Convergence of Adaptive FEM for some Elliptic Obstacle Problem

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CONVERGENCE OF ADAPTIVE FEM
FOR SOME ELLIPTIC OBSTACLE PROBLEM

M. PAGE AND D. PRAETORIUS

Abstract. In this work, we treat the convergence of adaptive lowest-order FEM for some elliptic obstacle problem with affine obstacle. For error estimation, we use a residual error estimator from [6]. We extend recent ideas from [12] for the unrestricted variational problem to overcome the lack of Galerkin orthogonality. The main result states that an appropriately weighted sum of energy error, edge residuals, and data oscillations satisfies a contraction property within each step of the adaptive feedback loop. This result is superior to a prior result from [6] in two ways: First, it is unnecessary to control the decay of the data oscillations explicitly. Second, our analysis avoids the use of a discrete local efficiency so that the local mesh-refinement is fairly arbitrary.

1. Introduction

1.1. Prior Work on Convergence of Adaptive FEM. Adaptive finite element methods for partial differential equations based on various types of a posteriori error estimators have been intensively studied and are now a standard tool in science and engineering, see e.g. the monographs [2, 23] and the references therein. As far as a posteriori error analysis for elliptic obstacle problems is concerned, we refer to [3, 5, 4, 13, 17, 20, 24].

In the case of elliptic boundary value problems, convergence of adaptive mesh-refining algorithms has first been proven in [14], followed by [18]. The latter works considered the residual error estimator for a P1-finite element discretization of the Poisson problem. In [18], the convergence analysis is based on reliability and the so-called discrete local efficiency of the residual error estimator, which relies on an interior node property for the local mesh-refinement. The main idea of the convergence proof then is to show that the error is contractive up to the data oscillations. This concept attracted quite some attention in the literature for various applications, e.g. the p-Laplacian [25], edge elements [8], mixed methods [9], nonconforming elements [10], and obstacle problems [6, 7].

For the Poisson problem, optimality of the adaptive algorithm from [18] was first shown in [22]. Recently, [12] presented a new convergence proof under weaker conditions. They showed that a weighted sum of error and error estimator satisfies a contraction property without requiring (discrete local) efficiency of the estimator. In particular, their proof avoided the interior node property of the local mesh-refinement, and they even proved optimality.

1.2. Contributions of Current Work. We consider the framework of [6], i.e. adaptive P1-finite elements for some elliptic obstacle problem with affine obstacle. However, to explain the differences to [6], we first recall their main result: Let $\varepsilon_\ell = J(U_\ell) - J(u) \geq 0$ denote the energy error, where $u$ is the exact solution of the obstacle problem and $U_\ell$ is the finite
element approximation in the \( \ell \)-th step of the adaptive algorithm. Based on a residual error estimator \( \varrho_\ell \) consisting of edge jumps and inspired by [18], [6, Theorem 3] states that the \( \varrho_\ell \)-steered adaptive mesh-refinement leads to

\[
\varepsilon_{\ell+1} \leq \kappa \varepsilon_\ell + C \text{osc}_\ell^2
\]

for all \( \ell \in \mathbb{N} \),

with \( \text{osc}_\ell \) being the data oscillations and with \( 0 < \kappa < 1 \) and \( C > 0 \) being \( \ell \)-independent constants. It is thus a consequence of elementary calculus that \( \text{osc}_\ell \to 0 \) as \( \ell \to \infty \). In [6, 18], however, the convergence \( \text{osc}_\ell \to 0 \) of the data oscillations has to be guaranteed by the implementation.

The main ingredients of the proof of [6, Theorem 3] are the reliability of the error estimator, its discrete local efficiency, and the marking strategy introduced by Dörfler [14] ensuring an appropriate selection of edges and elements for refinement. The discrete local efficiency, however, strongly relies on the interior node property of the local mesh-refinement, and thus the validity of the convergence analysis is constrained by the refinement strategy.

We follow a different convergence approach, inspired by [12]: Our main result (Theorem 8, Corollary 9) states that the adaptive algorithm steered by \( \eta_\ell^2 = \varrho_\ell^2 + \text{osc}_\ell^2 \), i.e. steered by edge jumps plus data oscillations, leads to

\[
\Delta_{\ell+1} \leq \kappa \Delta_\ell
\]

for all \( \ell \in \mathbb{N} \),

with a weighted sum \( \Delta_\ell = \varepsilon_\ell + \gamma \mu_\ell^2 \) and with \( 0 < \gamma, \kappa < 1 \) being \( \ell \)-independent constants. Here, \( \mu_\ell \) is a second error estimator introduced below which is equivalent to \( \eta_\ell \). Therefore, elementary calculus proves \( \varepsilon_\ell \to 0 \) as \( \ell \to \infty \) without any further assumptions. Moreover, our result is fairly independent of the chosen mesh-refinement and does not need the interior node property as does the analysis of [6].

The first step for our proof of (1) is to observe that the Dörfler marking for \( \eta_\ell \) implies the Dörfler marking for the auxiliary error estimator \( \mu_\ell \). The latter is sufficient to show that the sequence of the estimators \( \mu_\ell \) is contractive in the sense that

\[
\mu_{\ell+1}^2 \leq g \mu_\ell^2 + C \| U_{\ell+1} - U_\ell \|^2
\]

for all \( \ell \in \mathbb{N} \),

where \( C > 0 \) and \( g \in (0,1) \) are certain \( \ell \)-independent constants and \( \| \cdot \| \) denotes the energy norm. To show this, we exploit the definition of the error estimator \( \mu_\ell \), the marking strategy used, and basic properties of the local mesh-refinement. In addition and contrary to [12], our elementary analysis avoids to dominate the data oscillations \( \text{osc}_\ell \) by the element residuals \( \| h_\ell f \|_{L^2(\Omega)} \) and thus is slightly more accurate.

1.3. Outline of Current Work. In Section 2, we formulate the continuous and discrete obstacle problem, stated as energy minimization problems. Moreover, we recall the error estimator \( \eta_\ell \) from [6] which is lateron used to steer our adaptive algorithm, and state its reliability (Proposition 2). In Section 3.1, we introduce an error estimator \( \mu_\ell \) which is equivalent to \( \eta_\ell \), namely

\[
\eta_\ell \leq \mu_\ell \leq 4 \eta_\ell
\]

We then recall the marking strategy and the local mesh-refinement used. As a consequence, we prove that the estimator \( \mu_\ell \) satisfies an estimator reduction property (Proposition 3), cf. (2). One major part of our proof is to show that the edge data oscillations are, in fact, contractive (Lemma 6). Finally, Section 3.2 states our version of the \( \eta_\ell \)-steered adaptive mesh-refining algorithm (Algorithm 7) and proves the contraction result (1). In particular,
the generated sequence of discrete solutions $U_\ell$ converges, in fact, to the continuous solution $u$ (Theorem 8).

2. Model Problem

2.1. Continuous Formulation of Model Problem. Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with polygonal boundary $\Gamma := \partial \Omega$. We define an obstacle on $\Omega$ by the affine function $\chi$ with $\chi \leq 0$ on $\partial \Omega$. By $\mathcal{A} \subset H^1_0(\Omega)$, we denote the set of admissible functions

$$\mathcal{A} = \{ v \in H^1_0(\Omega) : v \geq \chi \text{ a.e. in } \Omega \},$$

which is convex, closed, and non-empty. For given $f \in L^2(\Omega)$, we consider the energy functional

$$J(v) = \frac{1}{2} \langle \langle v, v \rangle \rangle - \langle f, v \rangle,$$

where the energy scalar product reads

$$\langle \langle u, v \rangle \rangle = \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \text{for all } u, v \in H^1_0(\Omega)$$

and where

$$\langle f, v \rangle = \int_{\Omega} fv \, dx$$

denotes the $L^2$-scalar product. By $\| \cdot \|$, we denote the energy norm on $H^1_0(\Omega)$ induced by $\langle \langle \cdot, \cdot \rangle \rangle$. The minimization problem then reads as follows: Find $u \in \mathcal{A}$ such that

$$J(u) = \min_{v \in \mathcal{A}} J(v).$$

The following well-known abstract lemma, found e.g. in [16, Theorem II.2.1], states unique solvability of this problem and equivalence to some variational inequality.

