

ASC Report No. 01/2008

# **Adaptive Cell-centered Finite Volume Method**

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[www.asc.tuwien.ac.at](http://www.asc.tuwien.ac.at) ISBN 978-3-902627-00-1

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ISBN 978-3-902627-00-1

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# ADAPTIVE CELL-CENTERED FINITE VOLUME METHOD

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ABSTRACT. In our talk, we propose an adaptive mesh-refining strategy for the cell-centered FVM based on some a posteriori error control for the quantity  $\|\nabla_{\mathcal{T}}(u - \mathcal{I}u_h)\|_{L^2}$ . Here,  $u_h \in \mathcal{P}^0(\mathcal{T})$  denotes the FVM approximation of  $u$  and  $\mathcal{I}$  is a certain interpolation operator. As model example serves the Laplace equation with mixed boundary conditions, where our contributions extend a result of [NIC 05]. Moreover, this approach allows the coupling of finite volume schemes with the boundary element method, which is a new and fruitful combination of the FVM with ideas from [CAR 99-1, CAR 99-2].

## 1. INTRODUCTION AND ELLIPTIC MODEL PROBLEM

Let  $\Omega \subset \mathbb{R}^2$  be a bounded and connected domain with Lipschitz boundary  $\Gamma := \partial\Omega$ , which is divided into a closed Dirichlet boundary  $\Gamma_D \subseteq \Gamma$  with positive surface measure and a Neumann boundary  $\Gamma_N := \Gamma \setminus \Gamma_D$ . We consider the elliptic boundary value problem

$$(1.1) \quad \begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= u_D && \text{on } \Gamma_D \quad \text{and} \quad \partial u / \partial \mathbf{n} = g && \text{on } \Gamma_N. \end{aligned}$$

Here,  $f \in L^2(\Omega)$ ,  $u_D \in H^1(\Gamma_D)$ , and  $g \in L^2(\Gamma_N)$  are given data, and  $L^2(\cdot)$  and  $H^1(\cdot)$  denote the standard Lebesgue- and Sobolev-spaces equipped with the usual norms  $\|\cdot\|_{L^2(\cdot)}$  and  $\|\cdot\|_{H^1(\cdot)}$ . We aim to approximate the unique weak solution  $u \in H^1(\Omega)$  by a postprocessed finite volume scheme. Throughout,  $\mathcal{T}$  denotes a certain triangulation of  $\Omega$ , where  $\mathcal{N}$  and  $\mathcal{E}$  are the corresponding sets of nodes and edges, respectively. The set of nodes (edges) on the Dirichlet resp. Neumann boundary are  $\mathcal{N}_D$  resp.  $\mathcal{N}_N$  ( $\mathcal{E}_D$  resp.  $\mathcal{E}_N$ ). For brevity, we assume that the elements  $T \in \mathcal{T}$  are non-degenerate rectangles and refer to [ERA 07] for triangular elements. For  $T \in \mathcal{T}$ ,  $h_T := \text{diam}(T)$  denotes the Euclidean diameter, and for an edge  $E \in \mathcal{E}$ , we denote by  $h_E$  its length. We say that the triangulation  $\mathcal{T}$  is *almost regular*, if

- (i) the mixed boundary conditions are resolved, i.e. each edge  $E \in \mathcal{E}$  with  $E \cap \Gamma \neq \emptyset$  satisfies either  $E \subseteq \Gamma_D$  or  $E \subseteq \bar{\Gamma}_N$ .
- (ii) the intersection  $T_1 \cap T_2$  of two elements  $T_1, T_2 \in \mathcal{T}$  with  $T_1 \neq T_2$  is either empty or a node or an edge.
- (iii) the edge  $E \in \mathcal{E}$  contains an interior (i.e. hanging) node, there are two edges  $E_1, E_2 \in \mathcal{E}$  with  $E = E_1 \cup E_2$ , cf. Figure 2.1 for examples.

A node  $a \in \mathcal{N} \setminus (\mathcal{N}_D \cup \mathcal{N}_N)$  is a hanging node provided that there are elements  $T_1, T_2 \in \mathcal{T}$  such that  $a \in T_1 \cap T_2$  is a node of  $T_1$  but not of  $T_2$ . Let  $\mathcal{N}_H$  be the set of all hanging nodes.

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*Date:* **March 5, 2008 (revised version, accepted for publication).**

*Key words and phrases.* finite volume method, cell-centered method, diamond path, a posteriori error estimate, adaptive algorithm.

