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Collocation for solving DAEs with Singularities

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# Collocation Methods for Index 1 DAEs with a Singularity of the First Kind

O. Koch, R. März, D. Praetorius, and E. Weinmüller

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## Abstract

We study the convergence behavior of collocation schemes applied to approximate solutions of BVPs in linear index 1 DAEs which exhibit a critical point at the left boundary. Such a critical point of the DAE causes a singularity within the inherent ODE system. We focus our attention on the case when the inherent ODE system is singular with a *singularity of the first kind*, apply polynomial collocation to the original DAE system and consider different choices of the collocation points such as equidistant, Gaussian or Radau points. We show that for a well-posed boundary value problem for DAEs having a sufficiently smooth solution the global error of the collocation scheme converges with the so-called stage order, or equivalently, it is  $O(h^m)$ , where  $m$  is the number of collocation points. Superconvergence cannot be expected in general due to the singularity, not even for the differential components of the solution. The theoretical results are illustrated by numerical experiments.

## 1 Introduction

In recent years, a lot of effort has been put into the analysis and numerical treatment of BVPs in ODEs which can exhibit singularities. Such problems are typically given in the following form:

$$t^\alpha z'(t) = M(t)z(t) + f(t, z(t)), \quad t \in (0, 1], \quad (1a)$$

$$g(z(0), z(1)) = 0, \quad z \in C[0, 1], \quad (1b)$$

where  $\alpha \geq 1$ ,  $z$  is an  $n$ -dimensional real function,  $M$  is a smooth  $n \times n$  matrix, and  $f \in \mathbb{R}^n$  and  $g \in \mathbb{R}^p$  are smooth functions. For  $\alpha = 1$  the problem is called singular with a singularity of the first kind, for  $\alpha > 1$  it is essentially singular (singularity of the second kind). The search for efficient numerical methods to solve (1) is strongly motivated by numerous applications from physics, see [9], [10], [23], [43], chemistry, cf. [13], [41], [42], mechanics, [12], ecology, see [32] and [39], or economy [14], [15], [24]. Also, research activities in related fields, like the computation of connecting orbits in dynamical systems ([40]), or singular Sturm-Liouville problems ([6]), benefit from techniques developed for problems of the form (1). In this paper we will extend techniques developed in context of ODEs to DAEs.

Our first objective was to provide a sound theoretical basis and the implementation of an open domain MATLAB code for the numerical solution of BVPs with a singularity of the first kind,  $\alpha = 1$ . To compute the numerical solution of (1), we use collocation at an even<sup>1</sup> number of collocation points in the interior of a collocation interval. Our decision to use collocation was motivated by its advantageous convergence properties for (1). The convergence order is at least equal to the *stage order* of the method, while in the presence of a singularity other high order methods show order reductions and become inefficient (see for example [22]), cf. [21], [5] and [27]. The above convergence results mean that a collocation scheme with  $s$  inner collocation points constitutes a high order basic solver ( $O(h^s)$  uniformly in  $t$ ), robust with respect to the singularity of the first kind. Here, we denote by  $h$  the maximal stepsize in a (nonequidistant) grid.

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<sup>1</sup>The even number of points is motivated by the technical details of the error estimation procedure implemented in our code, cf. [25].

In order to solve the ODE systems efficiently the meshes have to be adapted to the solution behavior. For singular problems, we aim at meshes which are not affected by the steep direction field, staying coarse also close to the singularity when the solution is smooth in that region. To design a mesh adaptation procedure, we need an efficient asymptotically correct a posteriori estimate for the error of the numerical solution. Such a global error estimate was introduced in [4] and is based on the defect correction principle. We could show that for a collocation method of order  $O(h^s)$ , the error of the estimate (the difference between the exact global error and its estimate) is of order  $O(h^{s+1})$ , see [5] and [27]. This asymptotically correct error estimate yields a reliable basis for an efficient mesh selection procedure. Our grid adaptation procedure results in grids which adequately reflect the solution behavior. The final step was to implement the above algorithm and to provide an open domain MATLAB code for nonlinear problems with an error estimation routine and a grid selection strategy, see [2] and [3]. However, `sbvp 1.0`, in its original version, was designed to solve only first order systems in explicit form, so that many important applications which we knew about were not in scope of the code. This was a strong motivation for us to start to develop a new MATLAB code `bvpsuite` addressing these additional requirements, cf. [26]. This code is designed to solve implicit systems of ODEs which may have arbitrary variable order including zero. In particular, algebraic constraints are also permitted and therefore, DAEs are in scope of the code. Due to its advantageous convergence properties, collocation is a very robust and dependable solver *for singular ODEs with no serious competitors*. Consequently, we expected collocation to have similar properties in the context of singular DAEs and we will be able to confirm this hypothesis in this article.

In the past and also more recently, several authors successfully applied collocation to well-posed BVPs in index 1 DAEs with no singularities. In [33] and [11], nonlinear systems of DAEs with constant kernel of  $G_{x'}$  have been studied and superconvergence results for Gaussian and Lobatto points were given. First attempts to provide respective software go back to 1994, see [1]. In the scope of the collocation code COLDAE are semi-explicit problems. Collocation methods applied to solve linear and nonlinear BVPs in index 1 DAEs (without singularities) have been recently analyzed in [35] and [34], respectively. Here, the system is assumed to be given in a form of separated sets of equations involving derivatives and derivative free equations. Collocation at Gaussian points and Lobatto points is used to treat those separated equation subsets, respectively. According to our experience, in case of singularities, we need to restrict ourselves to Gaussian points, because the use of Radau or Lobatto points often results in a not correctly-posed discretized system, due to the unboundedness of the associated canonical projector, cf. [30].

Much progress has been made concerning DAE theory and applications, but there are still many questions left open. In particular, the numerical treatment of critical points and singularities is just emerging. With this paper we are giving the first insight into the behavior of polynomial collocation in the context of singular linear index 1 DAE systems.

Only a few years ago, the concept of DAEs *with properly stated leading term*, was introduced and studied in [7, 16, 36]. This enables a proper and natural description of the involved solution derivatives. In particular, one considers linear DAEs written in the form

$$A(t)(D(t)x(t))' + B(t)x(t) = g(t), \quad t \in [a, b], \quad (2)$$

with continuous coefficient matrices  $A(t) \in \mathbb{R}^{m \times n}$ ,  $D(t) \in \mathbb{R}^{n \times m}$ ,  $B(t) \in \mathbb{R}^{m \times m}$ , where  $A(t)$  and  $D(t)$  are assumed to be well-matched. One of the advantages of this precise description of the problem structure is that there exists an *inherent explicit regular ODE* (IERODE) uniquely determined by the problem data, see [16, 17, 18]. Under mild assumptions, DAEs in standard form can be reformulated to have properly stated leading terms. For DAEs with properly stated leading terms arising in applications, see [16]. Linear BVPs for regular DAEs with tractability index 1 are investigated in detail in [8].

In [38] linear index 1 DAEs with properly stated leading term and type 1A-critical points have been analyzed. This means that after decoupling the system using the matrix chain technique developed in [7] into the differential and algebraic components, the related inherent ODE exhibits a singularity of the first or second kind.

In [30], our experimental results showing the convergence of collocation schemes applied to solve (2) have been collected, and in [29] a convergence analysis for the above problem subject to initial conditions has been given. It turns out that for appropriately smooth problem data in (2) the stage order is retained. This means that the global error of the collocation scheme is  $O(h^s)$  uniformly in  $t$ . Here,  $h$  denotes the constant stepsize. In this article we will formulate the respective convergence results for BVPs in index 1 DAEs with a singularity of the first kind.

## 2 Convergence of Collocation Methods

### 2.1 Problem Specification

We investigate the convergence of collocation for index 1 DAEs, where the so-called *inherent ODE* may have a singularity of the first kind. In this section we analyze the error of collocation methods applied to a linear system of DAEs given in the following form:

$$A(t)(D(t)x(t))' + B(t)x(t) = g(t), \quad t \in (0, 1], \quad (3)$$

where  $A(t) \in \mathbb{R}^{m \times n}$ ,  $D(t) \in \mathbb{R}^{n \times m}$ ,  $B(t) \in \mathbb{R}^{m \times m}$ ,  $g(t)$ ,  $x(t) \in \mathbb{R}^m$ , with  $n \leq m$ . All data in (3), that is, the matrix functions  $A$ ,  $D$ , and  $B$ , and the function  $g$ , are assumed to be at least continuous on  $[0, 1]$ . Moreover, we require that

$$\ker A(t) = 0, \quad t \in (0, 1], \quad (4)$$

$$\mathcal{R}(D(t)) = \mathbb{R}^n, \quad t \in [0, 1]. \quad (5)$$

Condition (5) means that the matrix  $D(t)$  has constant rank on the closed interval. The structure (4) and (5) means that the system (3) has a properly stated leading term on  $(0, 1]$ , cf. [37]. In order to describe the boundary conditions which are necessary and sufficient for (3) to be well-posed, we decouple this system using techniques from [7]. To this end we define

$$N_0(t) := \ker A(t)D(t), \quad t \in (0, 1]. \quad (6)$$

Note that  $N_0(t) = \ker D(t)$ ,  $t \in [0, 1]$ .

Let us denote by  $Q_0$  a continuous pointwise projector function onto  $\ker D$ , i.e.  $Q_0(t)^2 = Q_0(t)$ ,  $\mathcal{R}(Q_0(t)) = \ker D(t)$ ,  $t \in [0, 1]$ , and let  $P_0 := I - Q_0$ . Next define

$$G_0(t) := A(t)D(t), \quad t \in [0, 1], \quad (7)$$

$$G_1(t) := G_0(t) + B(t)Q_0(t), \quad t \in [0, 1]. \quad (8)$$

In the following we discuss systems (3) which are regular with tractability index 1 on the interval  $(0, 1]$ . Consequently,  $G_1(t)$  is nonsingular for  $t \in (0, 1]$ . Finally, we introduce the pointwise generalized inverse  $D^-$  of  $D$  uniquely defined by the following requirements:

$$D^-DD^- = D^-, \quad DD^-D = D, \quad DD^- = I, \quad D^-D = P_0 \quad (9)$$

which need to hold pointwise on  $[0, 1]$ . Note that  $D^-$  is also continuous on  $[0, 1]$ .

We want to incorporate the case where the inherent ODE associated with (3) exhibits a singularity of the first kind, see [20].

It was demonstrated in [7] that with the above assumptions the solutions of the DAE (3) can be decoupled on  $(0, 1]$  into the *differential components*  $Dx$  and the *algebraic components*  $Q_0x$ . While  $u = Dx$  satisfies the explicit inherent ODE,

$$u'(t) + D(t)G_1^{-1}(t)B(t)D(t)^-u(t) = D(t)G_1^{-1}(t)g(t), \quad t \in (0, 1], \quad (10)$$

the algebraic components are given by

$$Q_0(t)x(t) = -Q_0(t)G_1^{-1}(t)B(t)D(t)^-u(t) + Q_0(t)G_1^{-1}(t)g(t) \quad (11)$$

and the solutions of (3) can be expressed as

$$x(t) = D(t)^-u(t) + Q_0(t)x(t), \quad t \in (0, 1].$$

We are interested in solutions being smooth on the whole interval  $[0, 1]$  since we intend to apply a high order collocation scheme for their approximation. However,  $G_1(t)$  becomes singular for  $t = 0$ , in general. The asymptotic behavior of (10) related to a singularity of the first kind arises when we assume that  $G_1(0)$  is singular but  $tG_1^{-1}(t)$  has a continuous extension on  $[0, 1]$ . Consequently, we can rewrite the matrix  $D(t)G_1^{-1}(t)B(t)D(t)^-$  and obtain

$$D(t)G_1^{-1}(t)B(t)D(t)^- =: -\frac{1}{t}M(t), \quad (12)$$

where  $M \in C[0, 1]$ . For the subsequent existence and uniqueness analysis we require  $M \in C^1[0, 1]$ , which means that the problem data needs to be appropriately smooth. Let us denote the right-hand side of (10) by  $f(t)$ , then we arrive at the inherent ODE of the form

$$u'(t) = \frac{1}{t}M(t)u(t) + f(t), \quad t \in (0, 1]. \quad (13)$$

As mentioned before, we are especially interested in smooth solutions  $x$  and therefore  $u$  needs to be at least in  $C[0, 1]$ . It turns out that the smoothness of  $u$  depends on the smoothness of  $f$  and the eigenstructure of  $M(0)$ . The boundary conditions associated with (13) have to be chosen such that a well-posed singular boundary value problem results. The theoretical background for this problem class, where  $f \in C[0, 1]$ , is discussed in detail in [20, 28], for example. In order to use this standard theory we assume that  $G_1^{-1}(t)g(t)$  and thus  $f(t)$  are in  $C[0, 1]$ .