**Lemma 1.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. For any closed, convex, and non-empty subset $\mathcal{A}$ of $\mathcal{H}$ and any linear functional $f \in \mathcal{H}^*$, there is a unique minimizer $u \in \mathcal{A}$ of

$$J(u) = \min_{v \in \mathcal{A}} J(v),$$

where the energy functional reads

$$J(v) = \frac{1}{2} \langle \langle v, v \rangle \rangle - f(v).$$

This minimizer is equivalently characterized in terms of the following variational inequality: Find $u \in \mathcal{A}$ such that

$$\langle \langle u, u - v \rangle \rangle \leq f(u - v)$$

for all $v \in \mathcal{A}$. □
2.2. Conforming Discretization. For the numerical solution of (8), we consider conforming and shape regular triangulations \( T_\ell \) of \( \Omega \) and denote the standard \( P_1 \)-finite element space of globally continuous and piecewise affine functions by \( S^1(T_\ell) \). The finite dimensional minimization problem then reads as follows: Find \( U_\ell \in A_\ell := A \cap S^1(T_\ell) \) such that

\[
\mathcal{J}(U_\ell) = \min_{V_\ell \in A} \mathcal{J}(V_\ell).
\]

(12)

Note that \( A_\ell \) is a non-empty, convex, and closed subset of \( S^1(T_\ell) \). With the same arguments as for the continuous problem, (12) admits a unique solution \( U_\ell \in A_\ell \).

2.3. Reliable Error Estimator. Now, let \( u \in A \) denote the continuous solution of (8) and \( U_\ell \in A_\ell \) be the discrete solution of (12) for some fixed triangulation \( T_\ell \). To steer the adaptive mesh-refinement, we use some residual-based error estimator

\[
\eta_\ell^2 := \sum_{E \in \mathcal{E}_\ell} (\eta(E))^2 + \sum_{T \in T_{\ell}\cap \Gamma} \text{osc}(T)^2
\]

from [6]: First, \( \eta(E)^2 \) denotes the weighted \( L^2 \)-norms of the normal jump

\[
\eta(E)^2 := h_E \| \partial_n U_\ell \|_{L^2(E)}^2 \quad \text{for } E \in \mathcal{E}_\ell
\]

with \( h_E = \text{diam}(E) \) the length of \( E \) and \([·] \) the jump over an interior edge \( E = T^+ \cap T^- \in \mathcal{E}_\ell \). Second, \( \text{osc}(E)^2 \) denotes the data oscillations of \( f \) over \( E \)

\[
\text{osc}(E)^2 := |\Omega_{\ell,E}| \| f - f_{\Omega_{\ell,E}} \|_{L^2(\Omega_{\ell,E})}^2 \quad \text{for } E \in \mathcal{E}_\ell
\]

with \( \Omega_{\ell,E} = T^+ \cup T^- \) the patch associated with \( E \) and \( f_{\Omega_{\ell,E}} := (1/|\Omega_{\ell,E}|) \int_{\Omega_{\ell,E}} f \, dx \) the corresponding integral mean of \( f \). Finally, for elements \( T \in T_{\ell}\cap \Gamma \) at the boundary, \( \eta_\ell \) involves the weighted element residuals

\[
\text{osc}(T)^2 := |T| \| f \|_{L^2(T)}^2 \quad \text{for } T \in T_{\ell}.
\]

The following proposition has essentially been shown in [6], where \( \text{osc}(T) \) is, however, weighted by \( \text{diam}(T)^2 \sim |T| \). We will discuss this, up to shape regularity, equivalent definition lateron, cf. Corollary 9 in Section 3.2.

**Proposition 2.** The estimator \( \eta_\ell \) from (13) is reliable in the sense that there holds

\[
\frac{1}{2} \| u - U_\ell \|^2 \leq \mathcal{J}(U_\ell) - \mathcal{J}(u) \leq C_1 \eta_\ell^2.
\]

The constant \( C_1 > 0 \) depends only on \( \Omega \) and the shape of the elements in \( T_\ell \).
Figure 1. For each triangle $T \in \mathcal{T}$, there is one fixed reference edge, indicated by the double line (left, top). Refinement of $T$ is done by bisecting the reference edge, where its midpoint becomes a new node. The reference edges of the son triangles are opposite to this newest vertex (left, bottom). To avoid hanging nodes, one proceeds as follows: We assume that certain edges of $T$, but at least the reference edge, are marked for refinement (top). Using iterated newest vertex bisection, the element is then split into 2, 3, or 4 son triangles (bottom).

Proof. The upper bound is stated in [6, Theorem 1]. To see the lower bound, we use the variational inequality (11). For $v = U_\ell$, this gives

$$
\|u - U_\ell\|^2 = \langle \langle u, u - U_\ell \rangle \rangle + \|U_\ell\|^2 - \langle \langle f, U_\ell \rangle \rangle
\leq \left( \frac{1}{2} \|U_\ell\| - \langle f, U_\ell \rangle \right) - \left( \frac{1}{2} \|u\| - \langle f, u \rangle \right)
+ \left( \frac{1}{2} \|U_\ell\| - \langle \langle U_\ell, u \rangle \rangle + \frac{1}{2} \|u\|^2 \right)
= \mathcal{J}(U_\ell) - \mathcal{J}(u) + \frac{1}{2} \|u - U_\ell\|^2
$$

and concludes the proof. □

3. A Convergent Adaptive Algorithm

3.1. Estimator Reduction. First, we define a slight modification of the error estimator $\eta_\ell$ introduced above. Recall that $\mathcal{E}_\ell^*$ is the set of all edges of $\mathcal{T}_\ell$. For the interior edges $E \in \mathcal{E}_\ell$, the edge data oscillations $\text{osc}_\ell(E)$ are defined in (15). For the boundary edges, we define

$$
\text{osc}_\ell(E)^2 := \text{osc}_\ell(T)^2 \quad \text{for all } E \in \mathcal{E}_\ell^* := \mathcal{E}_\ell^* \setminus \mathcal{E}_\ell \text{ and } T \in \mathcal{T}_\ell^\Gamma \text{ with } E \subset \partial T,
$$

where $\text{osc}_\ell(T)$ is defined in (16). We now consider

$$
\mu_\ell^2 := \sum_{E \in \mathcal{E}_\ell} \eta_\ell(E)^2 + \sum_{E \in \mathcal{E}_\ell^*} \text{osc}_\ell(E)^2 + \sum_{T \in \mathcal{T}_\ell^\Gamma} \text{osc}_\ell(T)^2.
$$

By definition, there holds

$$
\eta_\ell^2 \leq \mu_\ell^2 \leq 4 \eta_\ell^2,
$$

since one element $T \in \mathcal{T}_\ell^\Gamma$ may have at most three boundary edges. In particular, this proves that the reliability estimate (17) also holds with $\eta_\ell$ replaced by $\mu_\ell$.

For the moment, let us use the local contributions of $\mu_\ell$ to steer some adaptive mesh-refining algorithm. For marking, we use the marking strategy introduced by Dörfler [14]. Contrary to [14, 18] and [6], we use $\mathcal{E}_\ell^* \cup \mathcal{T}_\ell^\Gamma$ as index set for the marking criterion instead of only $\mathcal{E}_\ell$, and we mark simultaneously for $\eta_\ell(E)$ and data oscillations $\text{osc}_\ell(E)$ resp. $\text{osc}_\ell(T)$:
Given some parameter $\theta \in (0, 1)$, we seek a set $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell^* \cup \mathcal{T}_{\ell, \Gamma}$ of usually minimal cardinality such that
\begin{equation}
\theta \mu^2_\ell \leq \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} \eta(E)^2 + \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} \text{osc}_\ell(E)^2 + \sum_{T \in \mathcal{T}_{\ell, \Gamma} \cap \mathcal{M}_\ell} \text{osc}_\ell(T)^2.
\end{equation}

For the mesh-refinement, we use newest-vertex bisection, where we mark all edges $E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell$ for refinement. For all elements $T \in \mathcal{T}_{\ell, \Gamma} \cap \mathcal{M}_\ell$, we mark their reference edge. The refinement rules are shown in Figure 1, and the reader is also referred to [23, Chapter 4]. Besides uniform shape regularity of $\mathcal{T}_{\ell+1}$, there is a certain decay of the mesh-widths:

- Marked edges $E \in \mathcal{M}_\ell$ are split into two edges $E', E'' \in \mathcal{E}_{\ell+1}$ of half length.
- If at least one edge $E$ of an element $T \in \mathcal{T}$ is marked, $T$ is refined into up to four son elements $T' \in \mathcal{T}_{\ell+1}$ with $|T|/4 \leq |T'| \leq |T|/2$, cf. Figure 1.

These observations are essential to prove the following result.

**Proposition 3.** Suppose that the set $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell^* \cup \mathcal{T}_{\ell, \Gamma}$ satisfies (21) and that marked edges and marked elements are refined as stated before. Then, there holds
\begin{equation}
\mu^2_{\ell+1} \leq q \mu^2_\ell + C_2 \|U_{\ell+1} - U_\ell\|^2
\end{equation}
with some contraction constant $q \in (0, 1)$ which depends only on $\theta \in (0, 1)$. The constant $C_2 > 0$ additionally depends on the shape of the elements in $\mathcal{T}_0$.

For the convenience of the reader, the proof of Proposition 3 is split into three lemmata which estimate the decay of the different contributions of $\mu_\ell$ if the mesh $\mathcal{T}_\ell$ is locally refined.