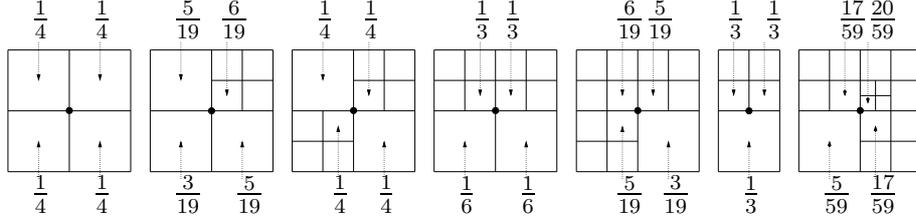


FIGURE 2.1. The weights  $\psi_T$  for an almost regular mesh of squares.

## 2. CELL-CENTERED FINITE VOLUME METHOD

We integrate the differential equation (1.1) over a control volume  $T \in \mathcal{T}$  and use the Gauss divergence theorem to obtain, with  $\mathcal{E}_T$  the set of edges of  $T$ ,

$$\int_T f \, dx = - \int_{\partial T} \frac{\partial u}{\partial \mathbf{n}_T} \, ds = - \sum_{E \in \mathcal{E}_T} \Phi_{T,E}(u) \, ds \quad \text{for all } T \in \mathcal{T}.$$

Here,  $\Phi_{T,E}(u) = \int_E \partial u / \partial \mathbf{n}_{T,E} \, ds$  is the diffusive flux and  $\mathbf{n}_{T,E}$  is the outer normal vector of  $T$  on  $E$ . Let  $u_h \in \mathcal{P}^0(\mathcal{T})$  be a piecewise constant approximation of  $u$ , namely  $u_T := u_h|_T \approx u(x_T)$ , where  $x_T$  denotes the center of an element  $T \in \mathcal{T}$ . For the cell-centered finite volume method,  $u_h$  is computed by replacing the diffusion flux  $\Phi_{T,E}(u)$  by a discrete diffusion flux  $F_{T,E}(u_h)$ . First,  $\Phi_{T,E}(u) = \int_E g \, ds$  is known for a Neumann edge  $E \in \mathcal{E}_N$ . One therefore defines

$$F_{T,E}(u_h) := \Phi_{T,E}(u_h) = \int_E g \, ds \quad \text{for } E \in \mathcal{E}_N.$$

Second, for a non-elementary edge with  $E = E_1 \cup E_2 \in \mathcal{E}$  and  $E_1, E_2 \in \mathcal{E}_E$ , where  $\mathcal{E}_E$  denotes the set of elementary edges (i.e. neither  $E_1$  nor  $E_2$  contain an interior node), there holds  $\Phi_{T,E}(u) = \Phi_{T,E_1}(u) + \Phi_{T,E_2}(u)$ , which leads to the definition

$$F_{T,E}(u_h) := F_{T,E_1}(u_h) + F_{T,E_2}(u_h) \quad \text{for } E = E_1 \cup E_2 \in \mathcal{E} \text{ with } E_1, E_2 \in \mathcal{E}_E.$$

Finally, it remains to define  $F_{T,E}(u_h)$  for elementary and Dirichlet edges  $E \in \mathcal{E}_E \cup \mathcal{E}_D$ . This is done by the *diamond path* method in the following way: For each node  $a \in \mathcal{N}$ , we define

$$(2.1) \quad u_a = \begin{cases} \sum_{T \in \tilde{\omega}_a} \psi_T(a) u_T, & \text{for all } a \in \mathcal{N} \setminus (\mathcal{N}_D \cup \mathcal{N}_N), \\ u_D(a), & \text{for all } a \in \mathcal{N}_D, \\ \bar{u}_a + \bar{g}_a, & \text{for all } a \in \mathcal{N}_N, \end{cases}$$

with certain weights  $\{\psi_T(a) \mid T \in \mathcal{T}, a \in \mathcal{N}_T\}$  for each element of the patch  $\tilde{\omega}_a := \{T \in \mathcal{T} \mid a \in \partial T\}$ . Here,  $\mathcal{N}_T$  denotes the set of nodes of  $T$ . For details on the computation of the weights, the reader is referred to [COU 00]. Figure 2.1 gives the precise values for almost regular triangulations of squares. Details on the computation of  $\bar{u}_a$  and  $\bar{g}_a$  are found in [ERA 07]. For an elementary edge  $E \in \mathcal{E}_E$ , let  $x_{E_p}$  and  $x_{E_q}$  be the starting and end point of

