ASC Report No. 40/2009

Convergence of Data Perturbed Adaptive Boundary Element Methods

Markus Aurada, Petra Goldenits, Dirk Praetorius

Institute for Analysis and Scientific Computing
Vienna University of Technology — TU Wien
www.asc.tuwien.ac.at ISBN 978-3-902627-02-5

Most recent ASC Reports

39/2009 *Ilona Gucwa, Peter Szmolyan*
Scaling in Singular Perturbation Problems: Blowing-up a Relaxation Oscillator

38/2009 *Anton Baranov, Harald Woracek*
Majorization in de Branges Spaces III. Division of Blaschke Products

37/2009 *Philipp Dörsek, Jens Markus Melenk*
Adaptive hp-FEM for the Contact Problem with Tresca Friction in Linear Elasticity: The Primal-dual Formulation and a Posteriori Error Estimation

36/2009 *Philipp Dörsek, Jens Markus Melenk*
Adaptive hp-FEM for the Contact Problem with Tresca Friction in Linear Elasticity: The Primal Formulation

35/2009 *Georg Kitzhofer, Othmar Koch, Gernot Pulverer, Christa Simon, Ewa B. Weinmüller*
BVPSUITE, A New MATLAB Solver for Singular Implicit Boundary Value Problems

34/2009 *M. Löhndorf, Jens Markus Melenk*
Mapping Properties of Helmholtz Boundary Integral Operators and their Application to the hp-BEM

33/2009 *Ansgar Jüngel, Stefan Krause, Paola Pietra*
Diffusive Semiconductor Moment Equations Using Fermi-Dirac Statistics

32/2009 *Ansgar Jüngel*
Diffusive and Nondiffusive Population Models

31/2009 *Jens Markus Melenk, Stefan Sauter*
Wave-Number Explicit Convergence Analysis for Galerkin Discretizations of the Helmholtz Equation (extended version)

30/2009 *Matthias Langer, Harald Woracek*
A Function Space Model for Canonical Systems with an Inner Singularity

Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstraße 8–10
1040 Wien, Austria

E-Mail: admin@asc.tuwien.ac.at
WWW: http://www.asc.tuwien.ac.at
FAX: +43-1-58801-10196

ISBN 978-3-902627-02-5

CONVERGENCE OF DATA PERTURBED ADAPTIVE BOUNDARY ELEMENT METHODS

M. AURADA, P. GOLDENITS, AND D. PRAETORIUS

Abstract. For the boundary integral formulation of a 2D Laplace equation with Dirichlet boundary conditions, we consider an adaptive Galerkin BEM based on an $(h-h/2)$-type error estimator which, contrary to prior works, includes the resolution of the perturbed Dirichlet data. Since the analysis of the error estimator depends crucially on the K-mesh property, we propose a local mesh-refinement strategy which preserves the K-mesh property and which is proven to be optimal with respect to the number of generated boundary elements. We then prove that the usual adaptive algorithm drives the underlying error estimator to zero. Under a saturation assumption for the non-perturbed problem which is observed empirically, the sequence of discrete solutions thus converges to the exact solution within the energy norm.

1. Introduction

Model Problem. We consider Symm's integral equation

$$V\phi = (K + 1/2)g \quad \text{on } \Gamma, \tag{1.1}$$

where $\Gamma := \partial\Omega$ is the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with $\mathrm{diam}(\Omega) < 1$. With the fundamental solution of the 2D Laplacian

$$G(z) = -\frac{1}{2\pi} \log |z|, \tag{1.2}$$

the simple-layer potential V and the double-layer potential K read formally

$$V\phi(x) := \int_\Gamma G(x-y)\phi(y)\, d\Gamma(y) \quad \text{and} \quad Kg(x) := \int_\Gamma \partial_{n(y)} G(x-y)\, g(y)\, d\Gamma(y). \tag{1.3}$$

Then, (1.1) is the equivalent integral formulation of the Laplace equation

$$-\Delta u = 0 \text{ in } \Omega \quad \text{with Dirichlet boundary conditions} \quad u = g \text{ on } \Gamma. \tag{1.4}$$

The solution of (1.1) is the normal derivative $\phi = \partial_n u \in H^{-1/2}(\Gamma)$ of the solution $u \in H^1(\Omega)$ of (1.4).

The operator $V : \mathcal{H} \to \mathcal{H}^*$ is an elliptic and symmetric isomorphism between the fractional order Sobolev space $\mathcal{H} := H^{-1/2}(\Gamma)$ and its dual $\mathcal{H}^* = H^{1/2}(\Gamma)$. It thus provides a scalar product defined by $\langle\!\langle \phi, \psi \rangle\!\rangle := \langle V\phi, \psi \rangle$, where the duality brackets $\langle \cdot, \cdot \rangle$ extend the $L^2(\Gamma)$-scalar product. We denote by $|||\phi||| := \langle\!\langle \phi, \phi \rangle\!\rangle^{1/2}$ the induced energy norm which is an equivalent norm on $\mathcal{H}$.

Galerkin Discretization. Given $g \in \mathcal{H}^*$, the unique solution $\phi \in \mathcal{H}$ of (1.1) solves the variational form

$$\langle\!\langle \phi, \psi \rangle\!\rangle = \langle (K + 1/2)g, \psi \rangle \quad \text{for all } \psi \in \mathcal{H}. \tag{1.5}$$

Date: October 20, 2009.

Let $\mathcal{T}_\ell$ be a partition of Γ into finitely many affine boundary elements. Then, the lowest-order Galerkin method is to find a $\mathcal{T}_\ell$-piecewise constant function $\phi_\ell^\star \in X_\ell := \mathcal{P}^0(\mathcal{T}_h)$ which solves

$$\langle\!\langle \phi_\ell^\star , \psi_\ell \rangle\!\rangle = \langle (K + 1/2)g , \psi_\ell \rangle \quad \text{for all } \psi_\ell \in X_\ell. \tag{1.6}$$

Since the double-layer potential Kg can hardly be evaluated numerically, we additionally approximate g by some appropriate $g_\ell := I_\ell g$, where $I_\ell : C(\Gamma) \to \mathcal{S}^1(\mathcal{T}_\ell)$ denotes the nodal interpolation operator onto the continuous and piecewise affine splines. Note that this requires additional regularity of the Dirichlet data $g \in H^1(\Gamma) \subset H^{1/2}(\Gamma)$ which often holds in practice. Let $\phi_\ell \in X_\ell$ be the unique solution of the perturbed Galerkin scheme

$$\langle\!\langle \phi_\ell , \psi_\ell \rangle\!\rangle = \langle (K + 1/2)g_\ell , \psi_\ell \rangle \quad \text{for all } \psi_\ell \in X_\ell. \tag{1.7}$$

We stress that all entries of the corresponding linear system can now be computed analytically. Moreover, another advantage is that the matrices which correspond to discrete integral operators, may now be easily approximated by, e.g., hierarchical matrix techniques [20] or the fast multipole method, cf. [25] and the references therein.

Adaptive Algorithm. To state the main idea of this work, let $\widehat{\mathcal{T}}_\ell$ be the uniform refinement of $\mathcal{T}_\ell$ with $\widehat{\phi}_\ell^\star, \widehat{\phi}_\ell \in \widehat{X}_\ell := \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ being the corresponding Galerkin solutions of (1.6) and (1.7), respectively, where X_ℓ is replaced by $\widehat{X}_\ell$. For the a posteriori error estimation, we then use $(h - h/2)$-based error estimators introduced in [13, 14, 18] for the non-perturbed Galerkin scheme (1.6), e.g.,

$$\mu_\ell := \| h_\ell^{1/2} (\widehat{\phi}_\ell - \phi_\ell) \|_{L^2(\Gamma)}, \tag{1.8}$$

where $h_\ell \in L^\infty(\Gamma)$ denotes the local mesh-size function defined by $h_\ell|_T := \mathrm{diam}(T)$ for all $T \in \mathcal{T}_\ell$. To control the approximation of g, we introduce the data oscillation

$$\mathrm{osc}_\ell := \| h_\ell^{1/2} (g - g_\ell)' \|_{L^2(\Gamma)}, \tag{1.9}$$

where $(\cdot)'$ denotes the arclength derivative. Then, there holds efficiency in the sense

$$C_{\mathrm{eff}}^{-1}\, \mu_\ell \le |\!|\!| \phi - \phi_\ell |\!|\!| + \mathrm{osc}_\ell. \tag{1.10}$$

Under the saturation assumption for the non-perturbed scheme (1.6)

$$|\!|\!| \phi - \widehat{\phi}_\ell^\star |\!|\!| \le C_{\mathrm{sat}}\, |\!|\!| \phi - \phi_\ell^\star |\!|\!|, \tag{1.11}$$

with some uniform constant $0 < C_{\mathrm{sat}} < 1$, there additionally holds reliability

$$C_{\mathrm{rel}}^{-1} |\!|\!| \phi - \phi_\ell |\!|\!| \le \mu_\ell + \mathrm{osc}_\ell. \tag{1.12}$$

Algorithm 4.1 formulates an adaptive mesh-refining strategy, which is steered by the local contributions of

$$\varrho_\ell := (\mu_\ell^2 + \mathrm{osc}_\ell^2)^{1/2}. \tag{1.13}$$

This provides a sequence of partitions $\mathcal{T}_\ell$ with nested spaces $X_\ell \subset X_{\ell+1}$, corresponding Galerkin solutions $\phi_\ell \in X_\ell$, and error estimators ϱ_ℓ. Moreover, Theorem 4.2 below guarantees that, independent of the saturation assumption (1.11), the adaptive algorithm leads to

$$\lim_{\ell\to\infty} \mu_\ell = 0 = \lim_{\ell\to\infty} \mathrm{osc}_\ell. \tag{1.14}$$

According to (1.12), the saturation assumption (1.11) thus yields convergence of the discrete Galerkin solutions $\phi_\ell \in X_\ell$ to the exact solution $\phi \in \mathcal{H}$.

The constants $C_{\rm eff}, C_{\rm rel} > 0$ of (1.10) and (1.12) as well as the proof of Theorem 4.2 depend on an upper bound of the K-mesh constant (or: local mesh-ratio)

$$\kappa(\mathcal{T}_\ell) := \max\{h_\ell|_T \,/\, h_\ell|_{T'} \;:\; T, T' \in \mathcal{T}_\ell \text{ neighbors}\}. \tag{1.15}$$

In particular, our adaptive algorithm includes some local mesh-refinement strategy which guarantees

$$\sup_{\ell\in\mathbb{N}} \kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0), \tag{1.16}$$

where $\mathcal{T}_0$ is the initial partition for the adaptive algorithm. Moreover, this local mesh-refinement rule is optimal in the sense of [3, 29], i.e., the number of elements in the ℓ-th adaptively generated partition $\mathcal{T}_\ell$ is controlled in terms of the initial partition $\mathcal{T}_0$ and the number of marked elements $\mathcal{M}_j$ in the preceding steps $j = 0, \dots, \ell - 1$ of the adaptive algorithm.

Further Outline of the Paper. Section 2 is concerned with optimal local mesh-refinement rules. We introduce two refinement strategies based on local bisection, which guarantee the uniform boundedness (1.16) for the K-mesh constant. In Section 3, we analyze the error estimator ϱ from (1.13) as well as some variants. Section 4 states our adaptive mesh-refining algorithm and proves the estimator convergence (1.14). Numerical experiments are found in Section 5, whereas Section 6 concludes the work with some remarks on generalizations and extensions of our results.

