1}{(2\pi\varepsilon)^d}\int_{\mathbb{R}^d}(F_0(1) + \varepsilon^2 F_2(1) + \varepsilon^4 F_4(1))\mathrm{d}p + O(\varepsilon^6) \\ &= \frac{e^A}{(\sqrt{2\pi}\varepsilon)^d}\bigg(1 + \frac{\varepsilon^2}{24}\Big(2\Delta A + |\nabla A|^2\Big) + \frac{\varepsilon^4}{5760}\Big(5|\nabla A|^4 + 20|\nabla A|^2\Delta A \\ &\quad + \|\nabla^2 A\|^2 + 20(\Delta A)^2 + 24\Delta^2 A + \frac{15}{2}\Delta|\nabla A|^2 + 33\nabla A\cdot\nabla\Delta A \\ &\quad + 12\nabla A\cdot\nabla|\nabla A|^2\Big)\bigg) + O(\varepsilon^6). \end{aligned}$$

To obtain an ε-expansion of A in terms of U, we insert the ansatz $A = A_0 + \varepsilon^2 A_2 + \varepsilon^4 A_4 + O(\varepsilon^6)$ in the above expression for u. Equating equal powers of ε yields the system

$$\begin{aligned} U &= \frac{e^{A_0}}{(\sqrt{2\pi})^d}, \quad 0 = A_2 + \frac{1}{24}\big(2\Delta A_0 + |\nabla A_0|^2\big), \\ 0 &= A_4 + \frac{1}{2}A_2^2 + \frac{1}{24}A_2\big(2\Delta A_0 + |\nabla A_0|^2\big) + \frac{1}{12}\big(\Delta A_2 + \nabla A_0\cdot\nabla A_2\big) \\ &\quad + \frac{1}{5760}\Big(5|\nabla A_0|^4 + 20|\nabla A_0|^2\Delta A_0 + \|\nabla^2 A_0\|^2 + 20(\Delta A_0)^2 \\ &\quad + 24\Delta^2 A_0 + \frac{15}{2}\Delta|\nabla A_0|^2 + 33\nabla A_0\cdot\nabla\Delta A_0 + 12\nabla A_0\cdot\nabla|\nabla A_0|^2\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} A_0 &= \log U + d\log(\sqrt{2\pi}), \quad A_2 = -\frac{1}{6}\frac{\Delta\sqrt{U}}{\sqrt{U}}, \\ A_4 &= \frac{1}{720}\Big(2\frac{\Delta^2 U}{U} - 3\frac{|\nabla U|^4}{u^4} + 4\nabla^2 U\nabla U\cdot\nabla U + 4\frac{\Delta U}{U}\frac{|\nabla U|^2}{U^2} - 4\frac{\nabla\Delta U}{U}\cdot\frac{\nabla U}{U} \\ &\quad - 2\Big(\frac{\Delta U}{U}\Big)^2 - \frac{\|\nabla^2 U\|^2}{U^2}\Big) = \frac{1}{360}\Big(\frac{1}{2}\|\nabla^2\log U\|^2 + \frac{1}{U}\nabla^2 : (U\nabla^2\log U)\Big). \end{aligned}$$

Finally, up to terms of order $O(\varepsilon^6)$, (56) becomes

$$\partial_t U = \Delta U - \frac{\varepsilon^2}{6}\operatorname{div}\Big(U\nabla\Big(\frac{\Delta\sqrt{U}}{\sqrt{U}}\Big)\Big) + \frac{\varepsilon^4}{360}\operatorname{div}\Big(U\nabla\Big(\frac{1}{2}\|\nabla^2\log U\|^2 + \frac{1}{U}\nabla^2 : (U\nabla^2\log U)\Big)\Big).$$

The second term on the right-hand side is the fourth-order operator of the DLSS equation. The sixth-order equation (3) is obtained by taking into account only the sixth-order expression and choosing $\varepsilon^4 = 360$.

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