**Lemma 4.** According to the refinement of marked elements $T \in \mathcal{T}_{\ell, \Gamma} \cap \mathcal{M}_\ell$, there holds
\begin{equation}
\sum_{T' \in \mathcal{T}_{\ell+1, \Gamma}} \text{osc}_{\ell+1}(T')^2 \leq \sum_{T \in \mathcal{T}_{\ell, \Gamma}} \text{osc}_{\ell}(T)^2 - \frac{1}{2} \sum_{T \in \mathcal{T}_{\ell, \Gamma} \cap \mathcal{M}_\ell} \text{osc}_{\ell}(T)^2.
\end{equation}

**Proof.** We define the set $\mathcal{M}_{T, \ell} := \{T' \in \mathcal{T}_{\ell+1} : \exists T \in \mathcal{T}_{\ell, \Gamma} \cap \mathcal{M}_\ell \land T' \subseteq T\}$ containing all elements obtained by refinement of marked elements. Then,
\begin{align*}
\sum_{T' \in \mathcal{T}_{\ell+1, \Gamma}} \text{osc}_{\ell+1}(T')^2 &\leq \sum_{T' \in \mathcal{T}_{\ell+1, \Gamma} \setminus \mathcal{M}_{T, \ell}} \|T'\|_{L^2(T')}^2 + \sum_{T' \in \mathcal{M}_{T, \ell}} \|T'\|_{L^2(T')}^2 \\
&\leq \sum_{T \in \mathcal{T}_{\ell, \Gamma} \setminus \mathcal{M}_\ell} \|T\|_{L^2(T)}^2 + \frac{1}{2} \sum_{T \in \mathcal{T}_{\ell, \Gamma} \cap \mathcal{M}_\ell} \|T\|_{L^2(T)}^2 \\
&= \sum_{T \in \mathcal{T}_{\ell, \Gamma} \setminus \mathcal{M}_\ell} \text{osc}_{\ell}(T)^2 + \frac{1}{2} \sum_{T \in \mathcal{T}_{\ell, \Gamma} \cap \mathcal{M}_\ell} \text{osc}_{\ell}(T)^2,
\end{align*}
where we have used that each father $T \in \mathcal{T}_{\ell}$ is the disjoint union of its sons $T' \in \mathcal{T}_{\ell+1}$. \hfill \Box

**Lemma 5.** According to the refinement of marked edges $E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell$, there holds
\begin{equation}
\sum_{E' \in \mathcal{E}_{\ell+1}} h_{E'} \|\partial_n U_{\ell}\|_{L^2(E')}^2 \leq \sum_{E \in \mathcal{E}_\ell} \eta_{\ell}(E)^2 - \frac{1}{2} \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} \eta_{\ell}(E)^2.
\end{equation}
Proof. We define the set \( \mathcal{M}_{\ell, t} := \{ E' \in \mathcal{E}_{t+1} : \exists E \in \mathcal{E}_t \cap \mathcal{M}_t \quad E' \subseteq E \} \) containing all edges obtained by refinement of marked edges. Then, one observes

\[
\sum_{E' \in \mathcal{E}_{t+1}} h_{E'} \| [\partial_n U_{t}] \|^2_{L^2(E')} = \sum_{E' \in \mathcal{E}_{t+1}} h_{E'} \| [\partial_n U_{t}] \|^2_{L^2(E')} + \sum_{E' \in \mathcal{E}_t \cap \mathcal{M}_t} h_{E'} \| [\partial_n U_{t}] \|^2_{L^2(E')}
\]

\[
\leq \sum_{E \in \mathcal{E}_t \setminus \mathcal{M}_t} h_{E} \| [\partial_n U_{t}] \|^2_{L^2(E)} + \frac{1}{2} \sum_{E \in \mathcal{E}_t \cap \mathcal{M}_t} h_{E} \| [\partial_n U_{t}] \|^2_{L^2(E)}
\]

\[
= \sum_{E \in \mathcal{E}_t \setminus \mathcal{M}_t} \eta_\ell(E)^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_t \cap \mathcal{M}_t} \eta_\ell(E)^2,
\]

where we have used that the jump \([\partial_n U_{t}]\) is zero on all edges \(E' \in \mathcal{E}_{t+1}\) which lie inside an element \(T \in \mathcal{T}_t\).

\[\square\]

Lemma 6. According to the refinement of marked edges \(E \in \mathcal{E}_t^* \cap \mathcal{M}_t\), there holds

\[
(25) \quad \sum_{E' \in \mathcal{E}_{t+1}} \text{osc}_{t+1}(E')^2 \leq \sum_{E \in \mathcal{E}_t^*} \text{osc}_{t}(E)^2 - \frac{1}{4} \sum_{E \in \mathcal{E}_t \cap \mathcal{M}_t} \text{osc}_{t}(E)^2.
\]

Proof. The proof of (25) is considerably longer than for the prior contributions in (23)–(24). The reason is that local mesh-refinement leads to additional edges inside of refined elements \(T \in \mathcal{T}_t\). This provides additional contributions which have to be controlled. To that end, we define for any edge \(E \in \mathcal{E}_t^* \cap \mathcal{M}_t\) and any element \(T \in \mathcal{T}_t\) with \(|T \cap \Omega_{t+1,E}| > 0\) the quantity

\[
\text{osc}_{t+1}(E|T)^2 = |\Omega_{t+1,E}| \| f - f_{\Omega_{t+1,E}} \|^2_{L^2(\Omega_{t+1,E} \cap T)},
\]

where we let \(\Omega_{t+1,E} := T'\) and \(f_{\Omega_{t+1,E}} := 0\) for a boundary edge \(E \in \mathcal{E}_{t+1, \Gamma} = \mathcal{E}_t^* \cap \mathcal{M}_t\) and \(T' \in \mathcal{T}_{t+1, \Gamma}\) the unique element with \(E \subset \partial T'\). Throughout the proof, \(f_\omega = (1/|\omega|) \int_\omega f \, dx\) denotes the integral mean of \(f\) over the measurable set \(\omega\). Note that the \(L^2\)-best approximation property of \(f_\omega\) yields

\[
\| f - f_\omega \|_{L^2(\omega)} \leq \| f - \alpha \|_{L^2(\omega)} \quad \text{for all } \alpha \in \mathbb{R},
\]

whence

\[
\| f - f_\omega \|_{L^2(\omega)} \leq \| f - f_\hat{\omega} \|_{L^2(\hat{\omega})} \quad \text{for all measurable sets } \hat{\omega} \supseteq \omega.
\]

For each element \(A \in \mathcal{T}_t\), only four cases occur: \(A\) is either not refined, i.e. \(A \in \mathcal{T}_t \cap \mathcal{T}_{t+1}\), or refined by either one, two, or three bisections, cf. Figure 2.
First, assume that an element \( A \in \mathcal{T}_l \cap \mathcal{T}_{l+1} \) is not refined. Let \( b, c, d \in \mathcal{E}_l \cap \mathcal{E}_{l+1} \) denote its three edges. We then define

\[
o_{\ell}(b|A)^2 = \text{osc}_{\ell+1}(b|A)^2, \quad o_{\ell}(c|A)^2 = \text{osc}_{\ell+1}(c|A)^2, \quad \text{and} \quad o_{\ell}(d|A)^2 = \text{osc}_{\ell+1}(d|A)^2.
\]

By definition, we obtain

\[
\sum_{E \in \mathcal{E}_{l+1}^* \setminus \{|A \cap \partial A_{l+1,\ell} > 0\}} \text{osc}_{\ell+1}(E|A)^2 \leq \sum_{E \in \mathcal{E}_l \setminus \partial A} o_{\ell}(E|A)^2
\]
even with equality.