In this paper we focus our attention on boundary value problems for singular ODE systems (13) which can equivalently be expressed as a well-posed initial value problem with initial conditions at  $t^* = 0$  or terminal conditions at  $t^* = 1$ . This means a restriction on the spectrum of the matrix  $M(0)$  from (12), see [28, 31], for a detailed explanation of this fact. The reason for the above assumption is that we intend to employ a shooting argument in the course of the analysis.

A singular initial value problem posed at  $t^* = 0$  for the differential equation (13) is well-posed if and only if the spectrum of  $M(0)$  contains no eigenvalues with positive real parts and the initial value satisfies  $u(0) = \gamma \in \ker M(0)$ . A singular terminal value problem posed at  $t^* = 1$  is well-posed if and only if the spectrum of  $M(0)$  contains no eigenvalues with negative real parts and the invariant subspace associated with the eigenvalue zero coincides with the nullspace of  $M(0)$ .

The inherent ODE (13) is augmented by the boundary conditions

$$B_a u(0) + B_b u(1) = \beta, \quad (14)$$

with appropriately chosen  $n \times n$  matrices  $B_a$ ,  $B_b$ , and right-hand side  $\beta$ . For the original DAE (3) this yields the boundary conditions

$$B_a D(0)x(0) + B_b D(1)x(1) = \beta. \quad (15)$$

For a well-posed boundary value problem (13), (14), the continuous solution can be represented in the form

$$u(t) = Ec + tF(t), \quad t \in [0, 1], \quad (16)$$

where the columns of  $E$  are a basis of  $\ker M(0)$  and  $F(t) \in C[0, 1]$ , cf. [20]. Consequently, the solution of the full problem (3) and (15) is continuous on  $[0, 1]$ , cf. (11), if the limit

$$\Theta_c := \lim_{t \rightarrow 0} Q_0(t)G_1^{-1}(t)B(t)D(t)^- Ec$$

exists, and hence we can set

$$\begin{aligned} x_0 &= x(0) = D(0)^- Ec - Q_0(0)\Theta_c \\ &+ Q_0(0) \left( -\lim_{t \rightarrow 0} tG_1^{-1}(t)B(t)D(t)^- F(t) + \lim_{t \rightarrow 0} G_1^{-1}(t)g(t) \right). \end{aligned} \quad (17)$$

In order to ensure that for any continuous solution  $u$  of the inherent ODE (13) there exists a continuous solution  $x$  of the DAE (3), we additionally assume that

$$\Theta := \lim_{t \rightarrow 0} Q_0(t)G_1^{-1}(t)B(t)D(t)^- R$$

exists, where  $R \in \mathbb{R}^{n \times n}$  is a projector onto  $\ker M(0)$ . Consequently, we have  $\Theta_c = \Theta Ec$ , and

$$\begin{aligned} x_0 &= D(0)^- Ec - Q_0(0)\Theta Ec \\ &+ Q_0(0) \left( -\lim_{t \rightarrow 0} tG_1^{-1}(t)B(t)D(t)^- F(t) + \lim_{t \rightarrow 0} G_1^{-1}(t)g(t) \right). \end{aligned} \quad (18)$$

It is important to note that in order to be able to set up a well-posed analytical problem, three (sufficient) conditions have been made. We have assumed that

$$tG_1^{-1}(t), \quad G_1^{-1}(t)g(t), \quad Q_0(t)G_1^{-1}(t)B(t)D(t)^- E \quad (19)$$

have continuous extensions on  $[0, 1]$ . The last assumption can also be expressed in terms of the so-called canonical projector,

$$Q_{can}(t) := Q_0(t)G_1^{-1}(t)B(t). \quad (20)$$

In general,  $Q_{can}(t)$  has to be continuous on  $[0, 1]$  to guarantee (19).

*Remark:* We exemplify the meaning of assumptions (19) by considering a special DAE system given by

$$tx_1'(t) + B_{11}(t)x_1(t) + B_{12}(t)x_2(t) = g_1(t), \quad (21a)$$

$$B_{21}(t)x_1(t) + B_{22}(t)x_2(t) = g_2(t), \quad (21b)$$

where  $B_{11}(t), B_{12}(t), B_{21}(t), B_{22}(t) \in \mathbb{R}^{m \times m}$ ,  $g_1(t), g_2(t) \in \mathbb{R}^m$ , and  $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^{2m}$ . It follows immediately that we can write the above system in the form (3), where

$$A(t) = \begin{pmatrix} tI \\ 0 \end{pmatrix}, \quad D(t) = (I \ 0), \quad B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}.$$

Let

$$Q_0(t) := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

then

$$G_1(t) = \begin{pmatrix} tI & B_{12}(t) \\ 0 & B_{22}(t) \end{pmatrix}, \quad G_1^{-1}(t) = \begin{pmatrix} \frac{1}{t}I & -\frac{1}{t}B_{12}(t)B_{22}^{-1}(t) \\ 0 & B_{22}^{-1}(t) \end{pmatrix},$$

and

$$Q_0(t)G_1^{-1}(t)B(t)D(t)^- = \begin{pmatrix} 0 \\ B_{22}^{-1}(t)B_{21}(t) \end{pmatrix}.$$

Let us assume that  $B_{22}(t)$  is nonsingular on  $[0, 1]$ . Then we can decouple (21), and (3) is an index 1 DAE on  $(0, 1]$ . Moreover,  $tG_1^{-1}(t)$  and  $Q_0(t)G_1^{-1}(t)B(t)D(t)^- R$  are continuous on  $[0, 1]$ .

The data of the corresponding inherent ODE (13) read:

$$M(t) = B_{11}(t) - B_{12}(t)B_{22}^{-1}(t)B_{21}(t), \quad f(t) = \frac{1}{t}(g_1(t) - B_{12}(t)B_{22}^{-1}(t)g_2(t)),$$

where  $M(t), f(t) = G_1^{-1}(t)g(t) \in C[0, 1]$ . Clearly, the differential component is  $u(t) = D(t)x(t) = x_1(t)$  and the algebraic component is

$$Q_0(t)x(t) = \begin{pmatrix} 0 \\ x_2(t) \end{pmatrix}, \quad x_2(t) = -B_{22}^{-1}(t)B_{21}(t)u(t) + B_{22}^{-1}(t)g_2(t).$$

If we only assume that  $B_{22}(t)$  is nonsingular on  $(0, 1]$  and  $B_{22}(0)$  is singular, then it is clear that we will require additional conditions to guarantee that the inherent ODE exhibits a singularity of the first kind, that it has a continuous solution  $u$ , and that this solution yields a smooth solution  $x$ . In particular, these requirements are satisfied if the matrices  $tB_{22}^{-1}(t)$ ,  $B_{22}^{-1}(t)B_{21}(t)$ , and  $B_{12}(t)B_{22}^{-1}(t)$  have continuous extensions on the closed interval  $[0, 1]$ .

For a high order method to work efficiently it is necessary that the analytical solution  $x$  is appropriately smooth. This means that the original problem data need to be smooth enough and additionally, all positive real parts of the eigenvalues of  $M(0)$  need to be sufficiently large, cf. [20].

In the next section, we apply polynomial collocation to approximate solutions of (3) by means of an enlarged system,

$$A(t)u'(t) + B(t)x(t) = g(t), \quad (22)$$

$$D(t)x(t) - u(t) = 0, \quad t \in (0, 1], \quad (23)$$

which can be brought into the standard form

$$\widehat{A}(t)(\widehat{D}(t)\widehat{x}(t))' + \widehat{B}(t)\widehat{x}(t) = \widehat{g}(t), \quad t \in (0, 1], \quad (24)$$

where  $\widehat{x}(t) = (x(t), u(t))^T$ ,  $\widehat{g}(t) = (g(t), 0)^T$ , and

$$\widehat{A}(t) = \begin{pmatrix} A(t) \\ 0 \end{pmatrix}, \quad \widehat{D}(t) = (0 \quad I) =: \widehat{D}, \quad \widehat{B}(t) = \begin{pmatrix} B(t) & 0 \\ D(t) & -I \end{pmatrix}.$$

Problem (24) is a regular DAE system with properly stated leading term and tractability index one on  $(0, 1]$ . To see this, note that  $\widehat{D}(t)$  is constant and therefore we define the related matrices  $\widehat{G}_0(t)$ ,  $\widehat{Q}_0$ , and  $\widehat{G}_1(t)$  as

$$\widehat{G}_0(t) := \widehat{A}(t)\widehat{D} = \begin{pmatrix} 0 & A(t) \\ 0 & 0 \end{pmatrix}, \quad \widehat{Q}_0 := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\widehat{G}_1(t) := \widehat{G}_0(t) + \widehat{B}(t)\widehat{Q}_0 = \begin{pmatrix} B(t) & A(t) \\ D(t) & 0 \end{pmatrix},$$

respectively. Moreover,

$$\ker \widehat{G}_1 = \{z \in \mathbb{R}^{m+n}, z = (z_1, z_2)^T; z_1 = Q_0 w, z_2 = Dw, w \in \ker G_1\}$$

for all  $t \in [0, 1]$  which means that  $\widehat{G}_1$  is nonsingular on  $(0, 1]$  and  $\widehat{G}_1(0)$  is singular, simultaneously with  $G_1$ . From the standard decoupling procedure applied to (24) the same inherent ODE (13) results, and for the solution  $\widehat{x}$  on  $(0, 1]$  the following representation holds:

$$\widehat{x}(t) = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} (I - Q_0(t)G_1^{-1}(t)B(t))D(t)^{-1}u(t) + Q_0(t)G_1^{-1}(t)g(t) \\ u(t) \end{pmatrix}. \quad (25)$$

It is easily seen that the matrices  $t\widehat{G}_1^{-1}(t)$  and  $\widehat{Q}_0(t)\widehat{G}_1^{-1}(t)\widehat{B}(t)\widehat{D}(t)^{-1}\widehat{R}$ , where  $\widehat{R} = R$ , are continuous on  $[0, 1]$ .

## 2.2 Collocation Methods

For the theoretical discussion of collocation methods, we define meshes

$$\Delta := (\tau_0, \tau_1, \dots, \tau_N),$$

and  $h_i := \tau_{i+1} - \tau_i$ ,  $i = 0, \dots, N-1$ ,  $\tau_0 = 0$ ,  $\tau_N = 1$ . For reasons of simplicity, we restrict the discussion to equidistant meshes,  $h_i = h$ ,  $i = 0, \dots, N-1$ . However, the results also hold for nonuniform meshes which have a limited variation in the stepsizes. For collocation,  $m$  distinct points  $t_{i,j} := \tau_i + h_i \rho_j$ ,  $j = 1, \dots, m$ ,



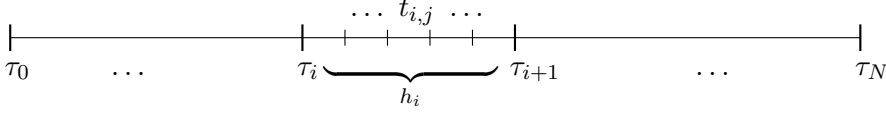


Figure 1: The computational grid

are inserted in each subinterval  $(\tau_i, \tau_{i+1})$ . Since we want to focus on Gaussian points, we restrict ourselves to interior collocation points, where  $\rho_1 > 0$  and  $\rho_m < 1$ . A grid with equidistant interior collocation points is illustrated in Figure 1.

Now, let us denote by  $\mathcal{B}_m$  the Banach space of continuous, piecewise polynomial functions  $q \in \mathbb{P}_m$  of degree  $\leq m$ ,  $m \in \mathbb{N}$ , equipped with the maximum norm  $\|\cdot\|_\infty$ . In the following, we denote by  $q$  the vector valued functions from  $\mathcal{B}_m$  independently of the number of their components.