2. Optimal Local Mesh-Refinement

Suppose that $\mathcal{T}_0 = \{T_1, \dots, T_N\}$ is a given initial partition of Γ into affine boundary segments T_j and that a sequence of meshes $\mathcal{T}_\ell$ is obtained inductively by local refinement, where

$$\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell) \tag{2.1}$$

is generated from $\mathcal{T}_\ell$ by refinement of (at least) certain marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$. Here, refinement of an element $T \in \mathcal{M}_\ell$ means that T is bisected into two elements $T_1, T_2 \in \mathcal{T}_{\ell+1}$ of half length, i.e., there holds $h_{\ell+1}|_T = \frac{1}{2}\,h_\ell|_T$. In addition, we aim to guarantee that the K-mesh constant $\kappa(\mathcal{T}_\ell)$ defined in (1.15) satisfies the uniform boundedness

$$\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0) \quad \text{for all } \ell \in \mathbb{N}. \tag{2.2}$$

Therefore, the set $\mathcal{R}_\ell := \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1} \subseteq \mathcal{T}_\ell$ of elements, which are refined by call of (2.1), actually satisfies $\mathcal{M}_\ell \subseteq \mathcal{R}_\ell$. Following [3, 29], we say that an algorithm (2.1) is *optimal* if, for any choice of the sets $\mathcal{M}_\ell$, there holds

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\rm mesh} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j, \tag{2.3}$$

where the constant $C_{\rm mesh} > 0$ only depends on $\mathcal{T}_0$ and where $\#\mathcal{U}$ denotes the cardinality of a set $\mathcal{U}$. In the following, we provide two local refinement strategies (2.1) which guarantee uniform boundedness (2.2) of $\kappa(\mathcal{T}_\ell)$ as well as optimality (2.3).

Remark 1. *To show that adaptive FEM converges with optimal order, one essential step is to prove* (2.3)*, cf.* [28, 11]*. Although optimality of adaptive BEM is far beyond the scope* *of this paper, it is nevertheless worth noting that the following mesh-refinement procedures fit into this mathematical framework.* □

Remark 2. *Clearly, the boundedness estimate (2.2) cannot be improved in general. For instance, let $\mathcal{T}_0$ be a uniform partition with $\#\mathcal{T}_0 > 1$ and $\#\mathcal{M}_0 = 1$. Provided that the obtained partition satisfies $\#\mathcal{T}_1 < 2\,\#\mathcal{T}_0$, i.e., the local refinement does not lead to a uniform refinement, there holds $\kappa(\mathcal{T}_0) = 1$, whereas $\kappa(\mathcal{T}_1) = 2$.* □

2.1. Level-Based Mesh-Refinement. To use the analytical techniques developed in [3, 29], we introduce the level of an element by induction: For $T \in \mathcal{T}_0$, let $\mathrm{level}(T) := 0$. If $T \in \mathcal{T}_\ell$ is bisected into two sons $T_1, T_2 \in \mathcal{T}_{\ell+1}$, we define $\mathrm{level}(T_1) := \mathrm{level}(T_2) := \mathrm{level}(T)+1$.

Algorithm 2.1. Input: *Partition $\mathcal{T}_\ell$, marked elements $\mathcal{M}_\ell^{(0)} := \mathcal{M}_\ell$, counter $i := 0$.*

(i) *Define $\mathcal{U}^{(i)} := \bigcup_{T\in\mathcal{M}_\ell^{(i)}} \{T' \in \mathcal{T}_\ell \backslash \mathcal{M}_\ell^{(i)}$ neighbor of $T \,:\, \mathrm{level}(T') < \mathrm{level}(T)\}$.*

(ii) *If $\mathcal{U}^{(i)} \neq \emptyset$, define $\mathcal{M}_\ell^{(i+1)} := \mathcal{M}_\ell^{(i)} \cup \mathcal{U}^{(i)}$, increase counter $i \mapsto i+1$, and goto* (i)*.*

(iii) *Otherwise, bisect all marked elements $T \in \mathcal{M}_\ell^{(i)}$ to obtain $\mathcal{T}_{\ell+1}$.*

Output: *Refined boundary partition $\mathcal{T}_{\ell+1}$ as well as $\mathcal{M}_\ell^{(i)} = \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$.* □

Note that Algorithm 2.1 is well-defined in the sense that it terminates for some counter $0 \le i \le \#\mathcal{T}_\ell - 1$. Moreover, the following lemma states that $\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0)$.

Lemma 2.2. *Assume that $\mathcal{T}_0$ is a given initial partition and that the partitions $\mathcal{T}_\ell$ are inductively generated by Algorithm 2.1, where the sets $\mathcal{M}_j \subseteq \mathcal{T}_j$ of marked elements are arbitrary. Then, neighboring elements satisfy*

$$|\mathrm{level}(T) - \mathrm{level}(T')| \le 1 \quad \textit{for all } T, T' \in \mathcal{T}_\ell \textit{ with } T \cap T' \neq \emptyset. \tag{2.4}$$

Moreover, there holds $\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0)$ for all $\ell \in \mathbb{N}$.

Proof. The estimate (2.4) easily follows from induction and the definition of the $\mathcal{U}^{(i)}$ in step (i) of Algorithm 2.1. Now, let $T, T' \in \mathcal{T}_\ell$ be neighbors, i.e., $T \neq T'$ and $T \cap T' \neq \emptyset$. Consequently, the unique ancestors $\widehat{T}, \widehat{T}' \in \mathcal{T}_0$ with $T \subseteq \widehat{T}$ and $T' \subseteq \widehat{T}'$ either coincide or are neighbors as well. Moreover, according to bisection, there hold $h_\ell|_T = 2^{-\mathrm{level}(T)} h_0|_{\widehat{T}}$ and $h_\ell|_{T'} = 2^{-\mathrm{level}(T')} h_0|_{\widehat{T}'}$. Together with (2.4), we obtain

$$\frac{h_\ell|_T}{h_\ell|_{T'}} = 2^{\mathrm{level}(T')-\mathrm{level}(T)} \frac{h_0|_{\widehat{T}}}{h_0|_{\widehat{T}'}} \le 2\kappa(\mathcal{T}_0).$$

Taking the supremum over all possible neighbors, we conclude $\kappa(\mathcal{T}_\ell) \le 2\kappa(\mathcal{T}_0)$. □

Theorem 2.3. *Assume that $\mathcal{T}_0$ is a given initial partition and that the partitions $\mathcal{T}_\ell$ are inductively generated by Algorithm 2.1, where the sets $\mathcal{M}_j \subseteq \mathcal{T}_j$ of marked elements are arbitrary. Then there holds optimality (2.3), where the constant C_{mesh} only depends on the initial partition $\mathcal{T}_0$.*

Proof. We aim to use the arguments from [29] to verify (2.3). In the latter work, the focus is on newest vertex bisection for simplicial meshes in $\mathbb{R}^d$ with $d \geq 2$. To adopt the notation of [29], note that the sets $\mathcal{M}_j$ are pairwise disjoint. Therefore, there holds $\#\mathcal{M} = \sum_{j=0}^{\ell-1} \#\mathcal{M}_j$ with $\mathcal{M} := \bigcup_{j=0}^{\ell-1} \mathcal{M}_j$. Finally, [29, Theorem 6.1] states the estimate

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \leq \#\mathcal{T}_\ell - \#(\mathcal{T}_\ell \cap \mathcal{T}_0) = \#(\mathcal{T}_\ell \backslash (\mathcal{T}_\ell \cap \mathcal{T}_0)) \lesssim \#\mathcal{M} = \sum_{j=0}^{\ell-1} \#\mathcal{M}_j,$$

where the notation $\lesssim$ suppresses the constant $C_{\rm mesh}$. From now on, our proof only aims to point out the modifications to ensure that the proof of [29, Theorem 6.1] applies to our case as well. — In our context, we call a partition $\mathcal{T}$ *conforming* provided that the level property (2.4) holds. It is easily observed that Algorithm 2.1 provides the coarsest conforming refinement $\mathcal{T}_{\ell+1}$ of the partition $\mathcal{T}_\ell$ such that all elements $T \in \mathcal{M}_\ell$ are refined. Moreover, we note that our refinement rule, i.e. refinement of an element by bisection, leads to a binary refinement tree as does newest vertex bisection in $\mathbb{R}^d$. Therefore, we can even call the refinement routine elementwise: Suppose that $\mathcal{T}' = \texttt{refine}(\mathcal{T}, \mathcal{M})$ is a realization of Algorithm 2.1 which applies refinement for the set $\mathcal{M} \cap \mathcal{T}$, where we define $\mathcal{T}' := \mathcal{T}$ in case of $\mathcal{M} \cap \mathcal{T} = \emptyset$. Suppose that $\mathcal{M}_\ell = \{T_1, \dots, T_m\}$. By induction, we may define

$$\mathcal{T}_\ell^{(0)} = \mathcal{T}_\ell \quad \text{and} \quad \mathcal{T}_\ell^{(i)} := \texttt{refine}(\mathcal{T}_\ell^{(i-1)}, \{T_i\}) \quad \text{for } i = 1, \dots, m.$$

Then, there holds $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell) = \mathcal{T}_\ell^{(m)}$. These observations provide the framework for the analysis of [29].

- First, we note that the definition of

$$d := \min_{\widehat{T} \in \mathcal{T}_0} \operatorname{diam}(\widehat{T}) \quad \text{and} \quad D := \max_{\widehat{T} \in \mathcal{T}_0} \operatorname{diam}(\widehat{T}),$$

leads to

$$2^{-\mathrm{level}(T)}\, d \leq \operatorname{diam}(T) \leq 2^{-\mathrm{level}(T)} D \quad \text{for all } T \in \mathcal{T}_\ell \text{ and } \ell \in \mathbb{N}_0, \tag{2.5}$$

which follows from the fact that $\operatorname{diam}(T) = 2^{-\mathrm{level}(T)} \operatorname{diam}(\widehat{T})$, where $\widehat{T} \in \mathcal{T}_0$ is the unique ancestor of $T \in \mathcal{T}_\ell$, i.e. $T \subseteq \widehat{T}$. This observation corresponds to [29, Equation (4.1)].

- Second, [29, Corollary 4.6] is satisfied due to Estimate (2.4).
- Third, suppose that $T' \in \mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell$ is generated by a call of $\texttt{refine}(\mathcal{T}_\ell, \{T\})$ for some $T \in \mathcal{M}_\ell$. By definition of Algorithm 2.1, there are some elements $T_0, \dots, T_r \in \mathcal{T}_\ell$ such that T_j is a neighbor of T_{j-1} with $\mathrm{level}(T_j) < \mathrm{level}(T_{j-1})$, $T_0 = T$, and $T' \subset T_r$. This implies $\mathrm{level}(T') = \mathrm{level}(T_r) + 1 < \mathrm{level}(T_0) + 1 = \mathrm{level}(T) + 1$ and verifies the analogon of [29, Theorem 5.1].
- Fourth, [29, Theorem 5.2] is a consequence of [29, Equation (4.1)] and [29, Theorem 5.1] and therefore holds in our case as well.
- Finally, the proof of [29, Theorem 6.1] only relies on [29, Theorem 5.1–5.2] and [29, Equation (4.1)] and therefore applies to our mesh-refinement as well. □

2.2. $\boldsymbol{\kappa}$-Based Mesh-Refinement. In this section, we use the preceding mesh-refinement to prove that the mesh-refinement proposed in [13, 14, 18] is also optimal. The advantage of this is that there is no need to compute or store the level function.

Algorithm 2.4. INPUT: *Partition* $\mathcal{T}_\ell$, *marked elements* $\mathcal{M}_\ell^{(0)} := \mathcal{M}_\ell$, *counter* $i := 0$.

(i) *Define* $\mathcal{U}^{(i)} := \bigcup_{T\in\mathcal{M}_\ell^{(i)}}\{T' \in \mathcal{T}_\ell\backslash\mathcal{M}_\ell^{(i)}$ *neighbor of* $T \,:\, h_\ell|_{T'} > \kappa(\mathcal{T}_0)\, h_\ell|_T\}$.

(ii) *If* $\mathcal{U}^{(i)} \neq \emptyset$, *define* $\mathcal{M}_\ell^{(i+1)} := \mathcal{M}_\ell^{(i)} \cup \mathcal{U}^{(i)}$, *increase counter* $i \mapsto i+1$, *and goto* (i).

(iii) *Otherwise, bisect all marked elements* $T \in \mathcal{M}_\ell^{(i)}$ *to obtain* $\mathcal{T}_{\ell+1}$.