Second, assume that an element \( A \in \mathcal{T}_l \) with edges \( b, c, d \in \mathcal{E}_l^* \) is refined by one bisection, cf. Figure 2, where the edge \( c \) is split into \( c_1, c_2 \in \mathcal{E}_{l+1}^* \) and where one additional edge \( a \in \mathcal{E}_{l+1}^* \) is created. Moreover, \( A \) is split into elements \( A_1, A_2 \in \mathcal{T}_{l+1} \) with area \(|A_1| = |A_2| = |A|/2\). Let \( B, C, D \in \mathcal{T}_l \) be the neighbours of \( A \) along the edges \( b, c, d \in \mathcal{E}_l^* \), where for instance \( B = \emptyset \) if \( b \in \mathcal{E}_l^* \setminus \mathcal{E}_l \) is a boundary edge. Then,

\[
\sum_{E \in \mathcal{E}_{l+1}^* \setminus \{|A \cap \partial A_{l+1,\ell} > 0\}} \text{osc}_{\ell+1}(E|A)^2
= \text{osc}_{\ell+1}(c_1|A)^2 + \text{osc}_{\ell+1}(c_2|A)^2 + \text{osc}_{\ell+1}(b|A)^2 + \text{osc}_{\ell+1}(d|A)^2 + \text{osc}_{\ell+1}(a|A)^2
= (|\Omega_{\ell+1,c_1} \cap C| + |A|/2) \|f - f_{\Omega_{\ell+1,c_1}}\|_{L^2(A_1)}^2 + (|\Omega_{\ell+1,c_2} \cap C| + |A|/2) \|f - f_{\Omega_{\ell+1,c_2}}\|_{L^2(A_2)}^2
+ (|\Omega_{\ell+1,b} \cap B| + |A|/2) \|f - f_{\Omega_{\ell+1,b}}\|_{L^2(A_1)}^2 + (|\Omega_{\ell+1,d} \cap D| + |A|/2) \|f - f_{\Omega_{\ell+1,d}}\|_{L^2(A_2)}^2
+ |A| \|f - f_A\|_{L^2(A)}^2.
\]

The last term belongs to the new edge \( a \in \mathcal{E}_{l+1}^* \). We define

\[
o_{\ell}(b|A)^2 = (|\Omega_{\ell+1,b} \cap B| + |A|/2) \|f - f_{\Omega_{\ell+1,b}}\|_{L^2(A_1)}^2 + (|A|/2) \|f - f_A\|_{L^2(A)}^2;
\]

\[
o_{\ell}(c|A)^2 = (|\Omega_{\ell+1,c_1} \cap C| + |A|/2) \|f - f_{\Omega_{\ell+1,c_1}}\|_{L^2(A_1)}^2
+ (|\Omega_{\ell+1,c_2} \cap C| + |A|/2) \|f - f_{\Omega_{\ell+1,c_2}}\|_{L^2(A_2)}^2;
\]

\[
o_{\ell}(d|A)^2 = (|\Omega_{\ell+1,d} \cap D| + |A|/2) \|f - f_{\Omega_{\ell+1,d}}\|_{L^2(A_2)}^2 + (|A|/2) \|f - f_A\|_{L^2(A)}^2;
\]

and observe that, by definition, (26) holds with equality.

Third, assume that an element \( A \in \mathcal{T}_l \) with edges \( b, c, d \in \mathcal{E}_l^* \) is refined by two bisections, cf. Figure 2, where the edges \( c, d \) are split into \( c_1, c_2, d_1, d_2 \in \mathcal{E}_{l+1}^* \), respectively, and two new edges \( a_1, a_2 \in \mathcal{E}_{l+1}^* \) are created. Moreover, \( A \) is split into elements \( A_1, A_2, A_3 \in \mathcal{T}_{l+1} \) with area \(|A_1| = |A|/2\) and \(|A_2| = |A_3| = |A|/4\). Let \( b, c, d \) and \( B, C, D \) be the same as in the
previous case. Then,

\[
\sum_{E \in \mathcal{E}_{\ell+1}^*} \text{osc}_{\ell+1}(E|A)^2 \\
= \text{osc}_{\ell+1}(c_1|A)^2 + \text{osc}_{\ell+1}(c_2|A)^2 + \text{osc}_{\ell+1}(d_1|A)^2 + \text{osc}_{\ell+1}(d_2|A)^2 + \text{osc}_{\ell+1}(b|A)^2 \\
+ \text{osc}_{\ell+1}(a_1|A)^2 + \text{osc}_{\ell+1}(a_2|A)^2 \\
= \left( |\Omega_{\ell+1,c_1} \cap C| + |A|/2 \right) \|f - \Omega_{\ell+1,c_1}\|_{L^2(A)}^2 + \left( |\Omega_{\ell+1,c_2} \cap C| + |A|/4 \right) \|f - \Omega_{\ell+1,c_2}\|_{L^2(A)}^2 \\
+ \left( |\Omega_{\ell+1,d_1} \cap D| + |A|/4 \right) \|f - \Omega_{\ell+1,d_1}\|_{L^2(A)}^2 + \left( |\Omega_{\ell+1,d_2} \cap D| + |A|/4 \right) \|f - \Omega_{\ell+1,d_2}\|_{L^2(A)}^2 \\
+ \left( |\Omega_{\ell+1,b} \cap B| + |A|/2 \right) \|f - \Omega_{\ell+1,b}\|_{L^2(A)}^2 \\
+ \left( |A|/2 \right) \|f - \Omega_{\ell+1,d_1}\|_{L^2(A)}^2 + \left( |A|/4 \right) \|f - \Omega_{\ell+1,d_2}\|_{L^2(A)}^2. \\
\]

The last two terms belong to the new edges \(a_1, a_2 \in \mathcal{E}_{\ell+1}^* \) and are roughly estimated by

\[
\left( |A|/4 \right) \|f - \Omega_{\ell+1,d_1}\|_{L^2(A)}^2 + \left( |A|/2 \right) \|f - \Omega_{\ell+1,d_2}\|_{L^2(A)}^2 \leq \left( |A|/4 \right) \|f - \Omega_{\ell+1}\|_{L^2(A)}^2. \\
\]

We define

\[
o_4(b|A)^2 = \left( |\Omega_{\ell+1,b} \cap B| + |A|/2 \right) \|f - \Omega_{\ell+1,b}\|_{L^2(A)}^2 + \left( |A|/2 \right) \|f - \Omega_{\ell+1}\|_{L^2(A)}^2, \\
o_4(c|A)^2 = \left( |\Omega_{\ell+1,c_1} \cap C| + |A|/2 \right) \|f - \Omega_{\ell+1,c_1}\|_{L^2(A)}^2 + \left( |A|/2 \right) \|f - \Omega_{\ell+1}\|_{L^2(A)}^2 + \left( |\Omega_{\ell+1,c_2} \cap C| + |A|/4 \right) \|f - \Omega_{\ell+1,c_2}\|_{L^2(A)}^2 + \left( |A|/4 \right) \|f - \Omega_{\ell+1}\|_{L^2(A)}^2, \\
o_4(d|A)^2 = \left( |\Omega_{\ell+1,d_1} \cap D| + |A|/4 \right) \|f - \Omega_{\ell+1,d_1}\|_{L^2(A)}^2 + \left( |A|/4 \right) \|f - \Omega_{\ell+1,d_1}\|_{L^2(A)}^2 + \left( |\Omega_{\ell+1,d_2} \cap D| + |A|/4 \right) \|f - \Omega_{\ell+1,d_2}\|_{L^2(A)}^2 + \left( |A|/2 \right) \|f - \Omega_{\ell+1}\|_{L^2(A)}^2. \\
\]

By definition, we again obtain (26).

Fourth, assume that an element \( A \in \mathcal{T}_\ell \) with edges \( b, c, d \in \mathcal{E}_\ell^* \) is refined by three bisections, cf. Figure 2, where the edges \( b, c, d \) are split into \( b_1, b_2, c_1, c_2, d_1, d_2 \in \mathcal{E}_{\ell+1}^* \), respectively, and three new edges \( a_1, a_2, a_3 \in \mathcal{E}_{\ell+1}^* \) are created. Moreover, \( A \) is split into elements \( A_1, A_2, A_3, A_4 \in \mathcal{T}_{\ell+1} \) with area \( |A_j| = |A|/4 \). For \( b, c, d \) and \( B, C, D \), we use the notation from the previous cases. Then,

\[
\sum_{E \in \mathcal{E}_{\ell+1}^*} \text{osc}_{\ell+1}(E|A)^2 \\
= \text{osc}_{\ell+1}(b_1|A)^2 + \text{osc}_{\ell+1}(b_2|A)^2 + \text{osc}_{\ell+1}(c_1|A)^2 + \text{osc}_{\ell+1}(c_2|A)^2 + \text{osc}_{\ell+1}(d_1|A)^2 + \text{osc}_{\ell+1}(d_2|A)^2 + \text{osc}_{\ell+1}(a_1|A)^2 + \text{osc}_{\ell+1}(a_2|A)^2 + \text{osc}_{\ell+1}(a_3|A)^2 \\
\leq \left( |\Omega_{\ell+1,b_1} \cap B| + |A|/4 \right) \|f - \Omega_{\ell+1,b_1}\|_{L^2(A)}^2 + \left( |\Omega_{\ell+1,b_2} \cap B| + |A|/4 \right) \|f - \Omega_{\ell+1,b_2}\|_{L^2(A)}^2 + \left( |\Omega_{\ell+1,c_1} \cap C| + |A|/4 \right) \|f - \Omega_{\ell+1,c_1}\|_{L^2(A)}^2 + \left( |\Omega_{\ell+1,c_2} \cap C| + |A|/4 \right) \|f - \Omega_{\ell+1,c_2}\|_{L^2(A)}^2 + \left( |\Omega_{\ell+1,d_1} \cap D| + |A|/4 \right) \|f - \Omega_{\ell+1,d_1}\|_{L^2(A)}^2 + \left( |\Omega_{\ell+1,d_2} \cap D| + |A|/4 \right) \|f - \Omega_{\ell+1,d_2}\|_{L^2(A)}^2 + \left( |A|/2 \right) \|f - \Omega_{\ell+1}\|_{L^2(A)}^2. \\
\]
Defining