By  $p \in \mathcal{B}_m$  we denote an approximation to the exact solution  $x$  of (3), (15), and by  $q \in \mathcal{B}_m$  an approximation to the exact solution  $u$  of the inherent ODE (13), cf. (25). The numerical scheme defining  $p$  and  $q$  has the form

$$A(t_{i,j})q'(t_{i,j}) + B(t_{i,j})p(t_{i,j}) = g(t_{i,j}), \quad (26)$$

$$D(t_{i,j})p(t_{i,j}) - q(t_{i,j}) = 0, \quad (27)$$

$$B_a D(0)p(0) + B_b D(1)p(1) = \beta, \quad (28)$$

where  $j = 1, \dots, m$  and  $i = 0, \dots, N - 1$ . It is clear by inspection of the number of unknowns and equations that further conditions will be necessary to close the system for the numerical treatment. Clearly, these additional conditions have to be consistent with the original DAEs. Various choices are possible, e.g.,

$$B(0)p(0) - g(0) \in \mathcal{R}(A(0)), \quad q(0) = D(0)p(0), \quad (29)$$

or

$$B(1)p(1) - g(1) \in \mathcal{R}(A(1)), \quad q(1) = D(1)p(1), \quad (30)$$

or combinations thereof. We will address this question below.

We first treat the case of boundary value problems which can equivalently be posed as initial value problems at  $t^* = 0$ . As we will demonstrate at the end of this section, it is sufficient to investigate the convergence for the IVP case. Consequently, we discuss the solution of

$$A(t)(D(t)x(t))' + B(t)x(t) = g(t), \quad (31)$$

$$D(0)x(0) = \gamma, \quad (32)$$

which is at least continuous on  $[0, 1]$  and satisfies, see (17),

$$Q_0(0)x(0) = Q_0(0)x_0. \quad (33)$$

Recall that in the initial condition (32) the vector  $\gamma$  can be freely chosen in  $\ker M(0)$ , and (33) is a consistent initial condition closing the system for the numerical treatment.

When applying collocation to (31)–(33), we are seeking piecewise polynomial functions  $p$  and  $q$  in  $\mathcal{B}_m$  which for  $j = 1, \dots, m$ , and  $i = 0, \dots, N - 1$  satisfy

$$A(t_{i,j})q'(t_{i,j}) + B(t_{i,j})p(t_{i,j}) = g(t_{i,j}), \quad (34)$$

$$D(t_{i,j})p(t_{i,j}) - q(t_{i,j}) = 0, \quad (35)$$

$$D(0)p(0) = \gamma, \quad (36)$$

$$Q_0(0)p(0) = Q_0(0)x_0, \quad P_0(0)p(0) = D(0)^- q(0). \quad (37)$$

The following arguments are very similar to those given in the analysis of collocation methods applied to ordinary differential equations with a singularity of the first kind. Thus, to keep the presentation concise we refer to [5, 21, 27] for many of the technical details. First, we study the existence and uniqueness of polynomial functions  $p, q \in \mathcal{B}_m$  satisfying (34)–(37). As in the analytical case discussed in Section 2.1, the system can be decoupled, yielding collocation equations for the inherent ODE,  $j = 1, \dots, m$ ,  $i = 0, \dots, N-1$ ,

$$q'(t_{i,j}) = \frac{1}{t_{i,j}} M(t_{i,j}) q(t_{i,j}) + f(t_{i,j}), \quad (38)$$

$$q(0) = \gamma \in \ker M(0), \quad (39)$$

and the value assignments,  $j = 1, \dots, m$ ,  $i = 0, \dots, N-1$ ,

$$p(t_{i,j}) = (I - Q_0(t_{i,j}) G_1^{-1}(t_{i,j}) B(t_{i,j})) D(t_{i,j})^{-1} q(t_{i,j}) + Q_0(t_{i,j}) G_1^{-1}(t_{i,j}) g(t_{i,j}), \quad (40)$$

$$p(0) = D(0)^{-1} \gamma + Q_0(0) x_0. \quad (41)$$

The results in [5, 21, 27] show that the collocation equations (38), (39) have a unique solution which satisfies

$$q(t) = Ec + t\varphi(t) = \gamma + t\varphi(t), \quad \varphi \in \mathcal{B}_m, \quad (42)$$

cf. (16). Conditions (40) and (41) uniquely define a piecewise polynomial  $p \in \mathcal{B}_m$ .

### 2.3 Error Analysis: IVPs, $t^* = 0$

We now analyze the error of the approximation  $p$  provided by (34)–(37). Let us introduce an error function  $\hat{e} \in \mathcal{B}_m$  defined by

$$\hat{e}'(t_{i,j}) = \hat{x}'(t_{i,j}) - \hat{p}'(t_{i,j}), \quad j = 1, \dots, m, \quad i = 0, \dots, N-1, \quad (43)$$

$$\hat{e}(0) = 0, \quad (44)$$

cf. [5, 21], where  $\hat{x}(t) = (x(t), u(t))^T$  is the exact solution, and  $\hat{p}(t) = (p(t), q(t))^T$  is its approximation by a collocation polynomial. Moreover,  $\hat{e}(t) := (e(t), e_u(t))^T$ . Trivially,

$$\hat{e}'(t) = \sum_{\mu=1}^m l_{\mu} \left( \frac{t - \tau_i}{h} \right) \hat{x}'(t_{i,\mu}) - \hat{p}'(t), \quad t \in (\tau_i, \tau_{i+1}), \quad (45)$$

where  $l_{\mu}$  denotes the  $\mu$ th Lagrange polynomial (of degree  $m-1$ ) associated with the abscissae  $\rho_1, \dots, \rho_m$  on the interval  $[0, 1]$ . In our analysis we wish to take into account that possibly the differential and the algebraic solution components have different smoothness. Thus, we assume that

$$u = Dx \in C^{\bar{k}+1}[0, 1], \quad x \in C^{\bar{l}+1}[0, 1],$$

and set

$$k := \min\{\bar{k}, m\}, \quad l := \min\{\bar{l}, m\}.$$

From standard results for interpolation, see [19], we can conclude that

$$\hat{e}'(t) = \hat{x}'(t) - \hat{p}'(t) + \begin{pmatrix} O(h^l) \\ O(h^k) \end{pmatrix}. \quad (46)$$

Due to (44), integration of (46) yields

$$\hat{e}(t) = \hat{x}(t) - \hat{p}(t) + t \begin{pmatrix} r(t) \\ s(t) \end{pmatrix}, \quad (47)$$

with  $r(t) = O(h^l)$  and  $s(t) = O(h^k)$ . Thus,  $\hat{e}$  satisfies the collocation scheme

$$A(t_{i,j}) e'_u(t_{i,j}) + B(t_{i,j}) e(t_{i,j}) = t_{i,j} B(t_{i,j}) r(t_{i,j}), \quad (48)$$

$$D(t_{i,j}) e(t_{i,j}) - e_u(t_{i,j}) = t_{i,j} (D(t_{i,j}) r(t_{i,j}) - s(t_{i,j})), \quad (49)$$

$$e(0) = 0, \quad e_u(0) = 0. \quad (50)$$

Again, we can reduce the problem to the collocation scheme applied to the inherent ODE related to (48), (49), cf. (38), and obtain

$$e'_u(t_{i,j}) = \frac{1}{t_{i,j}} M(t_{i,j}) e_u(t_{i,j}) - M(t_{i,j}) O(h^k), \quad (51)$$

$$e_u(0) = 0. \quad (52)$$

To prove this statement we multiply (48) by  $D(t_{i,j})G_1^{-1}(t_{i,j})$  and obtain

$$\begin{aligned} e'_u(t_{i,j}) + D(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j})D(t_{i,j})^{-1}e(t_{i,j}) \\ = t_{i,j}D(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j})r(t_{i,j}), \end{aligned} \quad (53)$$

taking into account the relations  $DG_1^{-1}A = I$ ,  $DG_1^{-1}BQ_0 = 0$ , and  $DG_1^{-1}B = DG_1^{-1}BD^{-1}D$ . We now express  $D(t_{i,j})e(t_{i,j})$  using (49),

$$D(t_{i,j})e(t_{i,j}) = e_u(t_{i,j}) + t_{i,j}D(t_{i,j})r(t_{i,j}) - t_{i,j}s(t_{i,j}),$$

and insert this into (53). Rearranging yields

$$\begin{aligned} e'_u(t_{i,j}) - \frac{1}{t_{i,j}} M(t_{i,j}) e_u(t_{i,j}) &= -M(t_{i,j}) s(t_{i,j}) \\ &+ (M(t_{i,j})D(t_{i,j}) + t_{i,j}D(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j}))r(t_{i,j}). \end{aligned} \quad (54)$$

The second term on the right-hand side vanishes, since

$$\begin{aligned} MD + tDG_1^{-1}B &= -tDG_1^{-1}BD^{-1}D + tDG_1^{-1}B \\ &= tDG_1^{-1}B(I - D^{-1}D) = tDG_1^{-1}BQ_0 = 0, \end{aligned}$$

and this completes the argument.

Note that the inhomogeneity in (51) remains uniformly bounded for  $t \in [0, 1]$ . Therefore, we can use [5, 21], see also (42), to conclude that

$$e_u(t) = tO(h^k). \quad (55)$$

Consequently, the error function  $e \in \mathcal{B}_m$  can be uniquely described by assigning its values,

$$\begin{aligned} e(t_{i,j}) &= (I - Q_0(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j}))D(t_{i,j})^{-1}e_u(t_{i,j}) \\ &+ Q_0(t_{i,j})G_1^{-1}(t_{i,j})t_{i,j}B(t_{i,j})r(t_{i,j}), \end{aligned} \quad (56)$$

$$e(0) = 0. \quad (57)$$

Finally, we have  $e(t_{i,j}) = O(h^{\min\{l,k\}})$  and since  $e \in \mathcal{B}_m$ , it immediately follows that  $e(t) = O(h^{\min\{l,k\}})$  and

$$x(t) - p(t) = O(h^{\min\{l,k\}}) \quad (58)$$

hold.

## 2.4 Error Analysis: IVPs, $t^* = 1$

Consider the case where the boundary value problem can equivalently be posed as an initial value problem at  $t^* = 1$ . The convergence proof given above requires a few modifications in that case. The collocation equations (34) and (35) are now augmented by the terminal conditions, cf. (11), (25),

$$\begin{aligned} D(1)p(1) &= \beta, \\ Q_0(1)p(1) &= -Q_0(1)G_1^{-1}(1)B(1)D(1)^{-1}\beta + Q_0(1)G_1^{-1}(1)g(1), \\ P_0(1)p(1) &= D(1)^{-1}q(1). \end{aligned}$$

Clearly, (44) is replaced by  $\widehat{e}(1) = 0$ , whence (47) becomes

$$\widehat{e}(t) = \widehat{x}(t) - \widehat{p}(t) + (1-t) \begin{pmatrix} r(t) \\ s(t) \end{pmatrix}. \quad (59)$$

Consequently, the error function satisfies the collocation scheme

$$A(t_{i,j})e'_u(t_{i,j}) + B(t_{i,j})e(t_{i,j}) = (1-t_{i,j})B(t_{i,j})r(t_{i,j}), \quad (60)$$

$$D(t_{i,j})e(t_{i,j}) - e_u(t_{i,j}) = (1-t_{i,j})D(t_{i,j})r(t_{i,j}) - (1-t_{i,j})s(t_{i,j}), \quad (61)$$

$$e(1) = 0, \quad e_u(1) = 0. \quad (62)$$

The collocation scheme for the associated inherent ODE reads

$$e'_u(t_{i,j}) = \frac{1}{t_{i,j}}M(t_{i,j})e_u(t_{i,j}) - \frac{1}{t_{i,j}}M(t_{i,j})(1-t_{i,j})O(h^k), \quad (63)$$

$$e_u(1) = 0. \quad (64)$$

This yields, see [27] for the proof,

$$e_u(t) = O(h^k). \quad (65)$$

Moreover,

$$\begin{aligned} e(t_{i,j}) &= (I - Q_0(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j}))D(t_{i,j})^{-1}e_u(t_{i,j}) \\ &\quad + (1-t_{i,j})Q_0(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j})r(t_{i,j}) = \frac{1}{t_{i,j}}O(h^{\min\{l,k\}}), \end{aligned} \quad (66)$$

$$e(1) = 0. \quad (67)$$

The error bound (58) follows under the additional assumption that the canonical projector (20),

$$Q_{can}(t) := Q_0(t)G_1^{-1}(t)B(t),$$

has a continuous extension on  $[0, 1]$ . Otherwise, we may have an order reduction.