OUTPUT: *Refined boundary partition* $\mathcal{T}_{\ell+1}$ *as well as* $\mathcal{M}_\ell^{(i)} = \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$. □

Remark 3. *For the implementation of Algorithm 2.4, it pays to sort the elements* $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ *by its diameter, i.e., one determines a permutation* π *such that* $\mathrm{diam}(T_{\pi(j)}) \le \mathrm{diam}(T_{\pi(j+1)})$ *for* $j = 1, \dots, N-1$. *Up to* $\mathcal{O}(N \log N)$ *for sorting, Algorithm 2.4 can then be realized in linear complexity.* □

We note that, by definition, Algorithm 2.4 provides the coarsest refinement $\mathcal{T}_{\ell+1}$ of a partition $\mathcal{T}_\ell$ with $\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0)$ such that all elements $T \in \mathcal{M}_\ell$ are refined and that there holds $\kappa(\mathcal{T}_{\ell+1}) \le 2\,\kappa(\mathcal{T}_0)$. The following theorem states optimality of the mesh-refinement from Algorithm 2.4. The proof will be achieved by comparison of Algorithm 2.4 with Algorithm 2.1.

Theorem 2.5. *Assume that* $\mathcal{T}_0$ *is a given initial partition and that the partitions* $\mathcal{T}_\ell$ *are inductively generated by Algorithm 2.4, where the sets* $\mathcal{M}_j \subseteq \mathcal{T}_j$ *of marked elements are arbitrary. Then there holds optimality* (2.3)*, where the constant* $C_{\rm mesh}$ *only depends on the initial partition* $\mathcal{T}_0$. *Moreover, there holds* $\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0)$ *for all* $\ell \in \mathbb{N}$.

Proof. First, we prove the uniform boundedness (2.2) of the K-mesh constant. To that end, let $T, T' \in \mathcal{T}_{\ell+1}$ be neighbors, i.e., $T \neq T'$ and $T \cap T' \neq \emptyset$. Consequently, the fathers $\widehat{T}, \widehat{T}' \in \mathcal{T}_\ell$ of T and T' either coincide or are neighbors as well. We aim to provide an upper bound for the quotient $h_{\ell+1}|_{T'}/h_{\ell+1}|_T$. In case of $\widehat{T} = \widehat{T}'$, there holds $h_{\ell+1}|_T = h_{\ell+1}|_{T'}$. Therefore, we may assume that $\widehat{T} \neq \widehat{T}'$. We now consider four cases:

(a) If $\widehat{T}, \widehat{T}'$ are both not refined, there holds $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}$ and $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}$.

(b) If $\widehat{T}, \widehat{T}'$ are both refined, there holds $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}/2$ and $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}/2$.

(c) If $\widehat{T}'$ is refined and $\widehat{T}$ is not, there holds $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}/2$ and $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}$.

(d) If $\widehat{T}'$ is not refined and $\widehat{T}$ is refined, there holds $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}$ and $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}/2$. Moreover, Algorithm 2.4 implies $h_\ell|_{\widehat{T}'} \le \kappa(\mathcal{T}_0) h_\ell|_{\widehat{T}}$.

In the cases (a)–(c), we thus observe $h_{\ell+1}|_{T'}/h_{\ell+1}|_T \le h_\ell|_{\widehat{T}'}/h_\ell|_{\widehat{T}} \le \kappa(\mathcal{T}_\ell)$. In case (d), there holds $h_{\ell+1}|_{T'}/h_{\ell+1}|_T = 2\,h_\ell|_{\widehat{T}'}/h_\ell|_{\widehat{T}} \le 2\,\kappa(\mathcal{T}_0)$. Altogether, this proves

$$\frac{h_{\ell+1}|_{T'}}{h_{\ell+1}|_T} \le \max\{\kappa(\mathcal{T}_\ell), 2\,\kappa(\mathcal{T}_0)\} \quad \text{for all neighboring elements } T, T' \in \mathcal{T}_{\ell+1},$$

whence $\kappa(\mathcal{T}_{\ell+1}) \le \max\{\kappa(\mathcal{T}_\ell), 2\,\kappa(\mathcal{T}_0)\}$. By induction, we conclude $\kappa(\mathcal{T}_{\ell+1}) \le 2\,\kappa(\mathcal{T}_0)$.

Second, the optimality (2.3) for the κ-based mesh-refinement is obtained via the estimate for the level-based mesh-refinement from the previous section. To that end, let $\widetilde{\texttt{refine}}$ denote the level-based mesh-refinement from Section 2.1. By induction, we now define an additional sequence of partitions by

$$\widetilde{\mathcal{T}}_{\ell+1} := \widetilde{\texttt{refine}}(\widetilde{\mathcal{T}}_\ell, \widetilde{\mathcal{M}}_\ell) \quad \text{with} \quad \widetilde{\mathcal{M}}_\ell := \mathcal{M}_\ell \cap \widetilde{\mathcal{T}}_\ell,$$

where $\widetilde{\mathcal{T}}_0 := \mathcal{T}_0$ and $\widetilde{\mathcal{M}}_0 := \mathcal{M}_0$. In the following, we prove that the partitions $\mathcal{T}_\ell$ generated by Algorithm 2.4 are coarser than the partitions $\widetilde{\mathcal{T}}_\ell$ generated by Algorithm 2.1 in the sense that each element $T \in \mathcal{T}_\ell$ is the union of elements from $\widetilde{\mathcal{T}}_\ell$, i.e.,

$$\forall \ell \in \mathbb{N}_0 \, \forall T \in \mathcal{T}_\ell \, \exists \mathcal{V}_\ell \subseteq \widetilde{\mathcal{T}}_\ell \qquad T = \bigcup_{\widetilde{T} \in \mathcal{V}_\ell} \widetilde{T}. \tag{2.6}$$

This implies $\#\mathcal{T}_\ell \leq \#\widetilde{\mathcal{T}}_\ell$. Moreover, there holds $\#\widetilde{\mathcal{M}}_\ell \leq \#\mathcal{M}_\ell$ by definition of the set $\widetilde{\mathcal{M}}_\ell$. Using the optimality (2.3) of the level-based refinement, we therefore infer optimality of the κ-based refinement

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \leq \#\widetilde{\mathcal{T}}_\ell - \#\widetilde{\mathcal{T}}_0 \lesssim \sum_{j=0}^{\ell-1} \#\widetilde{\mathcal{M}}_j \leq \sum_{j=0}^{\ell-1} \#\mathcal{M}_j.$$

Here, the symbol $\lesssim$ suppresses the constant $C_{\rm mesh}$ from Theorem 2.3. Altogether, it thus only remains to verify (2.6).

This is done by induction on $\ell \in \mathbb{N}_0$: The case $\ell = 0$ follows by definition $\mathcal{T}_0 = \widetilde{\mathcal{T}}_0$. Now, suppose that (2.6) holds for $\mathcal{T}_\ell$ and $\widetilde{\mathcal{T}}_\ell$ and consider an arbitrary element $T \in \mathcal{T}_{\ell+1}$. We have to distinguish certain cases:

- First, let $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}$. By the induction hypothesis, there is some $\mathcal{V} \subseteq \widetilde{\mathcal{T}}_\ell$ such that

$$T = \bigcup_{\widetilde{T} \in \mathcal{V}} \widetilde{T}$$

For any $\widetilde{T} \in \mathcal{V}$, there holds either $\widetilde{T} \in \widetilde{\mathcal{T}}_{\ell+1}$ or $\widetilde{T} = \widetilde{T}' \cup \widetilde{T}''$ for some $\widetilde{T}', \widetilde{T}'' \in \widetilde{\mathcal{T}}_{\ell+1}$. Consequently, this implies

$$T = \bigcup_{\widetilde{T} \in \widetilde{\mathcal{V}}} \widetilde{T} \quad \text{with} \quad \widetilde{\mathcal{V}} := \{\widetilde{T}' \in \widetilde{\mathcal{T}}_{\ell+1} \,:\, \exists \widetilde{T} \in \mathcal{V} \quad \widetilde{T}' \subseteq \widetilde{T}\}$$

- Second, let $T \in \mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell$, fix the unique $\widehat{T} \in \mathcal{T}_\ell$ with $T \subsetneqq \widehat{T}$, and assume that $\widehat{T} \in \mathcal{T}_\ell \backslash \widetilde{\mathcal{T}}_\ell$. By the induction hypothesis, there is some $\mathcal{V} \subseteq \widetilde{\mathcal{T}}_\ell$ such that

$$\widehat{T} = \bigcup_{\widetilde{T} \in \mathcal{V}} \widetilde{T}$$

Moreover, $\widehat{T} \in \mathcal{T}_\ell \backslash \widetilde{\mathcal{T}}_\ell$ implies $\mathcal{V} \subseteq \widetilde{\mathcal{T}}_{\ell+1}$. Now, recall that bisection leads to a binary refinement tree. Consequently, the two sons of $\widehat{T}$ have an analogous representation. In particular, this implies

$$T = \bigcup_{\widetilde{T} \in \widetilde{\mathcal{V}}} \widetilde{T} \quad \text{with} \quad \widetilde{\mathcal{V}} := \{\widetilde{T} \in \mathcal{V} \,:\, \widetilde{T} \subseteq T\} \subseteq \widetilde{\mathcal{T}}_{\ell+1}$$

- Finally, let $T \in \mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell$, fix the unique $\widehat{T} \in \mathcal{T}_\ell$ with $T \subsetneqq \widehat{T}$, and assume that $\widehat{T} \in \mathcal{T}_\ell \cap \widetilde{\mathcal{T}}_\ell$. In particular, $\widehat{T}$ is refined by the κ-based mesh-refinement from Algorithm 2.4. We now aim to show that $\widehat{T}$ will be marked for refinement by the level-based mesh-refinement from Algorithm 2.1 as well. To that end, we again consider all possible cases: • First, we note that $\widehat{T} \in \mathcal{M}_\ell$ implies $\widehat{T} \in \widetilde{\mathcal{M}}_\ell$ due to $\widehat{T} \in \mathcal{T}_\ell \cap \widetilde{\mathcal{T}}_\ell$. Therefore, we obtain $T \in \widetilde{\mathcal{T}}_{\ell+1}$. • Second, assume that $\widehat{T} \in \mathcal{T}_\ell \backslash \mathcal{M}_\ell$ has a marked neighbor $\widehat{T}' \in \mathcal{M}_\ell$ which leads to the additional marking of $\widehat{T}$, i.e., $h_\ell|_{\widehat{T}} > \kappa(\mathcal{T}_0) h_\ell|_{\widehat{T}'}$. Let $\widehat{T}_0, \widehat{T}_0' \in \mathcal{T}_0$ be the —not necessarily distinct— unique elements with $\widehat{T} \subseteq \widehat{T}_0$ and $\widehat{T}' \subseteq \widehat{T}_0'$. By definition of $\kappa(\mathcal{T}_0)$, there holds $h_0|_{\widehat{T}_0} \le \kappa(\mathcal{T}_0) h_0|_{\widehat{T}_0'}$. From the definition of the level-function, we infer $h_\ell|_{\widehat{T}} = 2^{-\mathrm{level}(\widehat{T})} h_0|_{\widehat{T}_0}$ and $h_\ell|_{\widehat{T}'} = 2^{-\mathrm{level}(\widehat{T}')} h_0|_{\widehat{T}_0'}$. Combining these relations, we obtain $2^{-\mathrm{level}(\widehat{T})} h_0|_{\widehat{T}_0} = h_\ell|_{\widehat{T}} > \kappa(\mathcal{T}_0) h_\ell|_{\widehat{T}'} = \kappa(\mathcal{T}_0) 2^{-\mathrm{level}(\widehat{T}')} h_0|_{\widehat{T}_0'}$ and end up with

$$\kappa(\mathcal{T}_0) \ge \frac{h_0|_{\widehat{T}_0}}{h_0|_{\widehat{T}_0'}} > \kappa(\mathcal{T}_0) 2^{\mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}')}$$

and hence $\mathrm{level}(\widehat{T}') > \mathrm{level}(\widehat{T})$. According to the induction hypothesis for $\widehat{T}' \in \mathcal{T}_\ell$ and the level-estimate (2.4), we infer that $\widehat{T}' \in \widetilde{\mathcal{T}}_\ell$. Consequently, $\widehat{T}' \in \mathcal{M}_\ell$ implies $\widehat{T}' \in \widetilde{\mathcal{M}}_\ell$ according to our first observation. Now, $\widehat{T}' \in \widetilde{\mathcal{M}}_\ell$ and $\mathrm{level}(\widehat{T}') > \mathrm{level}(\widehat{T})$ enforces refinement of $\widehat{T}$ by the level-based Algorithm 2.1. This and $\widehat{T} \in \widetilde{\mathcal{T}}_\ell$ imply $T \in \widetilde{\mathcal{T}}_{\ell+1}$. • Finally, for any element $\widehat{T} \in \mathcal{T}_\ell \backslash \mathcal{M}_\ell$ which is refined by Algorithm 2.4, we find a marked element $\widehat{T}^{(0)} \in \mathcal{M}_\ell$ and a chain of elements $\widehat{T}^{(1)}, \dots, \widehat{T}^{(i)} \in \mathcal{T}_\ell \backslash \mathcal{M}_\ell$ such that

$$\kappa(\mathcal{T}_0)\, h_\ell|_{\widehat{T}^{(j-1)}} < h_\ell|_{\widehat{T}^{(j)}} \quad \text{for } j = 1, \dots, i \quad \text{and} \quad \widehat{T}^{(i)} = \widehat{T}.$$