\[
\begin{align*}
\phi_t(b|A)^2 &= (|\Omega_{t+1,b} \cap B| + |A|/4) \| f - f_{\Omega_{t+1,b}} \|_{L^2(A_b)}^2 \\
&
+ (|\Omega_{t+1,b} \cap B| + |A|/4) \| f - f_{\Omega_{t+1,b}} \|_{L^2(A)}^2 + (|A|/2) \| f - f_A \|_{L^2(A)}^2 \\
\phi_t(c|A)^2 &= (|\Omega_{t+1,c} \cap C| + |A|/4) \| f - f_{\Omega_{t+1,c}} \|_{L^2(A)}^2 \\
&
+ (|\Omega_{t+1,c} \cap C| + |A|/4) \| f - f_{\Omega_{t+1,c}} \|_{L^2(A)}^2 + (|A|/2) \| f - f_A \|_{L^2(A)}^2 \\
\phi_t(d|A)^2 &= (|\Omega_{t+1,d} \cap D| + |A|/4) \| f - f_{\Omega_{t+1,d}} \|_{L^2(A)}^2 \\
&
+ (|\Omega_{t+1,d} \cap D| + |A|/4) \| f - f_{\Omega_{t+1,d}} \|_{L^2(A)}^2 + (|A|/2) \| f - f_A \|_{L^2(A)}^2,
\end{align*}
\]

we again guarantee (26).

Now, it only remains to show that for non-refined edges holds

\[
(27) \quad \sum_{T \in \mathcal{T}_e} \phi_t(E|T)^2 \leq \text{osc}_t(E)^2 \quad \text{for all } E \in \mathcal{E}_t^* \cap \mathcal{E}_{t+1}^*,
\]

whereas for edges which are refined, there holds

\[
(28) \quad \sum_{T \in \mathcal{T}_e} \phi_t(E|T)^2 \leq 3 \frac{\text{osc}_t(E)^2}{4} \quad \text{for all } E \in \mathcal{E}_t^* \setminus \mathcal{E}_{t+1}^*.
\]

Of course, there are quite some cases to be considered. Since all follow by direct calculation, we only consider some particular examples shown in Figure 3, while we refer to [21] for the consideration of all possible cases.

We first consider \( b := A \cap B \in \mathcal{E}_e \). According to our definitions, there holds

\[
\begin{align*}
\phi_t(b|B)^2 &= (|A|/2 + |B|) \| f - f_{A \cup B} \|_{L^2(B)}^2, \\
\phi_t(b|A)^2 &= (|B| + |A|/2) \| f - f_{A \cup B} \|_{L^2(A)}^2 + (|A|/2) \| f - f_A \|_{L^2(A)}^2.
\end{align*}
\]

This implies

\[
\begin{align*}
\phi_t(b|A)^2 + \phi_t(b|B)^2 &= (|A|/2 + |B|) \| f - f_{A \cup B} \|_{L^2(A \cup B)}^2 + (|A|/2) \| f - f_A \|_{L^2(A)}^2 \\
&\leq (|A|/2 + |B|) \| f - f_{A \cup B} \|_{L^2(A \cup B)}^2 + (|A|/2) \| f - f_{A \cup B} \|_{L^2(A \cup B)}^2 \\
&= \text{osc}_t(b)^2.
\end{align*}
\]
Next, we consider $d := A \cap D \in \mathcal{E}_\ell$. We have
\[
o_{\ell}(d|D)^2 = (|A|/4 + |D|/2) \| f - f_{A_2 \cup D_1} \|_{L^2(D_1)}^2 + (|A|/4 + |D|/2) \| f - f_{A_3 \cup D_2} \|_{L^2(D_2)},
\]
\[
o_{\ell}(d|A)^2 = (|D|/2 + |A|/4) \| f - f_{A_2 \cup D_1} \|_{L^2(A_2)}^2 + (|D|/2 + |A|/4) \| f - f_{A_3 \cup D_2} \|_{L^2(A_3)}^2 \]
+ (|A|/2) \| f - f_A \|_{L^2(A)}^2.
\]
This implies
\[
o_{\ell}(d|A)^2 + o_{\ell}(d|B)^2 = (|A|/4 + |D|/2) \| f - f_{A_2 \cup D_1} \|_{L^2(A_2 \cup D_1)}^2 \]
+ (|A|/4 + |D|/2) \| f - f_{A_3 \cup D_2} \|_{L^2(A_3 \cup D_2)}^2 \]
+ (|A|/2) \| f - f_{A_1 \cup D} \|_{L^2(A_1 \cup D)}^2 \]
\leq (|A|/4 + |D|/2) \| f - f_{A_1 \cup D} \|_{L^2(A_1 \cup D)}^2 \]
\leq \frac{3}{4} \text{osc}_\ell(d)^2.
\]
Finally, we consider the boundary edge $c := A \cap \Gamma \in \mathcal{E}_\ell^\ast$. In this case, there holds
\[
o_{\ell}(c|A)^2 = (|A|/2) \| f \|_{L^2(A_1)}^2 + (|A|/4) \| f \|_{L^2(A_3)}^2 + (|A|/4) \| f - f_A \|_{L^2(A)}^2 \leq \frac{3}{4} \text{osc}_\ell(c)^2,
\]
and we also observe the contraction property.

Having obtained (27)–(28), we may proceed as follows: We note that (26) provides
\[
\sum_{E \in \mathcal{E}_{\ell+1}} \text{osc}_{\ell+1}(E)^2 \leq \sum_{T \in \mathcal{T}_\ell} \sum_{E \in \mathcal{E}_{\ell+1}^{\ast} \cap \mathcal{M}_{\ell} \cap \mathcal{E}_{\ell}^{\ast}} \text{osc}_{\ell}(E|T)^2 \leq \sum_{T \in \mathcal{T}_\ell} \sum_{E \in \mathcal{E}_{\ell+1}^{\ast} \cap \mathcal{M}_{\ell} \cap \mathcal{E}_{\ell}^{\ast}} \text{osc}_{\ell}(E|T)^2 \leq \sum_{E \in \mathcal{E}_{\ell+1}^{\ast} \cap \mathcal{M}_{\ell} \cap \mathcal{E}_{\ell}^{\ast}} \text{osc}_{\ell}(E)^2.
\]
Therefore, (27)–(28) show
\[
\sum_{E \in \mathcal{E}_{\ell+1}} \text{osc}_{\ell+1}(E)^2 \leq \sum_{E \in \mathcal{E}_{\ell+1} \cap \mathcal{M}_{\ell} \cap \mathcal{E}_{\ell+1}} \text{osc}_{\ell}(E)^2 + \frac{3}{4} \sum_{E \in \mathcal{E}_{\ell+1} \setminus \mathcal{M}_{\ell} \cap \mathcal{E}_{\ell+1}} \text{osc}_{\ell}(E)^2 \]
\[
= \sum_{E \in \mathcal{E}_{\ell+1}} \text{osc}_{\ell}(E)^2 - \frac{1}{4} \sum_{E \in \mathcal{E}_{\ell+1} \setminus \mathcal{M}_{\ell} \cap \mathcal{E}_{\ell+1}} \text{osc}_{\ell}(E)^2.
\]
Observing that $\mathcal{E}_{\ell+1} \setminus \mathcal{M}_{\ell} \subseteq \mathcal{E}_{\ell+1} \setminus \mathcal{E}_{\ell+1}^{\ast}$, we conclude the proof. \hfill \square

**Proof of Proposition 3.** First, the triangle inequality in the sequence space $\ell_2$ proves
\[
\mu_{\ell+1} = \left( \sum_{T' \in \mathcal{T}_{\ell+1}, T'} \text{osc}_{\ell+1}(T')^2 + \sum_{E' \in \mathcal{E}_{\ell+1}^{\ast}, E'} \text{osc}_{\ell+1}(E')^2 + \sum_{E' \in \mathcal{E}_{\ell+1}^{\ast}, E'} h_{E'} \||\partial_n U_{\ell+1}||_{L^2(E')}^2 \right)^{1/2} \]
\[
\leq \left( \sum_{T' \in \mathcal{T}_{\ell+1}, T'} \text{osc}_{\ell+1}(T')^2 + \sum_{E' \in \mathcal{E}_{\ell+1}^{\ast}, E'} \text{osc}_{\ell+1}(E')^2 + \sum_{E' \in \mathcal{E}_{\ell+1}^{\ast}, E'} h_{E'} \||\partial_n U_{\ell+1}||_{L^2(E')}^2 \right)^{1/2} \]
\[
+ \left( \sum_{E' \in \mathcal{E}_{\ell+1}^{\ast}, E'} h_{E'} \||\partial_n (U_{\ell+1} - U_\ell)||_{L^2(E')}^2 \right)^{1/2}.
\]
In particular, the Young inequality proves for arbitrary \( \delta > 0 \)
\[
\mu_{\ell+1}^2 \leq (1 + \delta) \left( \sum_{T \in T_{\ell+1, \Gamma}} \text{osc}_{\ell+1}(T')^2 + \sum_{E' \in E_{\ell+1}'} \text{osc}_{\ell+1}(E')^2 + \sum_{E' \in E_{\ell+1}'} h_{E'} \| \partial_n U_{\ell} \|_{L^2(E')}^2 \right) \\
+ (1 + \delta^{-1}) \sum_{E' \in E_{\ell+1}'} h_{E'} \| \partial_n (U_{\ell+1} - U_{\ell}) \|_{L^2(E')}^2.
\] (29)