## 2.5 Error Analysis: BVPs

To conclude the error analysis, we give the shooting argument which demonstrates that the analysis of initial value problems given above is sufficient for the analysis of boundary value problems for the DAE (3). This argument is the same as the one given in [5, 21] for boundary value problems of singular ODEs, and carries over to our present analysis. Recall that  $tG_1^{-1}(t)$ ,  $G_1^{-1}(t)g(t)$ , and  $Q_0(t)G_1^{-1}(t)B(t)D(t)^{-1}E$  are assumed to have continuous extensions on  $[0, 1]$ . The columns of  $E$  form a basis for  $\ker M(0)$ .

The solution of the inherent ODE (13) can be written as

$$u(t) = \sum_{\mu=1}^r a_{\mu}u_{\mu}(t) + \tilde{u}(t),$$

where  $U = (u_{\mu})_{\mu=1,\dots,r}$  is the solution of the homogeneous matrix ODE with initial condition  $U(0) = E$ , and the particular solution  $\tilde{u}$  satisfies the inhomogeneous equation, albeit with homogeneous initial condition. It is easy to see from Section 2.1 that this implies a representation of  $x$  as

$$x(t) = \sum_{\mu=1}^r a_{\mu}x_{\mu}(t) + \tilde{x}(t),$$

where  $X = (x_{\mu})_{\mu=1,\dots,r}$  with

$$x_{\mu}(t) = (I - Q_0(t)G_1^{-1}(t)B(t))D(t)^{-1}u_{\mu}(t) \quad (68)$$

solves the matrix DAE

$$A(t)(D(t)X(t))' + B(t)X(t) = 0, \quad (69)$$

$$D(0)X(0) = E, \quad (70)$$

and  $\tilde{x} = (I - Q_0G_1^{-1}B)D^{-1}\tilde{u} + Q_0G_1^{-1}g$  satisfies

$$A(t)(D(t)\tilde{x}(t))' + B(t)\tilde{x}(t) = g(t), \quad (71)$$

$$D_0\tilde{x}(0) = 0. \quad (72)$$

Due to our assumptions,  $X$  and  $\tilde{x}$  are continuous on  $[0, 1]$ . An analogous representation holds for the collocation solution  $p$ ,

$$p(t) = \sum_{\mu=1}^r b_{\mu}p_{\mu}(t) + \tilde{p}(t), \quad (73)$$

where  $P = (p_{\mu})_{\mu=1, \dots, r}$  and  $\tilde{p}$  are the collocation solutions of the problems (69), (70) and (71), (72), respectively. Similarly as in the ODE case, see [5, 21], substitution of the solution representations (68) and (73) into (14) yields

$$a_{\mu} - b_{\mu} = O(h^k).$$

Consequently, the convergence result

$$\|u - q\|_{\infty} = O(h^k), \quad \|x - p\|_{\infty} = O(h^{\min\{l, k\}})$$

holds also for the solution of the boundary value problem (3), (15).

The considerations for problems posed at  $t^* = 1$  are analogous.

### 3 Numerical Experiments

Before continuing with the results of numerical experiments we stress that in the above theory, the important assumptions (19),

$$tG_1^{-1}(t) \in C[0, 1], \quad G_1^{-1}(t)g(t) \in C[0, 1], \quad Q_0(t)G_1^{-1}(t)B(t)D(t)^{-1}E \in C[0, 1] \quad (74)$$

have been used. They are sufficient for the stage order of the collocation scheme to hold. When they are violated, order reductions in the algebraic components have been observed. Especially, order reductions can be due to the behavior of the canonical projector  $Q_{can}(t) := Q_0(t)G_1^{-1}(t)B(t)$  for  $t \rightarrow 0$ , in the case when  $Q_{can}$  becomes unbounded in this limit. In the following, we consider two problem classes and try to illustrate important aspects of the theory by appropriately constructed model problems. All experiments have been carried out in MATLAB.

#### 3.1 Problem Class 1 - setting and analytical properties

The system of DAEs in this problem class has the form

$$\begin{pmatrix} t^k \\ 1 \end{pmatrix} (x_1(t) + \alpha(t)x_2(t))' + \begin{pmatrix} t^k\beta(t) & -t^k\alpha'(t) \\ 0 & \gamma(t) - \alpha'(t) \end{pmatrix} x(t) = \begin{pmatrix} t^kg_1(t) \\ g_2(t) \end{pmatrix} =: g(t), \quad (75)$$

where  $k \geq 0$ . Model problems in this class can be constructed in such a way that the inherent differential equation is regular or singular, with singularity of the first or of the second kind, see [30] for comprehensive experimental results on the convergence of collocation schemes applied to solve such problems.

For the models considered here,  $k = 0$  and the above system has the form

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} (x_1(t) + \alpha(t)x_2(t))' + \begin{pmatrix} \beta(t) & -\alpha'(t) \\ 0 & \gamma(t) - \alpha'(t) \end{pmatrix} x(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} =: g(t). \quad (76)$$

The inherent ordinary differential equation for  $u(t) := x_1(t) + \alpha(t)x_2(t)$  reads:

$$u'(t) = -\frac{\beta(t)(\gamma(t) - \alpha'(t))}{\varphi(t)}u(t) + \frac{1}{\varphi(t)}\left((\gamma(t) - \alpha'(t))g_1(t) + (\alpha(t)\beta(t) + \alpha'(t))g_2(t)\right), \quad (77)$$

where  $\varphi(t) := \gamma(t) + \alpha(t)\beta(t)$ . For the components  $x_1$  and  $x_2$  of the solution  $x(t)$  we then have

$$x_1(t) = \frac{\gamma(t)}{\varphi(t)}u(t) + \frac{\alpha(t)}{\varphi(t)}\left(g_1(t) - g_2(t)\right), \quad (78a)$$

$$x_2(t) = \frac{\beta(t)}{\varphi(t)}u(t) - \frac{1}{\varphi(t)}\left(g_1(t) - g_2(t)\right). \quad (78b)$$

The solution components  $x_1$  and  $x_2$  have to satisfy the consistency condition

$$\beta(t)x_1(t) - \gamma(t)x_2(t) = g_1(t) - g_2(t) \quad (79)$$

for any  $t \in (0, 1]$ .

We now discuss in more detail the data for the decoupling procedure of (76) which leads to (77) and the representation (78) of the solution of (76). With

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad D(t) = (1 \quad \alpha(t)), \quad B(t) = \begin{pmatrix} \beta(t) & -\alpha'(t) \\ 0 & \gamma(t) - \alpha'(t) \end{pmatrix},$$

system (76) has the form (3) and we have,

$$Q_0(t) = \begin{pmatrix} 0 & -\alpha(t) \\ 0 & 1 \end{pmatrix}, \quad P_0(t) = \begin{pmatrix} 1 & \alpha(t) \\ 0 & 0 \end{pmatrix}, \quad D^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G_0(t) := AD(t) = \begin{pmatrix} 1 & \alpha(t) \\ 1 & \alpha(t) \end{pmatrix}.$$

Moreover, since we assumed (76) to be an index 1 DAE, the matrix  $G_1(t) = G_0(t) + B(t)Q_0(t)$  is nonsingular,

$$G_1(t) = \begin{pmatrix} 1 & \alpha(t) \\ 1 & \alpha(t) \end{pmatrix} + \begin{pmatrix} \beta(t) & -\alpha'(t) \\ 0 & \gamma(t) - \alpha'(t) \end{pmatrix} \begin{pmatrix} 0 & -\alpha(t) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (\alpha(t) - \alpha(t)\beta(t) - \alpha'(t)) \\ 1 & \alpha(t) + \gamma(t) - \alpha'(t) \end{pmatrix}.$$

Consequently,  $\det G_1(t) = \gamma(t) + \alpha(t)\beta(t) \neq 0$  for  $t \in (0, 1]$  and

$$G_1^{-1}(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \alpha(t) + \gamma(t) - \alpha'(t) & -\alpha(t) + \alpha(t)\beta(t) + \alpha'(t) \\ -1 & 1 \end{pmatrix}. \quad (80)$$

Also, we have

$$D(t)G_1^{-1}(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)}(\gamma(t) - \alpha'(t) \quad \alpha(t)\beta(t) + \alpha'(t)),$$

and

$$D(t)G_1^{-1}(t)B(t)D^-(t) = \frac{\beta(t)(\gamma(t) - \alpha'(t))}{\alpha(t)\beta(t) + \gamma(t)}.$$

The inherent ODE related to (76) now reads

$$u'(t) + D(t)G_1^{-1}(t)B(t)D^-(t)u(t) = D(t)G_1^{-1}(t)g(t),$$

or equivalently,

$$u'(t) + \frac{\beta(t)(\gamma(t) - \alpha'(t))}{\alpha(t)\beta(t) + \gamma(t)}u(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)}((\gamma(t) - \alpha'(t))g_1(t) + (\alpha(t)\beta(t) + \alpha'(t))g_2(t)). \quad (81)$$

In order to specify  $x$  we need to calculate the canonical projector (20),

$$Q_{can}(t) = Q_0(t)G_1^{-1}(t)B(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \alpha(t)\beta(t) & -\alpha(t)\gamma(t) \\ -\beta(t) & \gamma(t) \end{pmatrix}. \quad (82)$$

Moreover, with

$$Q_0(t)G_1^{-1}(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \alpha(t) & -\alpha(t) \\ -1 & 1 \end{pmatrix},$$

we obtain

$$P_{can}(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \gamma(t) & \alpha(t)\gamma(t) \\ \beta(t) & \alpha(t)\beta(t) \end{pmatrix},$$

and finally, the explicit formula for the solution  $x$  of the DAE (76) follows from

$$x(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \gamma(t) \\ \beta(t) \end{pmatrix} u(t) + \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \alpha(t)g_1(t) - \alpha(t)g_2(t) \\ -g_1(t) + g_2(t) \end{pmatrix}. \quad (83)$$

It is important to note that the sign of the term

$$-\frac{\beta(t)(\gamma(t) - \alpha'(t))}{\varphi(t)},$$

cf. (77), with  $\beta(t)(\gamma(t) - \alpha'(t))$  evaluated at  $t = 0$ , is crucial and decides about the boundary conditions necessary for the solution  $u$  (and  $x$ ) to be at least in  $C[0, 1]$ , cf. [20] for the analysis of singular ODEs.

### Problem 1.1

We choose  $\alpha(t) \equiv -1$ , whence  $\varphi(t) = \gamma(t) - \beta(t)$ . With  $\gamma(t) = t + 2$  and  $\beta(t) = 2$ , we obtain  $\varphi(t) = t$  and the equation for  $u$  reads:

$$u'(t) = -\frac{2(t+2)}{t} u(t) + \frac{1}{t} \left( (t+2)g_1(t) - 2g_2(t) \right).$$

The requirement  $\varphi(t) \neq 0$  for  $t \in (0, 1]$  is satisfied because  $\gamma(t) - \beta(t) = t \neq 0$  for  $t \in (0, 1]$ . We additionally choose  $u(t) = te^{5t}$  and  $g_1(t) = -te^{5t}$  which yields

$$g_2(t) = -\frac{8t+7}{2} te^{5t}.$$

Finally, the solution of the problem is

$$x_1(t) = -\frac{6t+1}{2} e^{5t}, \quad x_2(t) = -\frac{8t+1}{2} e^{5t},$$

and the system of DAEs reads:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} (x_1(t) - x_2(t))' + \begin{pmatrix} 2 & 0 \\ 0 & t+2 \end{pmatrix} x(t) = \begin{pmatrix} -te^{5t} \\ -\frac{8t+7}{2} te^{5t} \end{pmatrix}.$$

The boundary conditions are

$$x_1(0) - x_2(0) = 0, \quad 2x_1(1) - 3x_2(1) = 6.5e^5.$$

Concerning the boundary conditions we note that here

$$\frac{1}{t}M(t) = -\frac{2(t+2)}{t}, \quad M(0) = -4,$$

and therefore we need to prescribe the value of  $u(0) = x_1(0) - x_2(0)$ . The second boundary condition is equivalent to

$$\beta(t)x_1(t) - \gamma(t)x_2(t) = g_1(t) - g_2(t),$$

cf. (79). For this example condition (74) is satisfied. From

$$\alpha(t) = -1, \quad \gamma(t) = t + 2, \quad \beta(t) = 2, \quad \alpha(t)\beta(t) + \gamma(t) = t,$$

and (80), we obtain

$$G_1^{-1}(t) = \frac{1}{t} \begin{pmatrix} t+1 & -1 \\ -1 & 1 \end{pmatrix} = O\left(\frac{1}{t}\right) \Rightarrow tG_1^{-1}(t) = O(1).$$

Moreover

$$G_1^{-1}(t)g(t) = \frac{1}{t} \begin{pmatrix} t+1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -te^{5t} \\ -\frac{8t+7}{2}te^{5t} \end{pmatrix} = \begin{pmatrix} t+1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -e^{5t} \\ -\frac{8t+7}{2}e^{5t} \end{pmatrix} \in C[0,1]$$

and  $E = 0$ . The associated canonical projector (20) is

$$Q_{can}(t) = \begin{pmatrix} -\frac{2}{t} & \frac{2+t}{t} \\ -\frac{2}{t} & \frac{2+t}{t} \end{pmatrix} = O\left(\frac{1}{t}\right).$$

We now describe the convergence behavior of the quantities

$$gex_{\text{tau}} := \|p - x\|_{\text{tau}}, \quad gex_{\text{tcol}} := \|p - x\|_{\text{tcol}}, \quad geu_{\text{tau}} := \|q - u\|_{\text{tau}}, \quad geu_{\text{tcol}} := \|q - u\|_{\text{tcol}}.$$

The first two expressions are the maximal values of the global errors of the solution  $x$  in the mesh points only (tau)<sup>2</sup> and in all grid points, including  $\tau_i$  and collocation points  $t_{i,j}$ , (tcol), respectively. The last two expressions are the respective global errors for the differential solution component  $u$ .