In particular, all these elements will be refined by call of Algorithm 2.4. Proceeding as in the previous step, we see that there holds $\widehat{T}^{(j)} \in \widetilde{\mathcal{T}}_\ell$ for all $j = 0, \dots, i$ as well as $\widehat{T}^{(0)} \in \widetilde{\mathcal{M}}_\ell$ and that all these elements will be refined by the level-based mesh-refinement as well. As above, we thus obtain $T \in \widetilde{\mathcal{T}}_{\ell+1}$. This concludes the proof. □

3. A Posteriori Error Estimation

We adopt the notation of the introduction. In particular, we assume additional regularity $g \in H^1(\Gamma)$ for the given Dirichlet data. Let $\mathcal{T}_\ell$ be a certain refinement of an initial partition $\mathcal{T}_0$ generated by the local mesh-refinements from the previous section. In particular, $\widehat{\mathcal{T}}_\ell = \mathtt{refine}(\mathcal{T}_\ell, \mathcal{T}_\ell)$ denotes the uniform refinement of $\mathcal{T}_\ell$, where all elements $T \in \mathcal{T}_\ell$ are bisected. We recall that ϕ_ℓ and $\widehat{\phi}_\ell$ denote the perturbed Galerkin solutions (1.7) with respect to $\mathcal{T}_\ell$ and $\widehat{\mathcal{T}}_\ell$, respectively, whereas $\phi_\ell^\star$ and $\widehat{\phi}_\ell^\star$ denote the non-perturbed Galerkin solutions (1.6). Moreover, $h_\ell \in L^\infty(\Gamma)$ denotes the local mesh-size function defined $\mathcal{T}_\ell$-elementwise by $h_\ell|_T := \mathrm{diam}(T)$.

3.1. Preliminaries. We recall the following two results of [18], namely [18, Proposition 1.1] and [18, Theorem 3.2], where the first one is an elementary consequence of the Galerkin orthogonality. Contrary, Lemma 3.2 is the interplay of an inverse estimate [19, Theorem 3.6] and an approximation result [7, Theorem 4.1] for the $H^{-1/2}(\Gamma)$-norm.

Lemma 3.1. *With $\eta_\ell^\star := |||\widehat{\phi}_\ell^\star - \phi_\ell^\star|||$, there holds*

$$\eta_\ell^\star \le |||\phi - \phi_\ell^\star|||. \tag{3.1}$$

Moreover, the estimate

$$|||\phi - \phi_\ell^\star||| \le C_{\mathrm{rel}}\, \eta_\ell^\star \tag{3.2}$$

is equivalent to the saturation assumption (1.11) *with* $C_{\mathrm{sat}} = (1 - C_{\mathrm{rel}}^{-2})^{1/2}$. □

Lemma 3.2. *We consider the a posteriori error estimators*

$$(3.3)\qquad \begin{aligned} \eta_\ell &:= |||\widehat{\phi}_\ell - \phi_\ell|||, & \mu_\ell &:= \|h_\ell^{1/2}(\widehat{\phi}_\ell - \phi_\ell)\|_{L^2(\Gamma)},\\ \widetilde{\eta}_\ell &:= |||(1-\Pi_\ell)\widehat{\phi}_\ell|||, & \widetilde{\mu}_\ell &:= \|h_\ell^{1/2}(1-\Pi_\ell)\widehat{\phi}_\ell\|_{L^2(\Gamma)}, \end{aligned}$$

where Π_ℓ denotes the L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_\ell)$. Then, there are constants $C_1, C_2 > 0$ such that

$$(3.4)\qquad \eta_\ell \le \widetilde{\eta}_\ell \le C_1\widetilde{\mu}_\ell \quad \textit{and} \quad \widetilde{\mu}_\ell \le \mu_\ell \le C_2\eta_\ell.$$

The constant $C_1 > 0$ depends only on Γ, whereas $C_2 > 0$ additionally depends on $\kappa(\mathcal{T}_0)$. □

The a posteriori control of the data approximation is done via an approximation result from [4, Thereom 1]. Our formulation in Equation (3.6), taken from [14, Lemma 2.2], is a consequence of the latter.

Lemma 3.3. *Let $\mathcal{K}_\ell$ denote the set of nodes of $\mathcal{T}_\ell$. For $z \in \mathcal{K}_\ell$, let $\varphi_z \in \mathcal{S}^1(\mathcal{T}_\ell) := \mathcal{P}^1(\mathcal{T}_\ell) \cap C(\Gamma)$ denote the ($\mathcal{T}_\ell$-piecewise affine and globally continuous) hat function associated with z. Then, the nodal interpolation operator reads*

$$(3.5)\qquad I_\ell : C(\Gamma) \to \mathcal{S}^1(\mathcal{T}_\ell), \quad I_\ell f := \sum_{z\in\mathcal{K}_\ell} f(z)\varphi_z.$$

With $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_\ell)$ the L^2-orthogonal projection, there holds $\Pi_\ell(f') = (I_\ell f)'$, where $(\cdot)'$ denotes the arclength derivative. Moreover, there holds the approximation result

$$(3.6)\qquad C_3^{-1}\,\|f - I_\ell f\|_{H^{1/2}(\Gamma)} \le \|h_\ell^{1/2}(f - I_\ell f)'\|_{L^2(\Gamma)} \le \|h_\ell^{1/2} f'\|_{L^2(\Gamma)} \quad \textit{for all } f \in H^1(\Gamma),$$

and the constant $C_3 > 0$ depends only on Γ and $\kappa(\mathcal{T}_0)$. □

3.2. A Posteriori Error Estimate. With the help of the preceding three lemmata, we may now provide some a posteriori error control for the perturbed Galerkin scheme (1.7).

Theorem 3.4. *For any error estimator $\tau_\ell \in \{\eta_\ell, \widetilde{\eta}_\ell, \mu_\ell, \widetilde{\mu}_\ell\}$, there holds*

$$(3.7)\qquad C_4^{-1}\tau_\ell \le |||\phi - \phi_\ell||| + \mathrm{osc}_\ell,$$

where the efficiency constant $C_4 > 0$ depends only on Γ and $\kappa(\mathcal{T}_0)$. Under the saturation assumption (1.11) *for the non-perturbed problem, there holds*

$$(3.8)\qquad C_5^{-1}\,|||\phi - \phi_\ell||| \le \tau_\ell + \mathrm{osc}_\ell,$$

where $C_5 > 0$ depends only on Γ, $\kappa(\mathcal{T}_0)$, and the saturation constant $C_{\rm sat}$.

Proof. To abbreviate notation, we use $\lesssim$ throughout to suppress generic constants in estimates. Since Galerkin solutions depend linearly and continuously on the data, the mapping properties of K yield

$$|||\widehat{\phi}_\ell^\star - \widehat{\phi}_\ell||| \lesssim \|(K+1/2)(g - g_\ell)\|_{H^{1/2}(\Gamma)} \lesssim \|g - g_\ell\|_{H^{1/2}(\Gamma)}.$$

According to the definition of $g_\ell = I_\ell g$, we may localize the $H^{1/2}$-norm by

$$\|g - g_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(g - g_\ell)'\|_{L^2(\Gamma)} = \mathrm{osc}_\ell,$$

cf. Lemma 3.3. The combination of the latter estimates proves

$$|||\widehat{\phi}_\ell^\star - \widehat{\phi}_\ell||| \lesssim \mathrm{osc}_\ell,$$

and the same upper bound holds also for $\|\phi_\ell^\star - \phi_\ell\|$. Because of Lemma 3.2, we may restrict to consider $\tau_\ell = \eta_\ell$. Then, the triangle inequality yields

$$\eta_\ell \le \eta_\ell^\star + \|\widehat{\phi}_\ell^\star - \widehat{\phi}_\ell\| + \|\phi_\ell^\star - \phi_\ell\| \lesssim \eta_\ell^\star + \mathrm{osc}_\ell \quad \text{as well as} \quad \eta_\ell^\star \lesssim \eta_\ell + \mathrm{osc}_\ell.$$

Consequently, there holds efficiency

$$\eta_\ell \lesssim \eta_\ell^\star + \mathrm{osc}_\ell \le \|\phi - \phi_\ell^\star\| + \mathrm{osc}_\ell \le \|\phi - \phi_\ell\| + \|\phi_\ell^\star - \phi_\ell\| + \mathrm{osc}_\ell \lesssim \|\phi - \phi_\ell\| + \mathrm{osc}_\ell.$$

Under the saturation assumption (1.11), we analogously observe

$$\|\phi - \phi_\ell\| \le \|\phi - \phi_\ell^\star\| + \|\phi_\ell^\star - \phi_\ell\| \lesssim \eta_\ell^\star + \mathrm{osc}_\ell \lesssim \eta_\ell + \mathrm{osc}_\ell.$$

This concludes the proof. □

Remark 4. *In numerical analysis, rather the L^2-orthogonal projection onto $\mathcal{S}^1(\mathcal{T}_\ell)$ than nodal interpolation is used, since it allows, for instance, the use of Aubin-Nietsche-type techniques, cf.* [26]*. Theorem 3.4 also holds if I_ℓ denotes the L^2-orthogonal projection. However, our proof of convergence of the adaptive BEM in Theorem 4.2 makes explicit use of nodal interpolation.* □

4. Adaptive Mesh-Refining Algorithm

The adaptive mesh-refinement introduced subsequently is steered by the refinement indicators

$$\varrho_\ell(T)^2 := \|h_\ell^{1/2}(1 - \Pi_\ell)\widehat{\phi}_\ell\|_{L^2(T)}^2 + \|h_\ell^{1/2}(g - g_\ell)'\|_{L^2(T)}^2 \tag{4.1}$$

defined for all $T \in \mathcal{T}_\ell$. Note that

$$\varrho_\ell^2 := \sum_{T \in \mathcal{T}_\ell} \varrho_\ell(T)^2 = \widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2. \tag{4.2}$$

These indicators are used to mark certain elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ for refinement. Based on some fixed parameter $\theta \in (0,1)$, the set $\mathcal{M}_\ell$ of marked elements is determined by use of the Dörfler marking (4.3) introduced in [12].

Algorithm 4.1. Input*: Initial partition $\mathcal{T}_0$, parameter $\theta \in (0,1)$, counter $\ell := 0$.*

(i) *Construct uniform refinement $\widehat{\mathcal{T}}_\ell$ of $\mathcal{T}_\ell$.*
(ii) *Compute Galerkin solution $\widehat{\phi}_\ell \in \widehat{X}_\ell := \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$.*
(iii) *Compute refinement indicators $\varrho_\ell(T)$ for all $T \in \mathcal{T}_\ell$.*
(iv) *Determine (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that*

$$\theta\, \varrho_\ell^2 \le \sum_{T \in \mathcal{M}_\ell} \varrho_\ell(T)^2. \tag{4.3}$$

(v) *Refine all marked elements $\mathcal{M}_\ell$ by use of Algorithm 2.4 (or Algorithm 2.1).*
(vi) *Increase counter $\ell \mapsto \ell + 1$ and go to* (i).