Second, we use the estimates (23), (24), and (25) to see
\[
\sum_{T \in T_{\ell+1, \Gamma}} \text{osc}_{\ell+1}(T')^2 + \sum_{E' \in E_{\ell+1}'} \text{osc}_{\ell+1}(E')^2 + \sum_{E' \in E_{\ell+1}'} h_{E'} \| \partial_n U_{\ell} \|_{L^2(E')}^2 \\
\leq \sum_{T \in T_{\ell, \Gamma}} \text{osc}_{\ell}(T)^2 + \sum_{E \in E_{\ell}^*} \text{osc}_{\ell}(E)^2 + \sum_{E' \in E_{\ell}'} h_{E} \| \partial_n U_{\ell} \|_{L^2(E)}^2 \\
- \frac{1}{2} \sum_{T \in T_{\ell, \Gamma} \cap M_{\ell}} \text{osc}_{\ell}(T)^2 - \frac{1}{4} \sum_{E \in E_{\ell}^* \cap M_{\ell}} \text{osc}_{\ell}(E)^2 - \frac{1}{2} \sum_{E' \in E_{\ell}'} h_{E} \| \partial_n U_{\ell} \|_{L^2(E)}^2 \\
\leq \mu_{\ell}^2 - \frac{1}{4} \left( \sum_{T \in T_{\ell, \Gamma} \cap M_{\ell}} \text{osc}_{\ell}(T)^2 + \sum_{E \in E_{\ell}^* \cap M_{\ell}} \text{osc}_{\ell}(E)^2 + \sum_{E' \in E_{\ell}'} h_{E} \| \partial_n U_{\ell} \|_{L^2(E)}^2 \right).
\]

Third, the Dörfler marking (21) yields
\[
\sum_{T \in T_{\ell+1, \Gamma}} \text{osc}_{\ell+1}(T')^2 + \sum_{E' \in E_{\ell+1}'} \text{osc}_{\ell+1}(E')^2 + \sum_{E' \in E_{\ell+1}'} h_{E'} \| \partial_n U_{\ell} \|_{L^2(E')}^2 \\
\leq (1 - \theta/4) \mu_{\ell}^2.
\]

Fourth, according to uniform shape regularity of the generated family \((T_{\ell})_{\ell \in \mathbb{N}}\), there holds
\[
\sum_{E' \in E_{\ell+1}'} h_{E'} \| \partial_n (U_{\ell+1} - U_{\ell}) \|_{L^2(E')}^2 \lesssim \| \nabla (U_{\ell+1} - U_{\ell}) \|_{L^2(\Omega)}^2 = \| U_{\ell+1} - U_{\ell} \|^2
\]
Plugging the last two estimates into (29), we prove (22), where we finally choose \( \delta > 0 \) sufficiently small to guarantee \( q := (1 + \delta)(1 - \theta/4) < 1 \).

**Remark.** Clearly, Lemma 4 and 5 also hold if certain elements \( T \in T_{\ell} \) are refined by five bisections, as is done in [6], or by so-called red-refinement. We refer to [23, Chapter 4] for details on different local mesh-refinements.

The same holds for Lemma 6 as well. In case of bisection-refinement this is easily seen as follows: We theoretically build an intermediate mesh \( \tilde{T}_{\ell+1} \), where elements marked for bisection are refined by three bisections. Then, Lemma 6 applies for the refinement from \( T_{\ell} \) to \( T_{\ell+1} \). To obtain \( T_{\ell+1} \), certain elements \( T' \in \tilde{T}_{\ell+1} \) have to be refined by bisection. Since (25) states, in particular, monotone decay of the oscillations, we conclude
\[
\text{osc}_{\ell+1}^2 \leq \tilde{\text{osc}}_{\ell+1}^2 \leq \sum_{E \in E_{\ell}^*} \text{osc}_{\ell}(E)^2 - \frac{1}{4} \sum_{E \in E_{\ell}^* \cap M_{\ell}} \text{osc}_{\ell}(E)^2.
\]

Finally, if certain elements of \( T_{\ell} \) are refined by red-refinement, the proof of (25) is obtained by similar elementary calculations as in the proof of Lemma 6. We refer to [21] for details.
Remark. Under further assumptions, the same analysis can be used to prove the estimator reduction of Proposition 3 for \( \eta_\ell \) instead of \( \mu_\ell \). For given \( \theta \in (0,1) \), let \( \mathcal{M}_\ell \subseteq \mathcal{E}_\ell \cup \mathcal{T}_{\ell,\Gamma} \) satisfy
\[
\theta \eta_\ell^2 \leq \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} (\eta(E)^2 + \text{osc}_E(T)^2) + \sum_{T \in \mathcal{T}_{\ell,\Gamma} \cap \mathcal{M}_\ell} \text{osc}_T(T)^2.
\]
First, suppose that each element \( T \in \mathcal{T}_\ell \) has at most one edge on the boundary \( \Gamma \). This can be guaranteed, for instance, by certain refinement of elements \( T \in \mathcal{T}_0 \) in the initial mesh. Let \( \mathcal{T}_{\ell,\Gamma}^* := \{ T \in \mathcal{T}_{\ell,\Gamma} : |T \cap \Gamma| = 0 \} \) denote the set of elements, which have some node on \( \Gamma \), but no edge. Note that —according to the assumption on the mesh— there is a one-to-one correspondence of the elements with edge on the boundary \( \mathcal{T}_{\ell,\Gamma} \setminus \mathcal{T}_{\ell,\Gamma}^* \), and the boundary edges \( \mathcal{E}_{\ell,\Gamma} = \mathcal{E}_\ell \setminus \mathcal{E}_\ell^* \). Therefore, the error estimator \( \eta_\ell \) can be rewritten in the form
\[
\eta_\ell^2 = \sum_{E \in \mathcal{E}_\ell^*} \eta(E)^2 + \sum_{E \in \mathcal{E}_\ell^*} \text{osc}_E(T)^2 + \sum_{T \in \mathcal{T}_{\ell,\Gamma}^*} \text{osc}_T(T)^2,
\]
and the set \( \mathcal{M}_\ell \) can be interpreted as some set \( \mathcal{M}_\ell^* \subseteq \mathcal{E}_\ell^* \cup \mathcal{T}_{\ell,\Gamma}^* \) with
\[
\theta \eta_\ell^2 \leq \sum_{E \in \mathcal{E}_\ell^* \cap \mathcal{M}_\ell^*} \eta(E)^2 + \sum_{E \in \mathcal{E}_\ell^* \cap \mathcal{M}_\ell^*} \text{osc}_E(T)^2 + \sum_{T \in \mathcal{T}_{\ell,\Gamma}^* \cap \mathcal{M}_\ell^*} \text{osc}_T(T)^2.
\]
However, if \( T \in \mathcal{T}_{\ell,\Gamma} \cap \mathcal{M}_\ell \) with \( |T \cap \Gamma| > 0 \) is marked for refinement, so far, only its reference edge is refined. Therefore, the actual refinement rule does not guarantee that the edge \( E = T \cap \Gamma \in \mathcal{E}_\ell^* \setminus \mathcal{E}_\ell \) is split. Put differently, the actual refinement rule does not guarantee that all edges \( E \in \mathcal{E}_\ell^* \cap \mathcal{M}_\ell^* \) are refined. One remedy is that marked elements \( T \in \mathcal{T}_{\ell,\Gamma} \cap \mathcal{M}_\ell \) are refined (at least) by three bisections (or by red-refinement). With this convention, all edges \( E \in \mathcal{E}_\ell^* \cap \mathcal{M}_\ell^* \) are refined. Therefore, Lemma 5 and Lemma 6 apply. Moreover, the proof of Lemma 4 carries over if \( \mathcal{T}_{\ell,\Gamma} \) is replaced by \( \mathcal{T}_{\ell,\Gamma}^* \). Altogether, this proves that the Dörfler marking (30) and the extended refinement strategy guarantee
\[
\eta_{\ell+1}^2 \leq q \eta_\ell^2 + C_2 \| U_{\ell+1} - U_\ell \|^2
\]
with some contraction constant \( q \in (0,1) \) which depends only on \( \theta \in (0,1) \) and the same constant \( C_2 > 0 \) as in Proposition 3. \( \square \)

3.2. Convergent Adaptive Algorithm. In this section, we formally state our version of the adaptive algorithm and prove that it generates a sequence of discrete solutions \( U_\ell \) which converge to the continuous minimizer \( u \).