According to Tables 1 and 2, since the solution  $x$  is appropriately smooth, we observe the following convergence behavior for the equidistant collocation points,

$$gex_{\text{tau}} = gex_{\text{tcol}} = geu_{\text{tau}} = geu_{\text{tcol}} = O(h^{m+1}) = O(h^4),$$

$$gex_{\text{tau}} = gex_{\text{tcol}} = geu_{\text{tau}} = geu_{\text{tcol}} = O(h^m) = O(h^4),$$

for  $m = 3$  and  $m = 4$ , respectively. Note that although the canonical projector is unbounded for  $t \rightarrow 0$ ,  $Q_{can}(t) = O(1/t)$ , no order reduction is observed. This is due to the fact that the involved inherent ODE is an initial value problem, and therefore an additional factor  $t$  helps to balance the unboundedness of  $Q_{can}(t) = O(1/t)$ , see (55) and (56). In general,

$$gex_{\text{tau}} = O(h^{m+1}), \quad gex_{\text{tcol}} = O(h^{m+1}), \quad geu_{\text{tau}} = O(h^{m+1}), \quad geu_{\text{tcol}} = O(h^{m+1}), \quad (84a)$$

$$gex_{\text{tau}} = O(h^m), \quad gex_{\text{tcol}} = O(h^m), \quad geu_{\text{tau}} = O(h^m), \quad geu_{\text{tcol}} = O(h^m), \quad (84b)$$

holds for  $m$  odd or even, respectively, see [30]. Moreover, note that the superconvergence behavior  $O(h^{2m})$  in  $\Delta$  does not hold in general, a well known fact in the context of singular ODEs.

### Problem 1.2

Here, we again have  $\alpha(t) \equiv -1$  and  $\varphi(t) = \gamma(t) - \beta(t)$ . With  $\gamma(t) = \sin t$  and  $\beta(t) = t$ , we obtain  $\varphi(t) = \sin t - t$  and the equation for  $u$  reads:

$$u'(t) = -\frac{\beta(t)\gamma(t)}{\gamma(t) - \beta(t)}u(t) + \frac{1}{\gamma(t) - \beta(t)}\left(\gamma(t)g_1(t) + \alpha(t)\beta(t)g_2(t)\right),$$

$$u'(t) = -\frac{t \sin t}{\sin t - t}u(t) + \frac{1}{\sin t - t}\left(\sin t g_1(t) - t g_2(t)\right).$$

The requirement  $\varphi(t) \neq 0$  for  $t \in (0, 1]$  is satisfied. We set  $u(t) = t - \sin t$  and  $g_1(t) = g_2(t)$ . Then

$$g_1(t) = g_2(t) = 1 - \cos t - t \sin t$$

---

<sup>2</sup>cf. the definition of  $\Delta$  in Section 2.2.



Uniform Mesh		Error for $\mathbf{x}$ at Grid $\mathbf{tau}$			Error for $\mathbf{x}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	2.963e-01			2.963e-01		
20	5.00e-02	2.089e-02	3.8	1.986e+03	2.089e-02	3.8	1.986e+03
40	2.50e-02	1.387e-03	3.9	2.570e+03	1.387e-03	3.9	2.570e+03
80	1.25e-02	8.941e-05	4.0	3.019e+03	8.941e-05	4.0	3.019e+03
160	6.25e-03	5.675e-06	4.0	3.324e+03	5.675e-06	4.0	3.324e+03
320	3.13e-03	3.575e-07	4.0	3.508e+03	3.575e-07	4.0	3.508e+03

Uniform Mesh		Error for $\mathbf{u}$ at Grid $\mathbf{tau}$			Error for $\mathbf{u}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	1.546e-02			1.546e-02		
20	5.00e-02	9.868e-04	4.0	1.443e+02	9.868e-04	4.0	1.443e+02
40	2.50e-02	6.200e-05	4.0	1.543e+02	6.200e-05	4.0	1.543e+02
80	1.25e-02	3.880e-06	4.0	1.576e+02	3.880e-06	4.0	1.576e+02
160	6.25e-03	2.426e-07	4.0	1.586e+02	2.426e-07	4.0	1.586e+02
320	3.13e-03	1.516e-08	4.0	1.589e+02	1.516e-08	4.0	1.589e+02

Uniform Mesh		Error for $\mathbf{x}$ at Grid $\mathbf{tau}$			Error for $\mathbf{x}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	1.687e-01			1.687e-01		
20	5.00e-02	1.199e-02	3.8	1.101e+03	1.199e-02	3.8	1.101e+03
40	2.50e-02	7.993e-04	3.9	1.450e+03	7.993e-04	3.9	1.450e+03
80	1.25e-02	5.161e-05	4.0	1.722e+03	5.161e-05	4.0	1.722e+03
160	6.25e-03	3.278e-06	4.0	1.907e+03	3.278e-06	4.0	1.907e+03
320	3.13e-03	2.066e-07	4.0	2.019e+03	2.066e-07	4.0	2.019e+03

Uniform Mesh		Error for $\mathbf{u}$ at Grid $\mathbf{tau}$			Error for $\mathbf{u}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	8.113e-05			6.495e-03		
20	5.00e-02	1.300e-06	6.0	7.461e+01	4.714e-04	3.8	3.953e+01
40	2.50e-02	5.283e-08	4.6	1.337e+00	3.166e-05	3.9	5.524e+01
80	1.25e-02	3.098e-09	4.1	1.896e-01	2.050e-06	3.9	6.715e+01
160	6.25e-03	1.880e-10	4.0	1.532e-01	1.304e-07	4.0	7.518e+01
320	3.13e-03	1.150e-11	4.0	1.442e-01	8.212e-09	4.0	8.080e+01

Table 1: Convergence results for Problem 1.1:  $m = 3$ , equidistant (top), and Gaussian collocation points (bottom).

follows and we obtain as a solution of the system

$$x_1(t) = -\sin t, \quad x_2(t) = -t.$$

The related system of DAEs is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} (x_1(t) - x_2(t))' + \begin{pmatrix} t & 0 \\ 0 & \sin t \end{pmatrix} x(t) = \begin{pmatrix} 1 - \cos t - t \sin t \\ 1 - \cos t - t \sin t \end{pmatrix}$$

and the boundary conditions are given by

$$x_1(1) - x_2(1) = -\sin(1) + 1, \quad x_1(1) - \sin(1)x_2(1) = 0 \Leftrightarrow x_2(1) = -1,$$

and it is clear that this problem must be treated as a terminal value problem from right ( $t = 1$ ) to left ( $t = 0$ ).

Since

$$\lim_{t \rightarrow 0} \left( -\frac{t \sin t}{\sin t - t} \right) = \lim_{t \rightarrow 0} \left( \frac{2}{t} \right) = \lim_{t \rightarrow 0} \left( \frac{\lambda}{t} \right), \quad \lambda > 0,$$

we prescribe the value of  $u$  at  $t = 1$ ,  $u(1) = x_1(1) - x_2(1)$ . The second boundary condition is again the consistency condition (79).

The condition (74) is not satisfied. From

$$\alpha(t) = -1, \quad \gamma(t) = \sin t, \quad \beta(t) = t, \quad \alpha(t)\beta(t) + \gamma(t) = \sin t - t,$$

Uniform Mesh		Error for $\mathbf{x}$ at Grid $\boldsymbol{\tau}$			Error for $\mathbf{x}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	3.675e-02			3.675e-02		
20	5.00e-02	2.321e-03	4.0	3.548e+02	2.321e-03	4.0	3.548e+02
40	2.50e-02	1.459e-04	4.0	3.627e+02	1.459e-04	4.0	3.627e+02
80	1.25e-02	9.155e-06	4.0	3.652e+02	9.155e-06	4.0	3.652e+02
160	6.25e-03	5.740e-07	4.0	3.677e+02	5.740e-07	4.0	3.677e+02
320	3.13e-03	3.382e-08	4.1	5.794e+02	3.382e-08	4.1	5.794e+02

Uniform Mesh		Error for $\mathbf{u}$ at Grid $\boldsymbol{\tau}$			Error for $\mathbf{u}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	1.672e-03			1.672e-03		
20	5.00e-02	1.032e-04	4.0	1.742e+01	1.032e-04	4.0	1.742e+01
40	2.50e-02	6.429e-06	4.0	1.674e+01	6.429e-06	4.0	1.674e+01
80	1.25e-02	4.015e-07	4.0	1.653e+01	4.015e-07	4.0	1.653e+01
160	6.25e-03	2.509e-08	4.0	1.645e+01	2.509e-08	4.0	1.645e+01
320	3.13e-03	1.572e-09	4.0	1.614e+01	1.572e-09	4.0	1.614e+01

Uniform Mesh		Error for $\mathbf{x}$ at Grid $\boldsymbol{\tau}$			Error for $\mathbf{x}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	1.367e-02			1.367e-02		
20	5.00e-02	8.633e-04	4.0	1.319e+02	8.633e-04	4.0	1.319e+02
40	2.50e-02	5.426e-05	4.0	1.349e+02	5.426e-05	4.0	1.349e+02
80	1.25e-02	3.405e-06	4.0	1.358e+02	3.405e-06	4.0	1.358e+02
160	6.25e-03	2.142e-07	4.0	1.338e+02	2.142e-07	4.0	1.338e+02
320	3.13e-03	1.347e-08	4.0	1.340e+02	1.347e-08	4.0	1.340e+02

Uniform Mesh		Error for $\mathbf{u}$ at Grid $\boldsymbol{\tau}$			Error for $\mathbf{u}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	1.140e-07			1.815e-04		
20	5.00e-02	2.479e-09	5.5	3.805e-02	6.356e-06	4.8	1.243e+01
40	2.50e-02	6.384e-11	5.3	1.830e-02	2.100e-07	4.9	1.597e+01
80	1.25e-02	1.974e-12	5.0	6.922e-03	6.748e-09	5.0	1.856e+01
160	6.25e-03	7.958e-13	1.3	6.150e-10	2.134e-10	5.0	2.053e+01
320	3.13e-03	1.080e-12	-0.4	8.506e-14	6.821e-12	5.0	1.896e+01

Table 2: Convergence results for Problem 1.1:  $m = 4$ , equidistant (top), and Gaussian collocation points (bottom).

and (80), it follows

$$G_1^{-1}(t) = \frac{1}{\sin t - t} \begin{pmatrix} \sin t - 1 & 1 - t \\ -1 & 1 \end{pmatrix} = O\left(\frac{1}{t^3}\right) \Rightarrow tG_1^{-1}(t) = O\left(\frac{1}{t^2}\right). \quad (85)$$

However,

$$G_1^{-1}(t)g(t) = \frac{1}{\sin t - t} \begin{pmatrix} \sin t - 1 & 1 - t \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 - \cos t - t \sin t \\ 1 - \cos t - t \sin t \end{pmatrix} = \begin{pmatrix} 1 - \cos t - t \sin t \\ 0 \end{pmatrix}$$

and  $E = 0$ . The canonical projector (20) has the form

$$Q_{can}(t) = \begin{pmatrix} -\frac{t}{\sin t - t} & \frac{\sin t}{\sin t - t} \\ -\frac{t}{\sin t - t} & \frac{\sin t}{\sin t - t} \end{pmatrix} = O\left(\frac{1}{t^2}\right). \quad (86)$$

Due to (85) and (86) we observe clear order reductions, see Tables 3 and 4. Here, we are dealing with a terminal value problem and there are no additional  $t$  factors to balance the behavior of  $Q_{can}(t)$ , see (65) and (66).