Output*: Sequences of partitions $\widehat{\mathcal{T}}_\ell$, solutions $\widehat{\phi}_\ell$, and estimators $\varrho_\ell = (\widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2)^{1/2}$.* □

Remark 5. *Algorithm 4.1 is formulated in the usual form*

```
Solve ↦ Estimate ↦ Mark ↦ Refine.
```

However, note that the Galerkin solution $\phi_\ell \in X_\ell$ which is controlled in terms of the error estimator ϱ_ℓ is not computed throughout. Instead, we only compute the improved Galerkin solution $\widehat{\phi}_\ell \in \widehat{X}_\ell$ with respect to a uniformly refined partition. From that point of view, our adaptive algorithm can be understood in the form

`Refine Uniformly` $\mapsto$ `Solve` $\mapsto$ `Estimate` $\mapsto$ `Mark` $\mapsto$ `Coarsen`.

Moreover, to compute $\widehat{\phi}_\ell$, the operators V and K are discretized with respect to $\widehat{\mathcal{T}}_\ell$. It is therefore obvious to use the improved data approximation $g_\ell := \widehat{I}_\ell g \in \mathcal{S}^1(\widehat{\mathcal{T}}_\ell)$ even on the right-hand side. Note that this only leads to minor modifications of osc_ℓ, *whereas the a posteriori analysis of Section 3 is not affected.* □

The following convergence result for Algorithm 4.1 is inspired by [2], where the concept of *estimator reduction* is introduced. In our case, the analysis is slightly extended to cover the fact that the Galerkin solutions ϕ_ℓ and ϕ_k are computed with respect to different right-hand sides due to $g_\ell \neq g_k$ in general.

Theorem 4.2. *Algorithm 4.1 guarantees convergence of the error estimator*

$$\lim_{\ell\to\infty} \varrho_\ell = 0. \tag{4.4}$$

Under the saturation assumption (1.11) *for the non-perturbed problem, we thus obtain convergence of the perturbed adaptive BEM,*

$$\lim_{\ell\to\infty} |||\phi - \phi_\ell||| = 0 = \lim_{\ell\to\infty} |||\phi - \widehat{\phi}_\ell|||. \tag{4.5}$$

The proof of Theorem 4.2 is based on the following a priori convergence result, which will be applied twice: First, for the approximate data $g_\ell' = (I_\ell g)' = \Pi_\ell g' \in H := L^2(\Gamma)$, second, for certain Galerkin solutions $\phi_{\ell,\infty} := \mathbb{G}_\ell \phi_\infty \in H := H^{-1/2}(\Gamma)$. A proof can be found in [23, 9, 2]

Lemma 4.3. *Suppose that H is a Hilbert space and $X_\ell \subseteq X_{\ell+1}$ is a sequence of closed subspaces of H. Let $\mathbb{P}_\ell : H \to X_\ell$ denote the orthogonal projection onto X_ℓ. Then, for any $x \in H$ and $x_\ell := \mathbb{P}_\ell x$, the limit $x_\infty := \lim_{\ell\to\infty} x_\ell \in H$ exists.* □

Proof of Theorem 4.2. We first recall that in 1D, there holds $g_\ell' = (I_\ell g)' = \Pi_\ell(g')$. Moreover, we stress that the projection Π_ℓ is even the $\mathcal{T}_\ell$-elementwise L^2-projection. For arbitrary $\delta > 0$, the Young inequality thus yields

$$\begin{aligned}
\varrho_{\ell+1}^2 &= \|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{\phi}_{\ell+1}\|_{L^2(\Gamma)}^2 + \|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})g'\|_{L^2(\Gamma)}^2 \\
&\le (1+\delta)\|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{\phi}_\ell\|_{L^2(\Gamma)}^2 + \|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})g'\|_{L^2(\Gamma)}^2 \\
&\quad + (1+\delta^{-1})\|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})(\widehat{\phi}_{\ell+1}-\widehat{\phi}_\ell)\|_{L^2(\Gamma)}^2 \\
&\le (1+\delta)\,\|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{\phi}_\ell\|_{L^2(\Gamma)}^2 + \|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})g'\|_{L^2(\Gamma)}^2 \\
&\quad + (1+\delta^{-1})\|h_{\ell+1}^{1/2}(\widehat{\phi}_{\ell+1}-\widehat{\phi}_\ell)\|_{L^2(\Gamma)}^2 \\
&\le (1+\delta)\,\|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{\phi}_\ell\|_{L^2(\Gamma)}^2 + \|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})g'\|_{L^2(\Gamma)}^2 \\
&\quad + (1+\delta^{-1})C_{\mathrm{inv}}^2\,|||\phi_{\ell+1}-\phi_\ell|||^2,
\end{aligned}$$

where we have finally used an inverse estimate, see [19, Theorem 3.6]. Note that the adaptive mesh-refining algorithm guarantees

$$h_{\ell+1}|_T = \frac{1}{2}\,h_\ell|_T \quad \text{as well as} \quad (\Pi_{\ell+1}\widehat{\phi}_\ell)|_T = \widehat{\phi}_\ell|_T \quad \text{for all } T \in \mathcal{M}_\ell.$$

Therefore,

$$\begin{aligned}\Lambda(T) &:= \|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{\phi}_\ell\|_{L^2(T)}^2 + \|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})g'\|_{L^2(T)}^2\\ &\le \frac{1}{2}\,\|h_\ell^{1/2}(1-\Pi_{\ell+1})g'\|_{L^2(T)}^2\\ &\le \frac{1}{2}\,\varrho_\ell(T)^2,\end{aligned}$$

for all $T \in \mathcal{M}_\ell$. Contrary, for $T \in \mathcal{T}_\ell\backslash\mathcal{M}_\ell$, there holds $h_{\ell+1}|_T \le h_\ell|_T$ and consequently $\Lambda(T) \le \varrho_\ell(T)^2$. Splitting the elements into marked and nonmarked elements, the marking criterion (4.3) implies

$$\begin{aligned}&\|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{\phi}_\ell\|_{L^2(\Gamma)}^2 + \|h_{\ell+1}^{1/2}(1-\Pi_{\ell+1})g_\ell'\|_{L^2(\Gamma)}^2\\ &\quad\le \frac{1}{2}\sum_{T\in\mathcal{M}_\ell}\varrho_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{M}_\ell}\varrho_\ell(T)^2\\ &\quad\le (1-\theta/2)\varrho_\ell^2.\end{aligned}$$

Altogether, we have thus seen

$$\varrho_{\ell+1}^2 \le (1+\delta)(1-\theta/2)\varrho_\ell^2 + (1+\delta^{-1})C_{\rm inv}^2\,|||\phi_{\ell+1}-\phi_\ell|||^2. \tag{4.6}$$

We now use Lemma 4.3 to see that the second term on the right-hand side vanishes as $\ell \to \infty$: According to Lemma 4.3, we first derive that the limit

$$G := \lim_{\ell\to\infty}\Pi_\ell g' \in L^2(\Gamma)$$

exists. For $\ell \ge k$, this implies

$$\begin{aligned}\|g_\ell - g_k\|_{H^{1/2}(\Gamma)} = \|(1-I_k)g_\ell\|_{H^{1/2}(\Gamma)} &\lesssim \|h_k^{1/2}[(1-I_k)g_\ell]'\|_{L^2(\Gamma)}\\ &= \|h_k^{1/2}(g_\ell' - g_k')\|_{L^2(\Gamma)}\\ &\lesssim \|g_\ell' - g_k'\|_{L^2(\Gamma)} \xrightarrow{k,\ell\to\infty} 0.\end{aligned}$$

Therefore, g_ℓ is a Cauchy sequence and converges to some limit $g_\infty \in H^{1/2}(\Gamma)$. Now, let $\phi_{\ell,\infty} \in X_\ell$ be the Galerkin solution for the continuous auxiliary problem $V\phi_\infty = (K+1/2)g_\infty$. According to Lemma 4.3, the limit

$$\phi_{\infty,\infty} := \lim_{\ell\to\infty}\phi_{\ell,\infty} \in H^{-1/2}(\Gamma)$$

exists. Moreover, stability of the Galerkin method and continuity of K prove

$$|||\phi_\ell - \phi_{\ell,\infty}||| \lesssim \|(K+1/2)(g_\ell - g_\infty)\|_{H^{1/2}(\Gamma)} \lesssim \|g_\ell - g_\infty\|_{H^{1/2}(\Gamma)} \xrightarrow{\ell\to\infty} 0.$$

Altogether, the triangle inequality thus proves a priori convergence of the perturbed Galerkin solutions ϕ_ℓ towards $\phi_{\infty,\infty}$, namely

$$|||\phi_\ell - \phi_{\infty,\infty}||| \le |||\phi_\ell - \phi_{\ell,\infty}||| + |||\phi_{\ell,\infty} - \phi_{\infty,\infty}||| \xrightarrow{\ell\to\infty} 0.$$

Finally, we choose $\delta > 0$ such that $\kappa := (1+\delta)(1-\theta/2) \in (0,1)$ is contractive. Together with the observed a priori convergence, (4.6) takes the form

$$\varrho_{\ell+1}^2 \le \kappa\, \varrho_\ell^2 + o(1) \quad \text{for } \ell \to \infty,$$

and elementary calculus concludes estimator convergence $\varrho_\ell^2 \to 0$ as $\ell \to \infty$. Details are left to the reader and can be found in [2, Lemma 1.1]. $\square$

Remark 6. *According to Theorem 4.2, there holds convergence*

$$\|g - g_\ell\|_{H^{1/2}(\Gamma)} \lesssim \mathrm{osc}_\ell \le \varrho_\ell \xrightarrow{\ell \to \infty} 0.$$

Even without the saturation assumption (1.11)*, this guarantees that the perturbed Galerkin solutions ϕ_ℓ converge to the same a priori limit as the non-perturbed Galerkin solutions $\phi_\ell^\star$, i.e., $\phi_{\infty,\infty} = \lim\limits_{\ell\to\infty} \phi_\ell = \lim\limits_{\ell\to\infty} \phi_\ell^\star =: \phi_\infty^\star$.* $\square$

Remark 7. *With minor modifications, the proof of Theorem 4.2 also holds if we replace the error estimator $\widetilde{\mu}_\ell$ in the definition* (4.1) *of ϱ_ℓ by μ_ℓ or by the local averaging estimators introduced in* [7]*. For the latter, we refer to* [2, Section 3.2]*, where the convergence analysis is given in case of a non-perturbed right-hand side.* $\square$

5. Numerical experiments

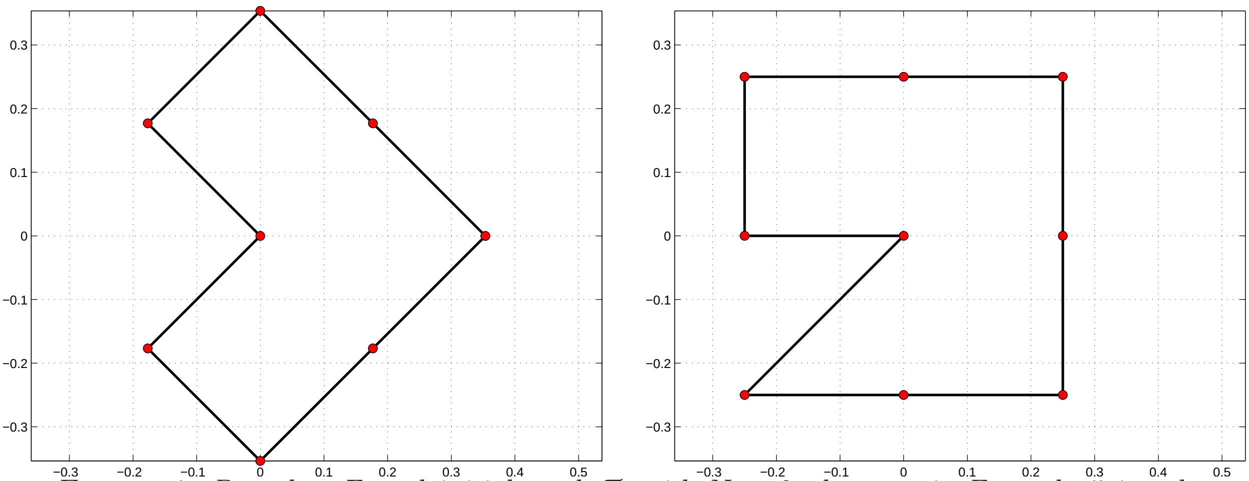

FIGURE 1. *Boundary Γ and initial mesh $\mathcal{T}_0$ with $N = 8$ elements in Example 5.1 and Example 5.3 (left) as well as boundary Γ and initial mesh $\mathcal{T}_0$ with $N = 9$ elements in Example 5.2 (right).*

All computations are performed by use of the MATLAB library `HILBERT` [1], which provides black-box implementations of the assembly of the Galerkin data as well as of error estimators and data oscillations.