Algorithm 7. Fix \( 0 < \theta < 1 \) and let \( \mathcal{T}_\ell \) with \( \ell = 0 \) be the initial triangulation. For each \( \ell = 0,1,2,\ldots \) do:

(i) Compute discrete solution \( U_\ell \in \mathcal{A}_\ell := \mathcal{A} \cap S^1(\mathcal{T}_\ell) \)
(ii) Compute indicators \( \eta(E)^2 + \text{osc}_E(T)^2 \) and \( \text{osc}_T(T)^2 \) for all \( E \in \mathcal{E}_\ell \) and all \( T \in \mathcal{T}_{\ell,\Gamma} \).
(iii) Determine set \( \mathcal{M}_\ell \subseteq \mathcal{E}_\ell \cup \mathcal{T}_{\ell,\Gamma} \) which satisfies
\[
\theta \eta_\ell^2 \leq \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} (\eta(E)^2 + \text{osc}_E(T)^2) + \sum_{T \in \mathcal{T}_{\ell,\Gamma} \cap \mathcal{M}_\ell} \text{osc}_T(T)^2.
\]
(iv) Mark all edges \( E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell \) and all reference edges of \( T \in \mathcal{T}_{\ell,\Gamma} \cap \mathcal{M}_\ell \) for refinement.
(v) Obtain new mesh \( \mathcal{T}_{\ell+1} \) by newest vertex bisection and increase counter \( \ell \mapsto \ell + 1 \).
Theorem 8. Let $\mu_\ell$ denote the error estimator introduced in Section 3.1 and suppose that, nevertheless, $\eta_\ell$ is used for marking (32) in Algorithm 7. Then, Algorithm 7 guarantees that the combined error quantity
\begin{equation}
\Delta_\ell := \mathcal{J}(U_\ell) - \mathcal{J}(u) + \gamma \mu_\ell^2
\end{equation}
satisfies the contraction property
\begin{equation}
\Delta_{\ell+1} \leq \kappa \Delta_\ell \quad \text{for all } \ell \in \mathbb{N}.
\end{equation}
The constants $0 < \gamma, \kappa < 1$ depend only on the parameter $\theta$ and the shape of the elements in $T_0$. In particular, there holds $\lim_{\ell \to \infty} \mathcal{J}(U_\ell) = \mathcal{J}(u)$ as well as $\lim_{\ell \to \infty} \|u - U_\ell\| = 0 = \lim_{\ell \to \infty} \eta_\ell$.

Proof. Note that the equivalence (20) of $\eta_\ell$ and $\mu_\ell$ and the Dörfler marking (32) for $\eta_\ell$ provide
\[ \frac{\theta}{4} \mu_\ell^2 \leq \theta \eta_\ell^2 \leq \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} (\eta_\ell(E)^2 + \operatorname{osc}_\ell(E)^2) + \sum_{T \in \mathcal{T}_\ell \cap \mathcal{M}_\ell} \operatorname{osc}_\ell(T)^2. \]
Consequently, $\mathcal{M}_\ell$ satisfies the Dörfler marking (21) for $\mu_\ell$ with modified parameter $\tilde{\theta} = \theta/4 \in (0, 1)$. According to Proposition 3, this implies
\[ \mu_{\ell+1}^2 \leq q \mu_\ell^2 + C_2 \|U_{\ell+1} - U_\ell\|^2 \]
with certain constants $0 < q < 1$ and $C_2 > 0$. Therefore,
\[ \Delta_{\ell+1} = \mathcal{J}(U_\ell) - \mathcal{J}(u) + \gamma \mu_{\ell+1}^2 - (\mathcal{J}(U_\ell) - \mathcal{J}(U_{\ell+1})) \]
\[ \leq \mathcal{J}(U_\ell) - \mathcal{J}(u) + \gamma q \mu_\ell^2 + \gamma C_2 \|U_{\ell+1} - U_\ell\|^2 - (\mathcal{J}(U_\ell) - \mathcal{J}(U_{\ell+1})). \]
Using the variational inequality (11) applied for $U_{\ell+1}$, we proceed as in the proof of Proposition 2 to see
\[ \frac{1}{2} \|U_{\ell+1} - U_\ell\|^2 \leq \mathcal{J}(U_\ell) - \mathcal{J}(U_{\ell+1}) \]
Choosing $\gamma$ sufficiently small to guarantee $\gamma C_2 - 1/2 \leq 0$, we then obtain
\[ \Delta_{\ell+1} \leq \mathcal{J}(U_\ell) - \mathcal{J}(u) + \gamma q \mu_\ell^2 + (\gamma C_2 - 1/2) \|U_{\ell+1} - U_\ell\|^2 \leq \mathcal{J}(U_\ell) - \mathcal{J}(u) + \gamma q \mu_\ell^2. \]
According to Proposition 2 and equivalence (20) of $\eta_\ell$ and $\mu_\ell$, there holds
\[ C_1^{-1} (\mathcal{J}(U_\ell) - \mathcal{J}(u)) \leq \eta_\ell^2 \leq \mu_\ell^2. \]
For $\varepsilon > 0$, we thus observe
\begin{equation}
\mathcal{J}(U_\ell) - \mathcal{J}(u) + \gamma q \mu_\ell^2 \leq (1 - \gamma \varepsilon C_1^{-1}) (\mathcal{J}(U_\ell) - \mathcal{J}(u)) + \gamma(q + \varepsilon) \mu_\ell^2 \leq \kappa \Delta_\ell
\end{equation}
with $\kappa := \max\{1 - \gamma \varepsilon C_1^{-1}, q + \varepsilon\}$. Since $q < 1$, we may choose $\varepsilon > 0$ sufficiently small to guarantee $q + \varepsilon < 1$. This choice leads to $\kappa < 1$, and we finally end up with (34). By induction, this implies
\[ \lim_{\ell \to \infty} \Delta_\ell = 0, \quad \text{whence } \lim_{\ell \to \infty} \mathcal{J}(U_\ell) = \mathcal{J}(u) \text{ and } \lim_{\ell \to \infty} \mu_\ell = 0. \]
With the estimate $\eta_\ell \leq \mu_\ell$ and reliability $\|u - U_\ell\| \lesssim \eta_\ell$, we thus conclude the proof. $\square$
In [6], the weighting \( h_T^2 = \text{diam}(T)^2 \) instead of \(|T|\) is used in the definition (16) of \( \text{osc}_\ell(T) \), i.e.,

\[
\eta_\ell^2 := \sum_{E \in \mathcal{E}_\ell} (\eta_\ell(E)^2 + \text{osc}_\ell(E)^2) + \sum_{T \in \mathcal{T}_\ell} \text{osc}_\ell(T)^2,
\]

where

\[
(37) \quad \text{osc}_\ell(T)^2 = h_T^2 \| f \|_{L^2(T)}^2 \quad \text{for} \ T \in \mathcal{T}_\ell.
\]

Note that this definition does not necessarily yield a contraction \( h_{T'} < h_T \) if an element \( T \in \mathcal{T}_{\ell,\Gamma} \cap \mathcal{M}_\ell \) is refined and \( T' \in \mathcal{T}_{\ell+1} \) is one of the resulting sons. Nevertheless, \( \eta_\ell \) leads to a convergent adaptive FEM in the sense of Theorem 8.