## 3.2 Problem Class 2

### 3.2.1 Problems with bounded canonical projector, $B_{22}$ nonsingular

The problems considered in the previous section exhibited scalar inherent ODEs with  $M(0) \in \mathbb{R}$ . Therefore, the associated BVP could be posed either as an *initial* value problem,  $M(0) < 0$ , or *terminal* value problem,  $M(0) > 0$ , see Problem 1.1 and Problem 1.2, respectively. In order to study the influence of

Uniform Mesh		Error for $\mathbf{x}_1$ at Grid $\mathbf{tau}$			Error for $\mathbf{x}_1$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	2.418e-02		2.366e-01	2.418e-02		2.366e-01
20	5.00e-02	1.217e-02	1.0	2.366e-01	1.217e-02	1.0	2.366e-01
40	2.50e-02	6.095e-03	1.0	2.417e-01	6.095e-03	1.0	2.417e-01
80	1.25e-02	3.049e-03	1.0	2.433e-01	3.049e-03	1.0	2.433e-01
160	6.25e-03	1.525e-03	1.0	2.438e-01	1.525e-03	1.0	2.438e-01
320	3.13e-03	7.623e-04	1.0	2.439e-01	7.623e-04	1.0	2.439e-01

Uniform Mesh		Error for $\mathbf{x}_2$ at Grid $\mathbf{tau}$			Error for $\mathbf{x}_2$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	2.423e-02		2.382e-01	2.423e-02		2.382e-01
20	5.00e-02	1.218e-02	1.0	2.382e-01	1.218e-02	1.0	2.382e-01
40	2.50e-02	6.096e-03	1.0	2.422e-01	6.096e-03	1.0	2.422e-01
80	1.25e-02	3.049e-03	1.0	2.434e-01	3.049e-03	1.0	2.434e-01
160	6.25e-03	1.525e-03	1.0	2.438e-01	1.525e-03	1.0	2.438e-01
320	3.13e-03	7.623e-04	1.0	2.439e-01	7.623e-04	1.0	2.439e-01

Uniform Mesh		Error for $\mathbf{u}$ at Grid $\mathbf{tau}$			Error for $\mathbf{u}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	3.631e-05		1.247e-02	3.631e-05		9.998e-03
20	5.00e-02	6.260e-06	2.5	1.247e-02	6.691e-06	2.4	1.247e-02
40	2.50e-02	1.538e-06	2.0	2.697e-03	1.605e-06	2.1	2.697e-03
80	1.25e-02	3.829e-07	2.0	2.519e-03	3.920e-07	2.0	2.519e-03
160	6.25e-03	9.563e-08	2.0	2.466e-03	9.681e-08	2.0	2.466e-03
320	3.13e-03	2.390e-08	2.0	2.453e-03	2.405e-08	2.0	2.453e-03

Table 3: Convergence results for Problem 1.2:  $m = 2$ , equidistant collocation points.

other eigen-structures of the matrix  $M(0)$ , we now consider the following higher dimensional semi-explicit problem posed on the interval  $(0, 1]$ ,

$$\begin{pmatrix} tI \\ 0 \end{pmatrix} \left( (I \ 0) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right)' + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}, \quad (87)$$

or explicitly,

$$tx_1'(t) + B_{11}x_1(t) + B_{12}x_2(t) = g_1(t), \quad (88a)$$

$$B_{21}x_1(t) + B_{22}x_2(t) = g_2(t), \quad (88b)$$

where  $B_{11}, B_{12}, B_{21}, B_{22} \in \mathbb{R}^{2 \times 2}$  are constant matrices,  $g_1(t), g_2(t) \in C[0, 1]$  and the matrix  $B_{22}$  is nonsingular. Due to the problem structure we can immediately rewrite (88) to decouple the inherent ODE system and the system of algebraic constraints.

We first express  $x_2(t)$  from (88b),

$$x_2(t) = B_{22}^{-1}(g_2(t) - B_{21}x_1(t)),$$

and rewrite (88a),

$$x_1'(t) = -\frac{1}{t} (B_{11}x_1(t) - B_{12}B_{22}^{-1}B_{21}x_1(t)) + \frac{1}{t} (g_1(t) - B_{12}B_{22}^{-1}g_2(t)).$$

Consequently, the inherent ODE system is singular with a singularity of the first kind and has the form

$$x_1'(t) = \frac{1}{t} Mx_1(t) + f(t), \quad t \in (0, 1], \quad (89)$$

where

$$M = B_{12}B_{22}^{-1}B_{21} - B_{11}, \quad f(t) = \frac{1}{t} (g_1(t) - B_{12}B_{22}^{-1}g_2(t)).$$

Problems 2.1 and 2.2 given below are specified in such a way that  $M$  has a special eigen-structure: Problem 2.1 has eigenvalues  $\lambda_1 = 0$ ,  $\lambda_2 = 5$ , and Problem 2.2,  $\lambda_1 = \lambda_2 = 0$ .

Uniform Mesh		Error for $\mathbf{x}_1$ at Grid $\tau$			Error for $\mathbf{x}_1$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	4.673e-02		4.620e-01	4.673e-02		4.620e-01
20	5.00e-02	2.344e-02	1.0	4.620e-01	2.344e-02	1.0	4.620e-01
40	2.50e-02	1.173e-02	1.0	4.672e-01	1.173e-02	1.0	4.672e-01
80	1.25e-02	5.867e-03	1.0	4.687e-01	5.867e-03	1.0	4.687e-01
160	6.25e-03	2.934e-03	1.0	4.692e-01	2.934e-03	1.0	4.692e-01
320	3.13e-03	1.467e-03	1.0	4.694e-01	1.467e-03	1.0	4.694e-01

Uniform Mesh		Error for $\mathbf{x}_2$ at Grid $\tau$			Error for $\mathbf{x}_2$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	4.676e-02		4.632e-01	4.676e-02		4.632e-01
20	5.00e-02	2.345e-02	1.0	4.632e-01	2.345e-02	1.0	4.632e-01
40	2.50e-02	1.173e-02	1.0	4.675e-01	1.173e-02	1.0	4.675e-01
80	1.25e-02	5.867e-03	1.0	4.689e-01	5.867e-03	1.0	4.689e-01
160	6.25e-03	2.934e-03	1.0	4.692e-01	2.934e-03	1.0	4.692e-01
320	3.13e-03	1.467e-03	1.0	4.694e-01	1.467e-03	1.0	4.694e-01

Uniform Mesh		Error for $\mathbf{u}$ at Grid $\tau$			Error for $\mathbf{u}$ at Mesh $\mathbf{tcol}$		
N	h	error	order	const.	error	order	const.
10	1.00e-01	2.450e-05		2.432e-02	2.450e-05		2.432e-02
20	5.00e-02	3.070e-06	3.0	2.432e-02	3.070e-06	3.0	2.432e-02
40	2.50e-02	3.840e-07	3.0	2.450e-02	3.840e-07	3.0	2.450e-02
80	1.25e-02	4.801e-08	3.0	2.456e-02	4.801e-08	3.0	2.456e-02
160	6.25e-03	6.001e-09	3.0	2.457e-02	6.001e-09	3.0	2.457e-02
320	3.13e-03	7.501e-10	3.0	2.458e-02	7.501e-10	3.0	2.458e-02

Table 4: Convergence results for Problem 1.2:  $m = 2$ , Gaussian collocation points.

The initial conditions which guarantee that for  $f \in C[0, 1]$  the solution  $x_1(t)$  of (89) is at least in  $C[0, 1]$  read  $Qx_1(0) = 0$ , where  $Q$  is a projection onto the subspace of  $\mathbb{R}^2$  spanned either by the eigenvector associated with the negative eigenvalue of  $M$  or by the principal vector associated with the eigenvalue zero (Problem 2.2). For uniqueness we need to prescribe the value of  $Px_1$ , where  $P = I - Q$ . Clearly,  $P = S + R$ , where  $S$  is a projection onto the subspace of  $\mathbb{R}^2$  spanned by the eigenvector associated with the positive eigenvalue of  $M$  (Problem 2.1). Recall that  $R$  is a projection onto the subspace of  $\mathbb{R}^2$  spanned by the eigenvector associated with the eigenvalue zero (Problems 2.1 and 2.2). According to theory, one needs to prescribe  $Sx_1$  at  $t = 1$ , but  $Rx_1$  can be specified at either  $t = 0$  or  $t = 1$ . We now give the full specification of Problems 2.1 and 2.2.

### Problem 2.1

The matrices  $B_{ij}$  are

$$B_{11} = \begin{pmatrix} 9 & 12 \\ -8 & -11 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

and the right-hand side is given by

$$g_1(t) = \begin{pmatrix} te^{5t}(5 \sin(t) + \cos(t)) + 8e^{5t} \sin(t) + 15 \cos(5t) + 2t^5 - 16.5 \\ -5t \sin(5t) - 7e^{5t} \sin(t) - 13 \cos(5t) - 2t^5 + 14 \end{pmatrix},$$

$$g_2(t) = \begin{pmatrix} e^{5t} \sin(t) + 2 \cos(5t) - 2.5 \\ 2.5e^{5t} \sin(t) + 3 \cos(5t) - 3 - t^5 \end{pmatrix}.$$

For the system (89) we have  $\lambda_1 = 0$ ,  $\lambda_2 = 5$  and

$$M = \begin{pmatrix} -10 & -15 \\ 10 & 15 \end{pmatrix}, \quad ev_0 = \begin{pmatrix} -1.5 \\ 1 \end{pmatrix}, \quad ev_5 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad R = \begin{pmatrix} 3 & 3 \\ -2 & -2 \end{pmatrix}, \quad S = \begin{pmatrix} -2 & -3 \\ 2 & 3 \end{pmatrix}.$$

Moreover,

$$f(t) = \frac{1}{t} \begin{pmatrix} te^{5t}(5 \sin(t) + \cos(t)) + 10e^{5t} \sin(t) + 15 \cos(5t) - 15 \\ -5t \sin(5t) + 10e^{5t} \sin(t) - 15 \cos(5t) + 15 \end{pmatrix},$$

and

$$x_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} e^{5t} \sin(t) - 1.5 + t^5 \\ \cos(5t) - t^5 \end{pmatrix}, \quad x_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} \cos(5t) - 1 \\ e^{5t} \sin(t) \end{pmatrix}.$$

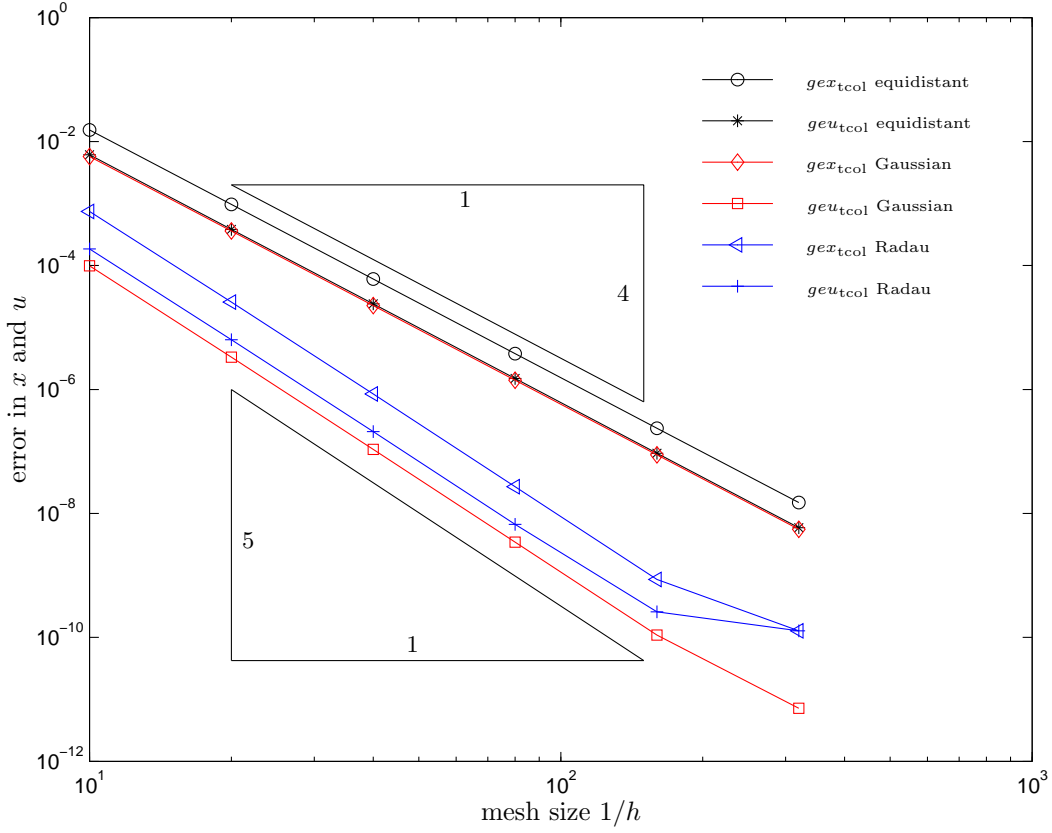


Figure 2: Errors  $gex_{tcol} = \|p - x\|_{tcol}$  and  $geu_{tcol} = \|q - u\|_{tcol}$  in Problem 2.1 for  $m = 4$  equidistant, Gaussian, and Radau collocation in dependence of  $1/h$ .