5.1. Dirichlet Problem on L-Shaped Domain. On the L-shaped domain from Figure 1, we consider the Laplace problem (1.4) with known exact solution $u \in H^1(\Omega)$ which reads in polar coordinates $u(r,\varphi) = r^{2/3}\cos(2\varphi/3)$. The normal derivative $\phi = \partial_n u$ has a generic singularity at the reentrant corner at $r = 0$. On the other hand, the Dirichlet data $g = u|_\Gamma$

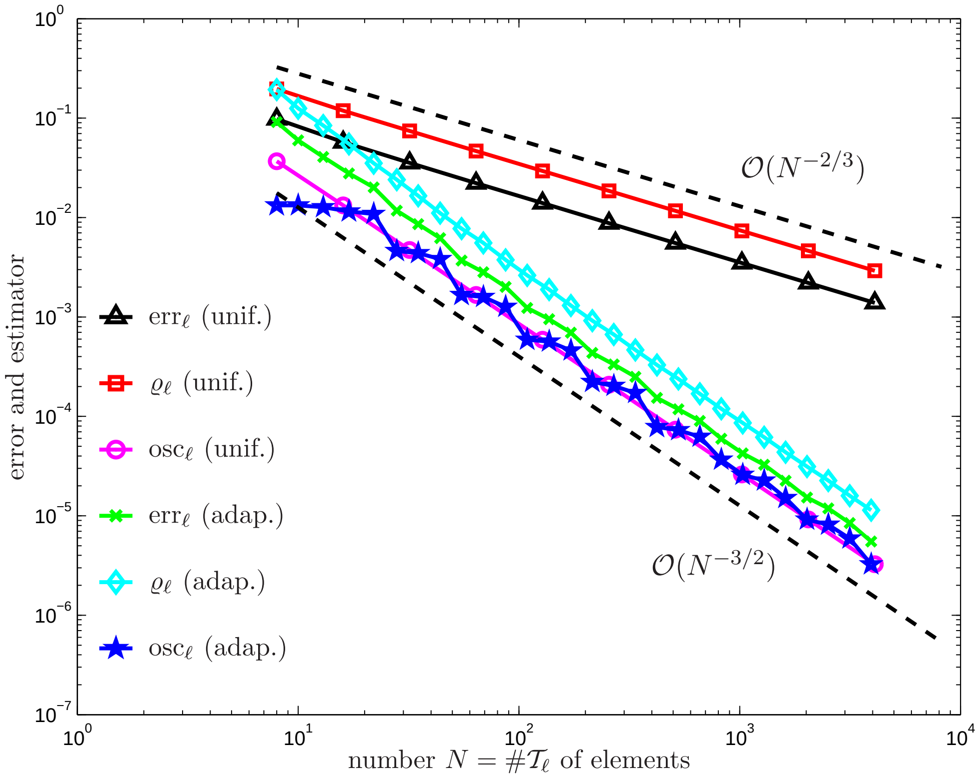

FIGURE 2. *Reliable error bound* err_ℓ *from* (5.1), *a posteriori error estimator* $\varrho_\ell = (\widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2)^{1/2}$, *and data oscillations* osc_ℓ *in Example 5.1 for uniform and* ϱ_ℓ*-steered adaptive mesh-refinement. The values are plotted over the number* $N = \#\mathcal{T}_\ell$ *of elements. For uniform mesh-refinement, the generic singularity of* ϕ *leads to a poor convergence rate of order* $\mathcal{O}(N^{-2/3})$, *whereas our adaptive strategy recovers the optimal order of convergence* $\mathcal{O}(N^{-3/2})$.

are smooth, and one observes optimal decay $\|h_\ell^{1/2}(g - g_\ell)'\|_{L^2(\Gamma)} = \mathcal{O}(h^{3/2})$ even for uniform mesh-refinement.

Since the exact solution is known and has additional regularity $\phi \in L^2(\Gamma)$, one can derive a reliable error bound as follows: First, the triangle inequality and the best approximation property of Galerkin projections yield

$$|||\phi - \phi_\ell||| \le |||\phi - \phi_\ell^\star||| + |||\phi_\ell - \phi_\ell^\star||| \le |||\phi - \Pi_\ell \phi||| + |||\phi_\ell - \phi_\ell^\star|||,$$

where $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_\ell)$ is the L^2-orthogonal projection. With the techniques from [18] and $|||\phi_\ell - \phi_\ell^\star||| \lesssim \mathrm{osc}_\ell$, we obtain

$$C_6^{-1}\, |||\phi - \phi_\ell||| \le \|h_\ell^{1/2}(\phi - \Pi_\ell \phi)\|_{L^2(\Gamma)} + \mathrm{osc}_\ell \le \|h_\ell^{1/2}(\phi - \phi_\ell)\|_{L^2(\Gamma)} + \mathrm{osc}_\ell =: \mathrm{err}_\ell, \tag{5.1}$$

where the constant $C_6 > 0$ only depends on Γ and $\kappa(\mathcal{T}_0)$. By use of numerical quadrature, the right-hand side err_ℓ is computable.

In the following, we restrict to uniform and ϱ_ℓ-steered adaptive mesh-refinement, where we use $\theta = 0.25$. Moreover, to avoid too many adaptive steps, we use the superset $\overline{\mathcal{M}}_\ell \supseteq \mathcal{M}_\ell$

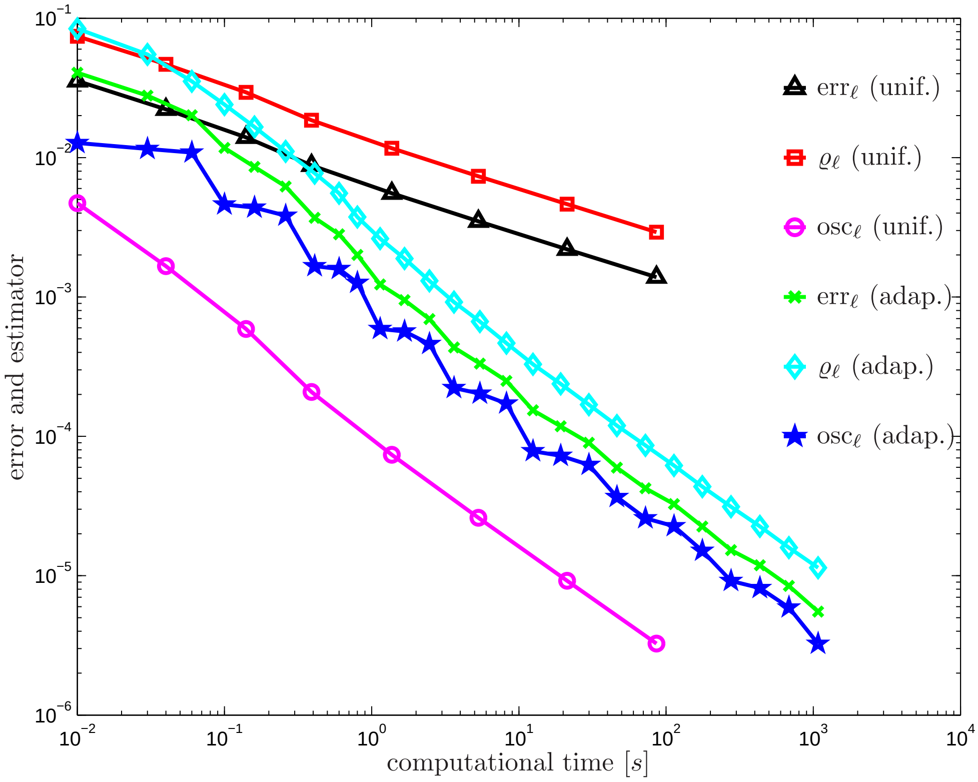

FIGURE 3. *Reliable error bound* err_ℓ *from* (5.1)*, a posteriori error estimator* $\varrho_\ell = (\widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2)^{1/2}$*, and data oscillations* osc_ℓ *in Example 5.1 for uniform and* ϱ_ℓ*-steered adaptive mesh-refinement. The values are plotted over the computational time. Even from this practical relevant point of view, we observe that our adaptive strategy beats uniform mesh-refinement for any prescribed accuracy* $\mathrm{tol} \le 10^{-2}$.

for refinement, where we guarantee

$$\#\overline{\mathcal{M}}_\ell \ge \frac{1}{4}\#\mathcal{T}_\ell \quad \text{as well as} \quad \varrho_\ell(T) \ge \varrho_\ell(T') \quad \text{for all } T \in \overline{\mathcal{M}}_\ell \text{ and } T' \in \mathcal{T}_\ell \backslash \overline{\mathcal{M}}_\ell,$$

i.e., we guarantee that (4.3) holds and that at least 25% of all elements are refined per step.

In Figure 2, we plot the error bound err_ℓ, the Dirichlet data oscillations osc_ℓ, and the error estimator $\varrho_\ell = (\widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2)^{1/2}$ over the number of coarse-mesh elements $N = \#\mathcal{T}_\ell$. We consider both uniform and adaptive mesh-refinement, where adaptivity is driven by Algorithm 4.1.

Recall that the optimal order of convergence is $\mathcal{O}(N^{-3/2})$ for lowest-order BEM. As can be predicted theoretically, uniform mesh-refinement for this example leads to a reduced order of convergence of $\mathcal{O}(N^{-2/3})$ which is due to the generic singularity of ϕ. On the other hand, the proposed adaptive strategy recovers the optimal order of convergence.

In any case, we see that the error estimator ϱ_ℓ behaves efficiently and reliably in the sense that the curves of ϱ_ℓ and err_ℓ stay parallel. Since ϱ_ℓ is —mathematically proven— a lower bound of the error quantity $|||\phi - \phi_\ell||| + \mathrm{osc}_\ell$, whereas err_ℓ is an upper bound, this gives empirical evidence for the saturation assumption.