**Corollary 9.** Suppose that \( \tilde{\eta}_\ell \) instead of \( \eta_\ell \) is used in Algorithm 7 for marking. Then, the modified algorithm still guarantees the contraction property (34). In particular, there holds

\[
\lim_{\ell \to \infty} \mathcal{J}(U_\ell) = \mathcal{J}(u) \text{ as well as } \lim_{\ell \to \infty} \| u - U_\ell \| = 0 = \lim_{\ell \to \infty} \tilde{\eta}_\ell.
\]

**Proof.** Note that there holds

\[
(38) \quad \text{osc}_\ell(T) \leq \tilde{\text{osc}}_\ell(T) \leq C_3 \text{osc}_\ell(T) \quad \text{for all } T \in \mathcal{T}_{\ell,\Gamma}
\]

with some constant \( C_3 \geq 1 \) which depends only on the shape regularity of the mesh \( \mathcal{T}_\ell \). Since newest vertex bisection leads to uniformly shape regular meshes, \( C_3 \) may be chosen independently of \( \ell \). The Dörfler marking (32) for \( \eta_\ell \) thus implies

\[
\theta \eta_\ell \leq \theta \tilde{\eta}_\ell \leq \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} (\eta_\ell(E)^2 + \text{osc}_\ell(E)^2) + \sum_{T \in \mathcal{T}_{\ell,\Gamma} \cap \mathcal{M}_\ell} \text{osc}_\ell(T)^2
\]

\[
\leq C_3 \left( \sum_{E \in \mathcal{E}_\ell \cap \mathcal{M}_\ell} (\eta_\ell(E)^2 + \text{osc}_\ell(E)^2) + \sum_{T \in \mathcal{T}_{\ell,\Gamma} \cap \mathcal{M}_\ell} \text{osc}_\ell(T)^2 \right).
\]

Put differently, the set \( \mathcal{M}_\ell \) satisfies the Dörfler marking (32) for \( (\theta, \eta_\ell) \) as well as the Dörfler marking (32) for \( (\tilde{\theta}, \tilde{\eta}_\ell) \), where \( \tilde{\theta} = \theta / C_3 \in (0,1) \). Therefore, Theorem 8 applies and (34) holds. In particular, \( \lim_{\ell \to \infty} \eta_\ell = 0 \) and the equivalence (38) also conclude \( \lim_{\ell \to \infty} \tilde{\eta}_\ell = 0 \). \( \square \)

**Remark.** Under the additional assumptions on the mesh-refinement, cf. the remark at the end of Section 3.1, one can prove contraposition (34) of \( \Delta_\ell := \mathcal{J}(U_\ell) - \mathcal{J}(u) + \gamma \eta_\ell^2 \) in Theorem 8 and Corollary 9, where \( \eta_\ell \) replaces \( \mu_\ell \) in the definition of \( \Delta_\ell \). \( \square \)

**Remark.** The estimator reduction (22) already implies convergence \( \lim_\ell \mu_\ell = 0 \), whence \( \lim_\ell \mathcal{J}(U_\ell) = \mathcal{J}(u) \) as well as \( \lim_\ell \| u - U_\ell \| = 0 \) according to Proposition 2. To see this, it remains to verify that the obstacle problem leads to a priori convergence \( \lim_\ell U_\ell = u_\infty \) with a certain limit \( u_\infty \in \mathcal{H} \). For linear problems, such a result is found in [1, 11, 19], and we refer to [21] for the proof of the a priori convergence in our non-linear setting. Then, (22) takes the form

\[
\mu_{\ell+1}^2 \leq q \mu_\ell^2 + \alpha_\ell
\]

with the zero sequence \( \alpha_\ell = C_2 \| U_{\ell+1} - U_\ell \|^2 \geq 0 \). Therefore, elementary calculus concludes \( \lim_\ell \mu_\ell = 0 \), cf. [1]. We stress that, contrary to [19], the estimator reduction concept from [1] avoids any use of discrete efficiency. It is only based on the precise definition of the error.
estimator, a uniform decay of the mesh-width locally on marked elements, and the observation that any kind of mesh-refinement will lead to a convergent sequence of discrete solutions. We stress, however, that the convergence results in Theorem 8 and Corollary 9 are stronger since they include even a contraction of some error quantity \( \Delta \epsilon \geq \epsilon \).

\[ \eta_{\ell} = J(U_{\ell}) - J(u). \]

\[ \Box \]

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Table 1. Numerical results for adaptive mesh-refinement with \( \theta = 0.6 \), where \( N = \#T_{\ell} \), \( \epsilon_{\ell} = J(U_{\ell}) - J(u) \), and \( \text{osc}_{\ell} \) from (43).

4. Numerical Experiment

In this section, we consider a numerical experiment from [6]. The conforming and shape regular mesh is adaptively generated by Algorithm 7. For the solution of the discrete obstacle problem at each level, the primal-dual active set strategy from [15] has been used. For the initial mesh \( T_0 \), we choose \( (U_0, \lambda_0) \equiv (0, 0) \) for the primal-dual pair as initial guesses for the iterative solver. For \( T_\ell \), we choose the prolongated discrete solutions associated with the previous mesh, i.e., \( U_\ell^{(0)} := U_{\ell-1} \) as well as \( \lambda_\ell^{(0)} := \lambda_{\ell-1} \). We stop the iterative solver if the difference of two consecutive solutions satisfies

\[
\| U_{\ell}^{(j)} - U_{\ell}^{(j-1)} \| \leq \tau N^{-1/2}
\]

for some tolerance \( \tau > 0 \), where \( N = \#T_{\ell} \) denotes the number of elements. We then define our discrete solution at \( T_{\ell} \) by \( U_{\ell} := U_{\ell}^{(j)} \) and \( \lambda_{\ell} := \lambda_{\ell}^{(j)} \).

While the numerical results are quite similar to those in [6], we stress that our approach theoretically includes the data oscillations into the estimator \( \eta_{\ell} \).
We consider the obstacle problem with constant obstacle $\chi \equiv 0$ on the L-shaped domain $\Omega := (-2, 2)^2 \setminus [0, 2) \times (-2, 0]$. The right-hand side is given in polar coordinates by

$$f(r, \varphi) := -r^{2/3} \sin(2\varphi/3)\left(\gamma'_1(r)/r + \gamma''_1(r)\right) - \frac{4}{3} r^{-1/3} \gamma'_1(r) \sin(2\varphi/3) - \gamma_2(r),$$

where $(\cdot)'$ denotes the radial derivative $d/dr$. Moreover, $\bar{r} := 2(r - 1/4)$ and

$$\gamma_1(r) = \begin{cases} 
1, & \bar{r} < 0, \\
-6\bar{r}^5 + 15\bar{r}^4 - 10\bar{r}^3 + 1, & 0 \leq \bar{r} < 1, \\
0, & \bar{r} \geq 1,
\end{cases}$$

$$\gamma_2(r) = \begin{cases} 
0, & r \leq 5/4, \\
1, & \text{else.}
\end{cases}$$

Then, the exact solution reads in polar coordinates

$$u(r, \varphi) = r^{2/3} \gamma_1(r) \sin(2\varphi/3).$$
and exhibits a corner singularity at the origin. We compare uniform and adaptive mesh-refinement, where we vary the adaptivity parameter $\theta \in \{0.2, 0.4, 0.6, 0.8\}$ in Algorithm 7.

Table 1 provides the experimental results for adaptive mesh-refinement with $\theta = 0.6$, where the energy error reads

$$
\varepsilon_{\ell} = \mathcal{J}(U_{\ell}) - \mathcal{J}(u), \quad (42)
$$

the overall data oscillations are defined by

$$
\text{osc}_{\ell}^2 = \sum_{E \in \mathcal{E}_{\ell}} \text{osc}_{\ell}(E)^2 + \sum_{T \in \mathcal{T}_{\ell,T}} \text{osc}_{\ell}(T)^2, \quad (43)
$$

and the error estimator $\eta_{\ell}$, defined in (13), includes oscillations and edge jumps.

In Figure 4, we plot $\sqrt{\varepsilon_{\ell}}$, $\eta_{\ell}$, and $\text{osc}_{\ell}$ over the number $N = \#T_{\ell}$ of elements for uniform and adaptive mesh-refinement with $\theta = 0.6$. Uniform mesh-refinement leads to a suboptimal convergence behaviour $\sqrt{\varepsilon_{\ell}} \approx O(N^{-5/12})$ with respect to the number $N = \#T_{\ell}$ of elements. Contrary, adaptive mesh-refinement regains the optimal order of convergence $\sqrt{\varepsilon_{\ell}} = O(N^{-1/2})$. We stress that the given data are smooth so that uniform as well as
adaptive mesh-refinement leads to $\text{osc}_e = \mathcal{O}(N^{-1})$, which corresponds to second-order convergence with respect to a uniform mesh-width. For both mesh-refinements, we see that the
curves of $\eta$ and $\sqrt{\varepsilon}$ are parallel. This experimentally confirms the reliability of $\eta$ from Proposition 2 and indicates that $\eta$ is also efficient.

Figure 5 provides the experimental comparison for different values of $\theta \in \{0.2, 0.4, 0.6, 0.8\}$. We see that each choice of $\theta$ leads to optimal order of convergence and that the corresponding curves essentially coincide. Since achievement of a prescribed precision takes much longer with uniform refinement, the benefits of adaptive refinement are clearly visible. Additionally, we stress that also the convergence rate itself is improved.

Figure 6 displays the adaptively generated meshes $T_5$ and $T_{11}$, respectively for $\theta = 0.6$. As expected, refinement is basically restricted to the inactive zone. Due to the data oscillation terms in the estimator $\eta$ — and hence contrary to [6] — we also observe certain refinement within the active zone. Finally, Figure 7 shows the discrete solution on $T_8$.

References


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