Here,  $Q = 0$  and the solution is continuous. We just need to specify boundary conditions which are necessary for uniqueness. We prescribe

$$(Sx_1(1))_2 = 2x_{11}(1) + 3x_{12}(1) = 2e^5 \sin(1) + 3 \cos(5) - 4,$$

and for the second condition we specify  $(Rx_1(0))_1$ , which means

$$x_{11}(0) + x_{12}(0) = -0.5.$$

For  $x_2$ , we prescribe the consistency condition  $B_{22}x_2(t) + B_{21}x_1(t) = g_2(t)$  at  $t = 0$ ,

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} x_2(0) + \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} x_1(0) = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}.$$

### Problem 2.2

For this model we define the matrices  $B_{ij}$  as

$$B_{11} = \begin{pmatrix} -9 & -15 \\ 8 & 13 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix},$$

and the right-hand side is given by

$$g_1(t) = \begin{pmatrix} te^{4t}(4 \sin(t) + \cos(t)) - 10e^{4t} \sin(t) - 12 \cos(4t) + 9 \\ -4t \sin(4t) + 9e^{4t} \sin(t) + 11 \cos(4t) - 9 \end{pmatrix},$$

$$g_2(t) = \begin{pmatrix} e^{4t} \sin(t) + 2 \cos(4t) - 3 \\ \frac{7}{3} e^{4t} \sin(t) + 3 \cos(4t) - 3 \end{pmatrix}.$$

In (89) we now have  $\lambda_1 = \lambda_2 = 0$  and

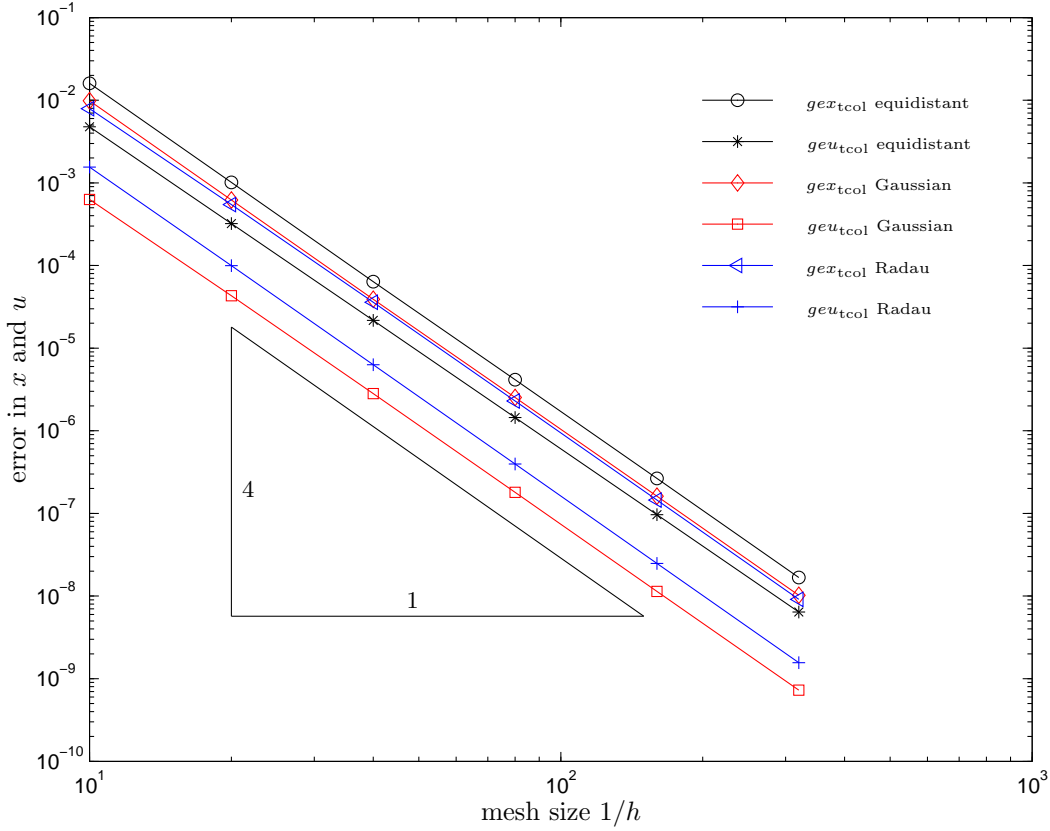


Figure 3: Errors  $gex_{tcol} = \|p - x\|_{tcol}$  and  $geu_{tcol} = \|q - u\|_{tcol}$  in Problem 2.2 for  $m = 3$  equidistant, Gaussian, and Radau collocation in dependence of  $1/h$ .

$$M = \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}, ev_0 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, pv_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, R = \begin{pmatrix} 3 & 3 \\ -2 & -2 \end{pmatrix}, Q = \begin{pmatrix} -2 & -3 \\ -2 & -3 \end{pmatrix}.$$

Moreover,

$$f(t) = \frac{1}{t} \begin{pmatrix} te^{4t}(4\sin(t) + \cos(t)) - 6e^{4t}\sin(t) - 9\cos(4t) + 9 \\ -4t\sin(4t) + 4e^{4t}\sin(t) + 6\cos(4t) - 6 \end{pmatrix},$$

and

$$x_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} e^{4t}\sin(t) - 3 \\ \cos(4t) + 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} \cos(4t) - 1 \\ e^{4t}\sin(t) \end{pmatrix}.$$

We require

$$(Qx_1(0))_1 = 0 \Leftrightarrow 2x_{11}(0) + 3x_{12}(0) = 0$$

for the solution  $x_1$  to be continuous on  $[0, 1]$ . As a second condition we choose to specify  $(Rx_1(1))_1$ , which means

$$x_{11}(1) + x_{12}(1) = e^4 \sin(1) + \cos(4) - 2.$$

The consistency conditions for  $x_2$  read  $B_{22}x_2(0) + B_{21}x_1(0) = g_2(0)$ , or equivalently,

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} x_2(0) + \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} x_1(0) = \begin{pmatrix} -3 \\ -6 \end{pmatrix}.$$

For Problems 2.1 and 2.2, condition (74) is satisfied due to the Remark on page 5.

Numerical results for Problems 2.1 and 2.2 can be found in Figures 2 and 3, respectively. In [30] we have also considered other eigen-structures of the matrix  $M$  and it is worth mentioning that collocation at Gaussian points proved robust for all situations. The experiments for this problem class show convergence rates which do not differ from those given in (84). Here, we have also used Radau points as collocation points, Lobatto points cannot be used because of the singularity. As for Gaussian points,

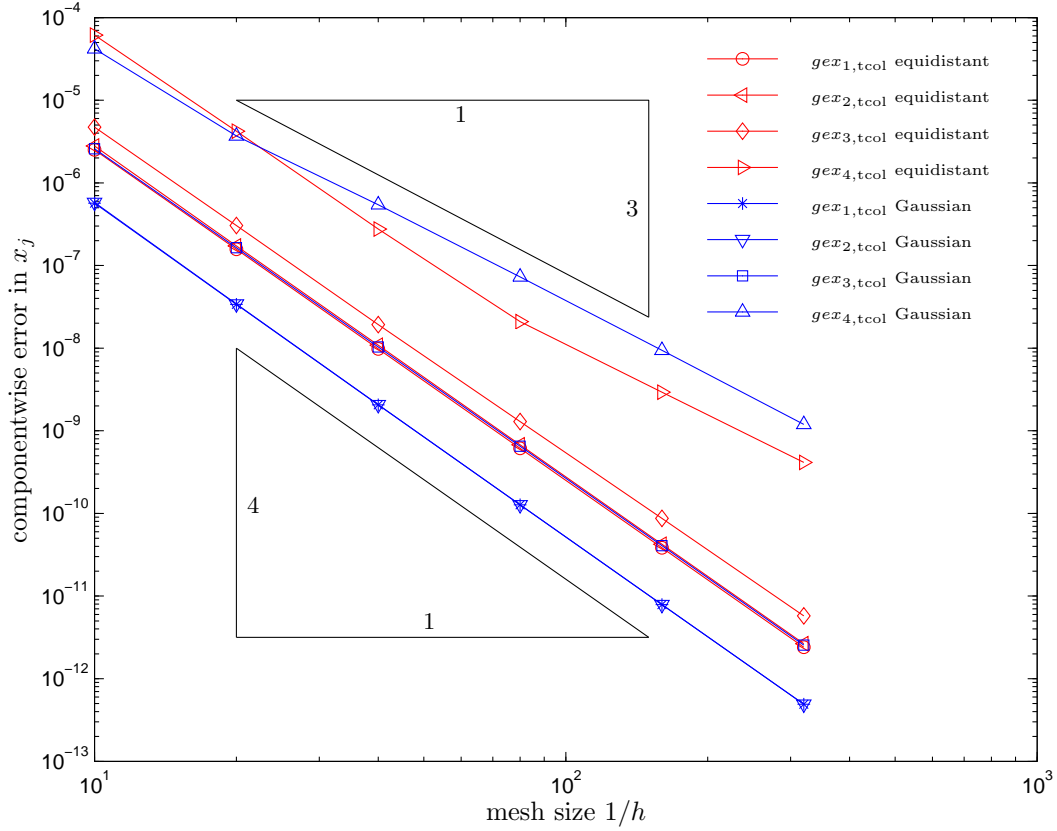


Figure 4: Componentwise errors  $gex_{j,tcol} = \|p_j - x_j\|_{tcol}$ , for  $j = 1, \dots, 4$  in Problem 2.3 for  $m = 3$  equidistant and Gaussian collocation in dependence of  $1/h$ .

the superconvergence order does not always hold, cf. Figure 2.