From a theoretical point of view, the outcome of Figure 2 indicates that adaptive mesh-refinement pays in the sense that the optimal order of convergence is recovered. In Figure 3,

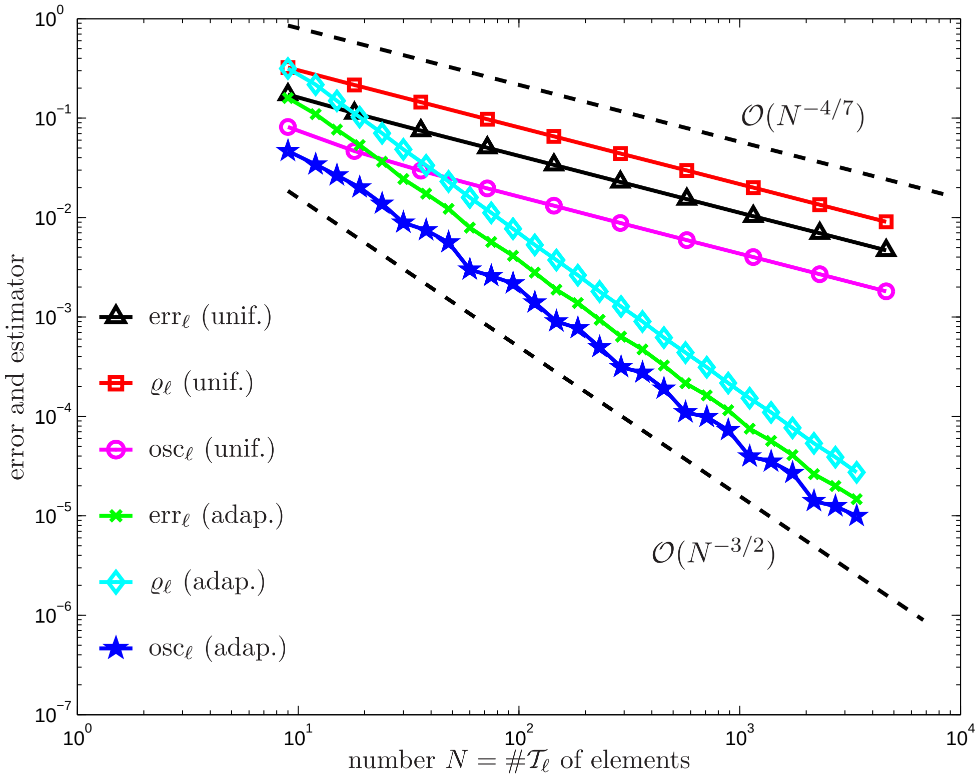

FIGURE 4. *Reliable error bound* err_ℓ *from* (5.1), *a posteriori error estimator* $\varrho_\ell = (\widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2)^{1/2}$, *and data oscillations* osc_ℓ *in Example 5.2 for uniform and* ϱ_ℓ*-steered adaptive mesh-refinement. The values are plotted over the number* $N = \#\mathcal{T}_\ell$ *of elements. For uniform mesh-refinement, the generic singularities of* ϕ *and* g *lead to a poor convergence rate of order* $\mathcal{O}(N^{-4/7})$, *whereas our adaptive strategy recovers the optimal order of convergence* $\mathcal{O}(N^{-3/2})$.

we plot err_ℓ, osc_ℓ, and ϱ_ℓ over the computational time. To be precise, we define the computational time as follows:

- For uniform mesh-refinement $t_\ell^{(\mathrm{unif})}$ is the time elapsed for ℓ uniform mesh-refinements of the initial mesh $\mathcal{T}_0$ to obtain $\mathcal{T}_\ell$, the assembly of the Galerkin data corresponding to $\mathcal{P}^0(\mathcal{T}_\ell)$, and the computation of $\phi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$.

For adaptive mesh-refinement, we stress that ϕ_ℓ depends on the entire history. Therefore, the computational time is defined in an inductive manner:

- We define $t_{-1}^{(\mathrm{adap})} := 0$.
- For $\ell \geq 0$, $t_\ell^{(\mathrm{adap})}$ is the sum of the previous step $t_{\ell-1}^{(\mathrm{adap})}$ plus the time elapsed for the uniform refinement of $\mathcal{T}_\ell$ to obtain $\widehat{\mathcal{T}}_\ell$, the assembly of the Galerkin data for $\mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$, the computation of $\widehat{\phi}_\ell \in \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$, the computation of the local contributions of the error estimator $\widetilde{\mu}_\ell$ and the data oscillations osc_ℓ, the marking step to determine $\mathcal{M}_\ell$, and the local refinement of $\mathcal{T}_\ell$ by Algorithm 2.4 to obtain $\mathcal{T}_{\ell+1}$.

At first glance, the definitions of $t_\ell^{(\mathrm{unif})}$ and $t_\ell^{(\mathrm{adap})}$ seem to favour uniform mesh-refinement. We observe, however, that even for a low accuracy $\mathrm{tol} \approx 10^{-2}$, adaptive mesh-refinement

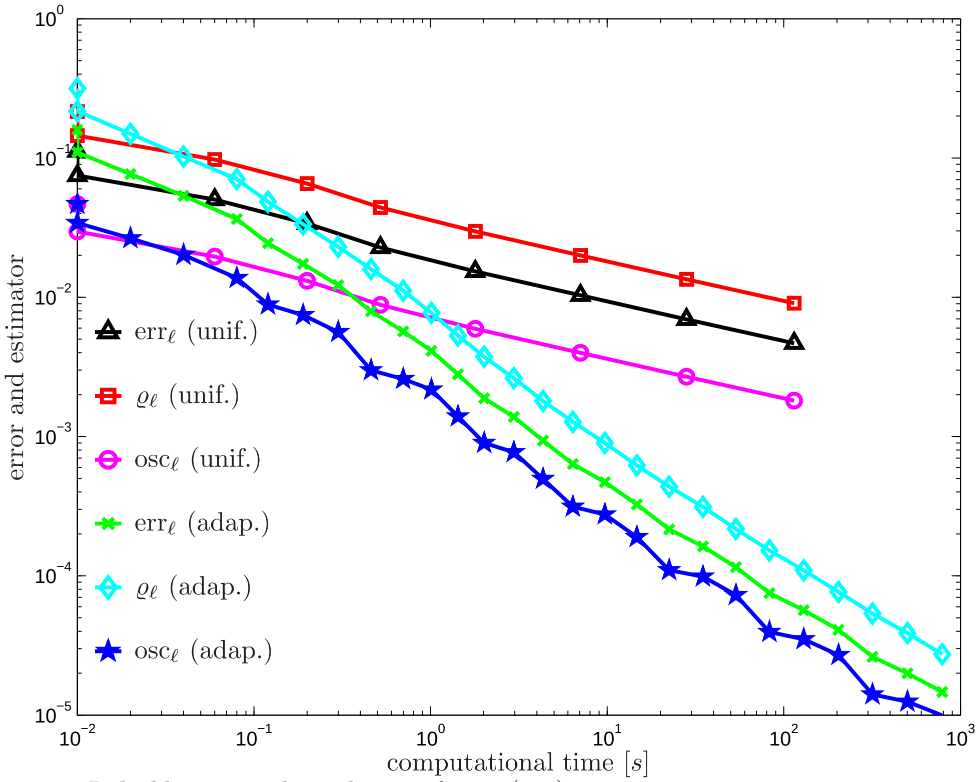

FIGURE 5. *Reliable error bound* err_ℓ *from* (5.1), *a posteriori error estimator* $\varrho_\ell = (\widetilde{\mu}_\ell^2 + \text{osc}_\ell^2)^{1/2}$, *and data oscillations* osc_ℓ *in Example 5.2 for uniform and* ϱ_ℓ*-steered adaptive mesh-refinement. Even with respect to the computational time, we observe that our adaptive strategy beats uniform mesh-refinement for any prescribed accuracy* $\text{tol} \le 10^{-2}$.

is superior to uniform mesh-refinement in the sense that it takes less time to compute an approximation with accuracy 10^{-2}.

5.2. Dirichlet Problem on Z-Shaped Domain. On the Z-shaped domain from Figure 1, we consider the Laplace problem (1.4) with known exact solution $u \in H^1(\Omega)$ which reads in polar coordinates $u(r,\varphi) = r^{4/7}\cos(4\varphi/7)$. Now, the normal derivative $\phi = \partial_n u$ as well as the Dirichlet data $g = u|_\Gamma$ show generic singularities at the reentrant corner at $r = 0$.

In Figure 4 and Figure 5, we plot err_ℓ, osc_ℓ, ϱ_ℓ over the number of coarse mesh-elements and the computational time, respectively. The observations are similar to those of the previous example and underline the effectiveness of the proposed adaptive scheme.

5.3. Dirichlet Problem with Singular Dirichlet Data. On the L-shaped domain from Figure 1, we consider the Laplace problem (1.4). With $v_\alpha(x) := \text{Re}((x_1 + x_2)^\alpha)$, the known exact solution $u \in H^1(\Omega)$ reads $u(x) = v_{2/3}(x) + v_{7/8}(x - z)$, where z is the uppermost corner of Γ. Note that $\phi = \partial_n u$ has a generic singularity at the reentrant corner, but is piecewise H^1 along the remaining boundary. In addition, the Dirichlet data have a singularity at z in the sense that $g = u|_\Gamma$ is not piecewise H^2 locally around z. At the reentrant corner, however, g is smooth. Therefore, the adaptive algorithm has to resolve precisely two different singularities, located at two different nodes.

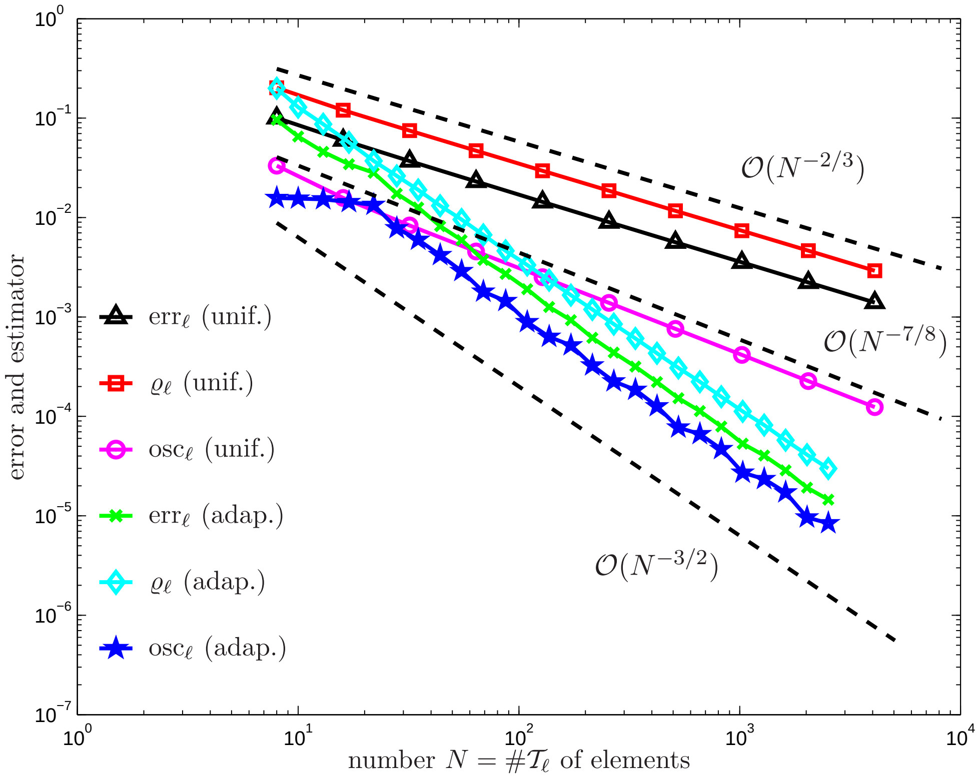

FIGURE 6. *Reliable error bound* err_ℓ, *a posteriori error estimator* $\varrho_\ell = (\widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2)^{1/2}$, *and data oscillations* osc_ℓ *in Example 5.3 for uniform and* ϱ_ℓ*-steered adaptive mesh-refinement. The values are plotted over the number* $N = \#\mathcal{T}_\ell$ *of elements.*

In Figure 6 and Figure 7, we plot err_ℓ, osc_ℓ, and ϱ_ℓ over the number of coarse mesh-elements and the computational time, respectively. In contrast to Experiment 5.1, we observe improved decay of the Dirichlet data oscillations comparing uniform and adaptive mesh-refinement due to the resolution of the additional singularity.

6. Conclusion

In this paper, we proposed and analyzed an adaptive mesh-refining algorithm for the weakly-singular integral equation associated with the 2D Laplacian. To the best of our knowledge, this is the first time that the resolution of the given (Dirichlet) data g is included into the a posteriori error estimation and corresponding adaptive scheme, whereas prior works only focussed on the singularity of the approximated solution, cf. [4, 5, 6, 7, 8, 10, 13, 14, 15, 16, 17, 18, 21, 24, 27]. Using recent ideas, we succeeded to prove convergence of the adaptive algorithm in terms of the so-called estimator reduction introduced in [2].

One additional benefit of our approach is that it decouples the singularities of the integral operator K and possible singularities of g. In addition, the operator K is now applied to discrete functions only. One important aspect for future research is to incorporate the approximation of the (now discrete) integral operator by use of, e.g., $\mathcal{H}$-matrices [20] or the fast multipole method.

The same results can probably be achieved for the lowest-order Galerkin BEM of the hypersingular integral equation, where the given Neumann data are then approximated by

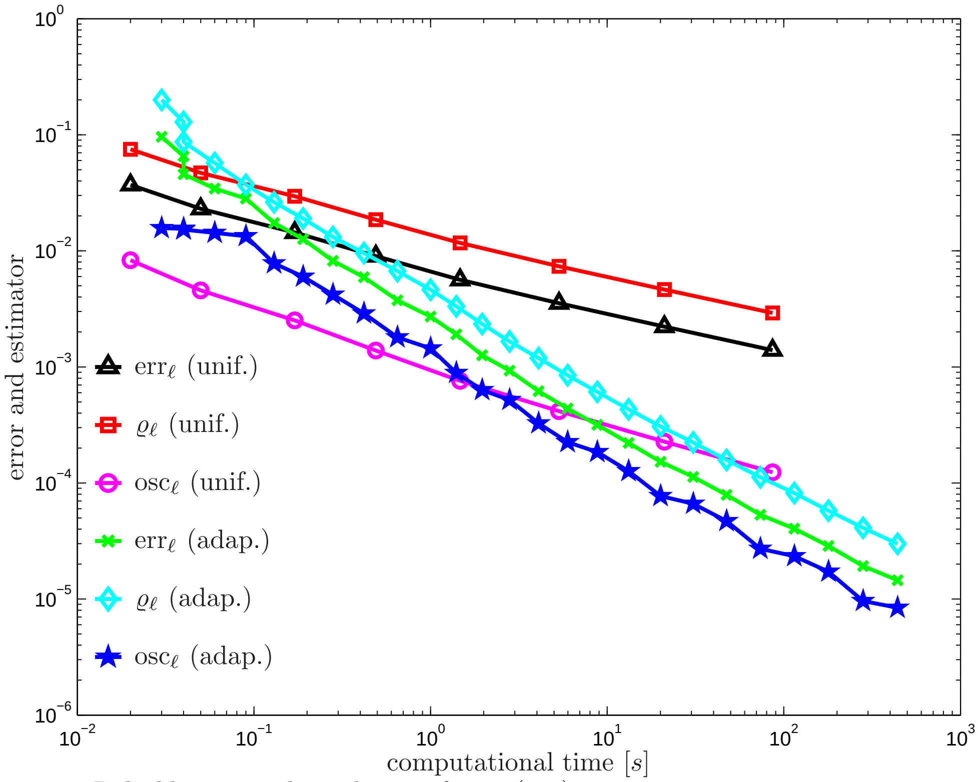

FIGURE 7. *Reliable error bound* err_ℓ *from* (5.1), *a posteriori error estimator* $\varrho_\ell = (\widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2)^{1/2}$, *and data oscillations* osc_ℓ *in Example 5.3 for uniform and* ϱ_ℓ*-steered adaptive mesh-refinement. Even with respect to the computational time, we observe that our adaptive strategy beats uniform mesh-refinement for any prescribed accuracy* $\mathrm{tol} \le 10^{-2}$.

use of the L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_\ell)$. This is, however, work in progress and postponed to a forthcoming paper.

We stress that the a posteriori and convergence analysis for the Dirichlet data oscillations is, so far, restricted to 2D in the following sense: The results in Section 3 hold for 2D and 3D with I_ℓ the L^2-orthogonal projection onto $\mathcal{S}^1(\mathcal{T}_\ell)$. For 2D, we prefer, however, the nodal interpolation operator I_ℓ since this allows to analyze the convergence of the adaptive scheme in Section 4. We stress that our analysis is based on the a priori convergence of the approximated Dirichlet data $g_\ell := I_\ell g \in \mathcal{S}^1(\mathcal{T}_\ell)$ towards some limit $g_\infty \in H^{1/2}(\Gamma)$. In the proof of Theorem 4.2, this could be established by use of the identity $(I_\ell g)' = \Pi_\ell(g')$ with Π_ℓ the L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_\ell)$. This identity, however, only holds for 2D BEM, i.e., 1D manifolds. Moreover, nodal interpolation is not well-defined for 3D BEM and Dirichlet data $g \in H^1(\Gamma)$.

Next, the necessary a priori convergence of $g_\ell = I_\ell g$ in case of the L^2-orthogonal projection I_ℓ remains mathematically open. Altogether, this prohibits to establish a convergence Theorem analogously to Theorem 4.2 for 3D BEM.

Finally, it is a major contribution of this paper to introduce and analyze optimal local mesh-refinement for 2D BEM meshes. Optimality of the local mesh-refinement is a necessary ingredient to prove optimal convergence of adaptive FEM, and it must be expected that this also holds for adaptive BEM.

The κ-based mesh-refinement from Algorithm 2.4 has roughly been proposed in [13, 14, 18]. To the best of our knowledge, all available a posteriori error estimators for 2D BEM depend crucially on the boundedness of the K-mesh constant, cf. [4, 5, 6, 7, 8, 10, 13, 14, 16, 18, 21, 24, 27], which is now algorithmically guaranteed. In addition, 3D BEM a posteriori error estimates usually depend on the boundedness of the shape regularity constant, which is a much stronger assumption than the uniform K-mesh property [8, 15, 17, 18]. In this case, mesh-refinement based on newest vertex bisection is known to be of optimal complexity [29]. Optimal convergence behaviour of adaptive mesh-refinement in 3D BEM computations are, however, only observed for anisotropic mesh-refinement, which is excluded by newest-vertex bisection.

References

[1] M. Aurada, M. Ebner, S. Ferraz-Leite, P. Goldenits, M. Karkulik, M. Mayr, D. Praetorius: *HILBERT — A Matlab Library for Adaptive Galerkin Boundary Element Computations*, in preparation 2009.

[2] M. Aurada, S. Ferraz-Leite, D. Praetorius: *Estimator Reduction and Convergence of Adaptive FEM and BEM*, ASC Report **27/2009**, Institute for Analysis and Scientific Computing, Vienna University of Technology 2009.

[3] P. Binev, W. Dahmen, R. DeVore: *Adaptive Finite Element Methods with Convergence Rates*, Numer. Math. **97** (2004), 219–268.

[4] C. Carstensen: *An A Posteriori Error Estimate for a First-Kind Integral Equation*, Math. Comp. **66** (1997), 69–84.

[5] C. Carstensen, B. Faermann: *Mathematical Foundation of A Posteriori Error Estimates and Adaptive Mesh-Refining Algorithms for Boundary Integral Equations of the First Kind*, Eng. Anal. Bound. Elem. **25** (2001), 497–509.

[6] C. Carstensen, D. Praetorius: *A Posteriori Error Control in Adaptive Qualocation Boundary Element Analysis for a Logarithmic-Kernel Integral Equation of the First Kind*, SIAM J. Sci. Comput. **25** (2004), 259–283.

[7] C. Carstensen, D. Praetorius: *Averaging Techniques for the Effective Numerical Solution of Symm's Integral Equation of the First Kind*, SIAM J. Sci. Comp. **27** (2006), 1226–1260.

[8] C. Carstensen, D. Praetorius: *Averaging Techniques for A Posteriori Error Control in Finite Element and Boundary Element Analysis*, Springer Lect. Notes Appl. Comput. Mech. **29** (2007), 29–59.

[9] C. Carstensen, D. Praetorius: *Convergence of Adaptive Boundary Element Methods*, ASC Report **15/2009**, Institute for Analysis and Scientific Computing, Vienna University of Technology 2009.

[10] C. Carstensen, E. Stephan: *Adaptive Boundary Element Methods for Some First Kind Integral Equations*, SIAM J. Numer. Anal. **33** (1996), 1106–1124.

[11] J. Cascon, C. Kreuzer, R. Nochetto, K. Siebert: *Quasi-Optimal Convergence Rate for an Adaptive Finite Element Method*, SIAM J. Numer. Anal. **46** (2008), 2524–2550.

[12] W. Dörfler: *A Convergent Adaptive Algorithm for Poisson's Equation*, SIAM J. Numer. Anal. **33** (1996), 1106–1124.

[13] C. Erath, S. Ferraz-Leite, S. Funken, D. Praetorius: *Energy Norm Based A Posteriori Error Estimation for Boundary Element Methods in Two Dimensions*, Appl. Numer. Math. **59** (2009), 2713–2734.

[14] C. Erath, S. Funken, P. Goldenits, D. Praetorius: *Simple Error Estimators for the Galerkin BEM for Some Hypersingular Integral Equation in 2D*, ASC Report **20/2009**, Institute for Analysis and Scientific Computing, Vienna University of Technology 2009.

[15] V. Ervin, N. Heuer: *An Adaptive Boundary Element Method for the Exterior Stokes Problem in Three Dimensions*, IMA J. Numer. Anal. **26** (2006), 337–357.

[16] B. Faermann: *Localization of the Aronszajn-Slobodeckij Norm and Application to Adaptive Boundary Element Methods, Part I: The Two-Dimensional Case*, IMA J. Numer. Anal. **20** (2000), 203–234.

[17] B. Faermann: *Localization of the Aronszajn-Slobodeckij Norm and Application to Adaptive Boundary Element Methods, Part II: The Three-Dimensional Case*, Numer. Math. **92** (2002), 297–325.
[18] S. Ferraz-Leite, D. Praetorius: *Simple A Posteriori Error Estimators for the h-Version of the Boundary Element Method*, Computing **83** (2008), 135–162.
[19] I. Graham, W. Hackbusch, S. Sauter: *Finite Elements on Degenerate Meshes: Inverse-Type Inequalities and Applications*, IMA J. Numer. Anal. **25** (2005), 379–407.
[20] W. Hackbusch: *Hierarchische Matrizen*, Springer, Heidelberg, 2009.
[21] N. Heuer, M. Mellado, E. Stephan: *hp-Adaptive Two-Level Methods for Boundary Integral Equations on Curves*, Computing **67** (2001), 305–334.
[22] M. Maischak: *The Analytical Computation of the Galerkin Elements for the Laplace, Lamé and Helmholtz Equation in 2D BEM*, Technical Report, Institute for Applied Mathematics, University of Hannover (1999).
[23] P. Morin, K. Siebert, A. Veeser: *A Basic Convergence Result for Conforming Adaptive Finite Elements*, Math. Models Methods Appl. Sci. **18** (2008), 707–737.
[24] P. Mund, E. Stephan, J. Weisse: *Two-Level Methods for the Single Layer Potential in* $\mathbb{R}^3$, Computing **60** (1998), 243–266.
[25] G. Of, O. Steinbach, P. Urthaler: *Fast Evaluation of Newton Potentials in the Boundary Element Method*, Preprint **2008/3**, Institute for Numerical Mathematics, Graz University of Technology, 2008.
[26] O. Steinbach: *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements*, Springer, New York, 2008.
[27] H. Schulz, O. Steinbach: *A New A Posteriori Error Estimator in Adaptive Direct Boundary Element Methods. The Dirichlet Problem*, Calcolo **37** (2000), 79–96.
[28] R. Stevenson: *Optimality of a Standard Adaptive Finite Element Method*, Found. Comput. Math. **7** (2007), 245–269.
[29] R. Stevenson: *The Completion of Locally Refined Simplicial Partitions Created by Bisection*, Math. Comp. **77** (2008), 227–241.

Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstrasse 8-10, A-1040 Wien, Austria

E-mail address: Markus.Aurada@tuwien.ac.at

E-mail address: Petra.Goldenits@tuwien.ac.at

E-mail address: Dirk.Praetorius@tuwien.ac.at (corresponding author)