### 3.2.2 Problems with unbounded canonical projector, $B_{22}$ singular

Here, we continue the studies of the behavior of the canonical projector  $Q_{can}(t)$ . The following examples can be written as a modified system (87) and have the form

$$\begin{pmatrix} t^\alpha I \\ 0 \end{pmatrix} \left( (I \ 0) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right)' + \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}. \quad (90)$$

Here again,

$$D = (I \ 0), \quad D^- = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Moreover,  $B_{22}(t) := t^\beta B_{22}$ ,  $\beta > 0$ ,  $\alpha + \beta = 1$ , and  $B_{22}$  nonsingular. In this setting

$$G_1(t) = \begin{pmatrix} t^\alpha I & B_{12}(t) \\ 0 & B_{22}(t) \end{pmatrix},$$

$$G_1^{-1}(t) = \begin{pmatrix} \frac{1}{t^\alpha} I & -\frac{1}{t^{\alpha+\beta}} B_{12}(t) B_{22}^{-1} \\ 0 & \frac{1}{t^\beta} B_{22}^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{t^\alpha} I & -\frac{1}{t} B_{12}(t) B_{22}^{-1} \\ 0 & \frac{1}{t^\beta} B_{22}^{-1} \end{pmatrix},$$

which means that  $tG_1^{-1}$  has a continuous extension to  $[0, 1]$  in the case where all involved matrices are continuous on  $[0, 1]$ . However, the canonical projector

$$Q_{can}(t) = Q_0 G_1^{-1}(t) B(t) = \begin{pmatrix} 0 & 0 \\ \frac{1}{t^\beta} B_{22}^{-1} B_{21}(t) & I \end{pmatrix} \quad (91)$$

is unbounded on  $[0, 1]$  for  $\beta > 0$ . Moreover,

$$DG_1^{-1}(t)B(t)D^- = \frac{1}{t^\alpha}B_{11}(t) - \frac{1}{t}B_{12}(t)B_{22}^{-1}B_{21}(t) = \frac{1}{t}(t^\beta B_{11}(t) - B_{12}(t)B_{22}^{-1}B_{21}(t)),$$

and hence

$$M(t) = -t^\beta B_{11}(t) + B_{12}(t)B_{22}^{-1}B_{21}(t), \quad M(0) = B_{12}(0)B_{22}^{-1}B_{21}(0).$$

For the experiments, we choose the matrices  $B_{12}(t), B_{21}(t)$  to be constant and the matrices  $B_{22}, B_{21}$  to be nonsingular. Moreover, with the choice  $B_{12} = B_{21}^{-1}$  and  $B_{22}^{-1} = \text{diag}(\lambda_1, \lambda_2)$ ,  $\lambda_i \neq 0$ , the eigenvalues of  $M(0)$  are  $\lambda_1$  and  $\lambda_2$ . Consequently, since  $M(0)$  is nonsingular,  $R = 0$ , and the matrix  $Q_0G_1^{-1}(t)B(t)D^-R = 0$  has a continuous extension on  $[0, 1]$ .

### Problem 2.3

The matrices  $B_{ij}$  are

$$B_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad B_{22}(t) = \begin{pmatrix} t^\beta & 0 \\ 0 & \frac{t^\beta}{5} \end{pmatrix}.$$

Moreover,

$$B_{22} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix}, \quad B_{22}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \quad M(0) = B_{12}B_{22}^{-1}B_{21} = \begin{pmatrix} -7 & -12 \\ 8 & 13 \end{pmatrix}.$$

System (90) is subject to boundary conditions

$$\begin{aligned} x_1(1) &= \sin(1), & x_2(1) &= e^1, \\ x_1(1) + x_2(1) + x_3(1) &= \sin(1) + e^1 + \cos(1), \\ 2x_1(1) + 3x_2(1) + \frac{1}{5}x_4(1) &= 2\sin(1) + 3e^1 + \frac{1}{5}e^{-1}. \end{aligned}$$

Since the eigenvalues of  $M(0)$  are both positive, we prescribe the values of the differential components  $x_1(t)$  and  $x_2(t)$  at  $t = 1$ . The remaining two conditions are consistent boundary conditions for the algebraic components. Note that we solve a terminal value problem which generically shows order reductions when  $Q_{can}(t)$  becomes unbounded for  $t \rightarrow 0$ .

For the right-hand side given by

$$g(t) = \begin{pmatrix} t^\alpha \sin(t) + (3 + t^{\alpha+1}) \cos(t) - t^\ell e^{-t} \\ -2 \cos(t) + t^\ell e^{-t} + (t^\alpha + t^{\alpha+1})e^t \\ t \sin(t) + t^\beta \cos(t) + te^t \\ 2t \sin(t) + \frac{t^{\beta+\ell}}{5}e^{-t} + 3te^t \end{pmatrix}, \quad \ell \geq \alpha.$$

the problem has a solution of the form

$$x(t) = \begin{pmatrix} t \sin(t) \\ te^t \\ \cos(t) \\ t^\ell e^{-t} \end{pmatrix}.$$

We finally set the parameters,  $\alpha = 0$ ,  $\beta = 1$ , and  $\ell = 3$ .

The numerical results for this example can be found in Figure 4. For the case where the differential solution components  $u(t)$  are smooth, no order reduction is observed, although the projection matrix (91) is unbounded for  $t \rightarrow 0$  ( $\beta = 1$ )



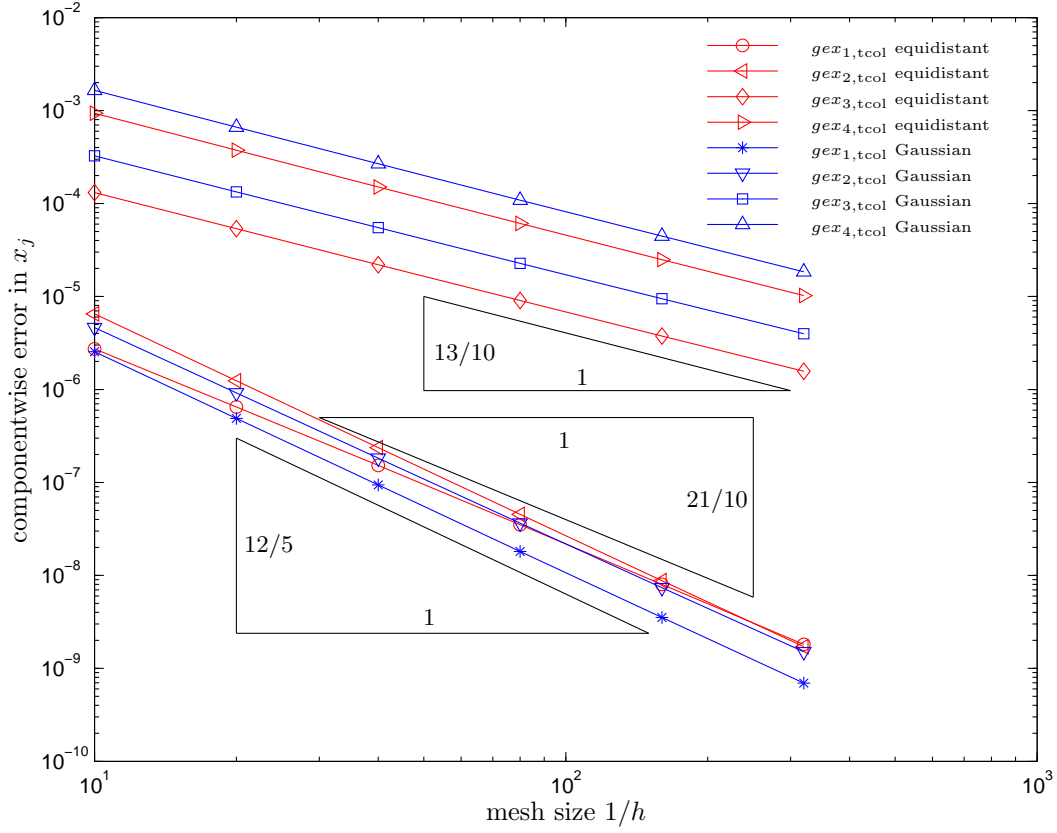


Figure 5: Componentwise errors  $gex_{j,tcol} = \|p_j - x_j\|_{tcol}$ , for  $j = 1, \dots, 4$  in Problem 2.4 for  $m = 5$  equidistant and Gaussian collocation in dependence of  $1/h$ .

### Problem 2.4

We use the same data as in Problem 2.3 except for the right-hand side,

$$g(t) = \begin{pmatrix} \gamma t^{\alpha+\gamma-1} \sin(t) + (3 + t^{\alpha+\gamma}) \cos(t) - t^\ell e^{-t} \\ -2 \cos(t) + t^\ell e^{-t} + (t^{\alpha+\delta} + \delta t^{\alpha+\delta-1}) e^t \\ t^\gamma \sin(t) + t^\beta \cos(t) + t^\delta e^t \\ 2t^\gamma \sin(t) + \frac{t^{\beta+\ell}}{5} e^{-t} + 3t^\delta e^t \end{pmatrix},$$

with  $\ell \geq \alpha \geq 0$ ,  $\alpha + \beta = 1$ ,  $\beta > 0$ ,  $\delta \geq 1$ ,  $\gamma \geq 0$ ,  $\gamma + \alpha \geq 0$ .

The boundary conditions for the resulting terminal value problem are given by

$$\begin{aligned} x_1(1) &= \sin(1), & x_2(1) &= e^1, \\ x_1(1) + x_2(1) + x_3(1) &= \sin(1) + e^1 + \cos(1), \\ 2x_1(1) + 3x_2(1) + \frac{1}{5}x_4(1) &= 2\sin(1) + 3e^1 + \frac{1}{5}e^{-1}. \end{aligned}$$

Again, since the eigenvalues of  $M(0)$  are both positive, we prescribe the values of the differential components  $x_1(t)$  and  $x_2(t)$  at  $t = 1$ . The remaining two conditions are consistent boundary conditions for the algebraic components. Note that (74) is again satisfied. The solution now has the form

$$x(t) = \begin{pmatrix} t^\gamma \sin(t) \\ t^\delta e^t \\ \cos(t) \\ t^\ell e^{-t} \end{pmatrix},$$

and the differential components  $x_1$  and  $x_2$  may become unsmooth. We set  $\alpha = 0$  and  $\beta = 1$ , and the remaining parameters are specified as  $\ell = \frac{5}{2}$ ,  $\gamma = \frac{6}{5}$ ,  $\delta = \frac{5}{2}$ .

The experiments for this example are reported in Figure 5. Now we observe order reductions due to the fact that the canonical projector (91) is unbounded for  $t \rightarrow 0$  ( $\beta = 1$ ). One would expect to see the convergence order  $O(h^{2.5})$  owing to the properties of  $x$ , especially the differential components. However, one loses approximately one additional power of  $h$  which can be attributed to the  $O(1/t)$  behavior of  $Q_{can}(t)$ .

## 4 Summary

We investigated the convergence behavior of collocation schemes applied to solve BVPs in linear index 1 DAEs with a singularity of the first kind. We have considered a very general analytical problem setting, linear index 1 DAE systems with *properly stated leading term*, which were required to be well-posed and have sufficiently smooth solutions. For the discussion of the analytical problem and in the convergence analysis of the collocation scheme, we utilized a theoretical decoupling of the system to derive the explicit inherent ODE and the algebraical constraints. We could show that the global error of the collocation scheme for  $m$  equidistant, Gaussian and Radau collocation points is  $O(h^m)$  uniformly in  $t$ . More precisely,

$$\begin{aligned} ge_{x_{\text{tau}}} &= O(h^{m+1}), & ge_{x_{\text{tcol}}} &= O(h^{m+1}), & ge_{u_{\text{tau}}} &= O(h^{m+1}), & ge_{u_{\text{tcol}}} &= O(h^{m+1}), \\ ge_{x_{\text{tau}}} &= O(h^m), & ge_{x_{\text{tcol}}} &= O(h^m), & ge_{u_{\text{tau}}} &= O(h^m), & ge_{u_{\text{tcol}}} &= O(h^m), \end{aligned}$$

holds for  $m$  odd and even, respectively. By means of experiments we could illustrate the fact that conditions on the problem data which are sufficient for the analytical problem to be well-posed turn out to be necessary for the numerical scheme to have the stage order  $m$ . The superconvergence does not hold in general, due to the singularity in the inherent ODE.

Similar convergence results hold also for systems of DAEs with no singularities in the inherent ODE. According to [30], for appropriately smooth solution  $x$  we observe the above convergence behavior for equidistant collocation points. For Gaussian points this result cannot be improved in general, superconvergence cannot be expected to hold. Clearly, when the solution of the problem is not sufficiently smooth, corresponding order reductions are observed, in line with classical collocation theory. For Radau collocation superconvergence,

$$ge_{x_{\text{tau}}} = O(h^{2m-1}), \quad ge_{u_{\text{tau}}} = O(h^{2m-1})$$

is observed, and the uniform convergence behavior is

$$ge_{x_{\text{tcol}}} = O(h^{m+1}), \quad ge_{u_{\text{tcol}}} = O(h^{m+1}),$$

for all values of  $m$ , except for the case  $m = 1$ , corresponding to the backward Euler rule showing the expected linear convergence.

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