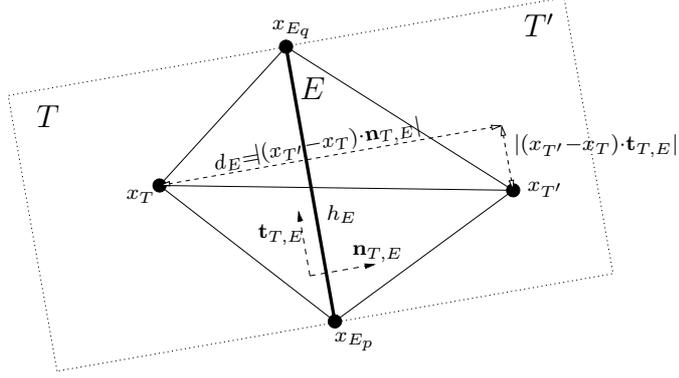


FIGURE 2.2. Notation for the diamond path.

$E \in \mathcal{E}_E \cup \mathcal{E}_D$  and  $T, T' \in \mathcal{T}$  the unique elements with  $E = T \cap T'$ , cf. Figure 2.2. Then,

$$(2.2) \quad F_{T,E}(u_h) := h_E \left( \frac{u_{T'} - u_T}{d_E} - \alpha_E \frac{u_{E_q} - u_{E_p}}{h_E} \right)$$

with  $\alpha_E = \frac{(x_{T'} - x_T) \cdot \mathbf{t}_{T,E}}{(x_{T'} - x_T) \cdot \mathbf{n}_{T,E}}, \quad d_E = (x_{T'} - x_T) \cdot \mathbf{n}_{T,E}.$

Here, the additional unknowns  $u_{E_p}$  and  $u_{E_q}$  are located at the nodes  $x_{E_p}$  and  $x_{E_q}$  and are computed by (2.1). The tangential vector  $\mathbf{t}_{T,E}$  is chosen orthogonal to  $\mathbf{n}_{T,E}$  in mathematically positive sense. For a Dirichlet edge  $E \in \mathcal{E}_D$ , we compute  $F_{T,E}(u_h)$  by (2.2), where  $x_{T'}$  is now replaced by the midpoint  $x_{E_m}$  of  $E$  and  $u_{T'}$  becomes  $u_D(x_{E_m})$ . Altogether, the discrete problem reads: Find  $u_h \in \mathcal{P}^0(\mathcal{T})$  such that

$$- \sum_{E \in \mathcal{E}_T} F_{T,E}(u_h) = \int_T f \, dx, \quad \text{for all } T \in \mathcal{T}.$$

Note, that the conservativity of the continuous flux  $\Phi_{T,E}(u) = -\Phi_{T',E}(u)$  also holds for the discrete flux  $F_{T,E}(u_h) = -F_{T',E}(u_h)$ .

**Remark.** We stress, that even for an admissible triangular mesh  $\mathcal{T}$  in the sense of [EYM 00, Definition 9.1], local mesh-refinement is nontrivial, since admissibility necessarily implies that all angles are strictly less than  $\pi/2$ . For rectangular meshes, local mesh-refinement cannot avoid hanging nodes and thus contradicts the admissibility condition.

### 3. A POSTERIORI ERROR ESTIMATE

For a non-degenerate rectangular element  $T = \text{conv}\{a_1, a_2, a_3, a_4\} \subset \mathbb{R}^2$  with edges  $E_j = \text{conv}\{a_j, a_{j+1}\}$  and  $a_5 := a_1$ , we define  $\mathcal{P}_T = \mathcal{P}_2 \oplus \text{span}\{x^3 - 3xy^2, y^3 - 3yx^2\}$  and  $\Sigma_T = (S_1, \dots, S_8)$ , where

$$S_j(p) = p(a_j), \quad S_{j+4}(p) = \int_{E_j} \frac{\partial p}{\partial \mathbf{n}_{T,E_j}} \, ds \quad \text{for } j = 1, \dots, 4 \quad (p \in \mathcal{P}_T),$$

cf. [NIC 05, Section 4.2]. Then, the Morley-type element  $(T, \mathcal{P}_T, \Sigma_T)$  is a nonconforming finite element. The corresponding Morley interpolant  $\mathcal{I}u_h$  is now defined  $\mathcal{T}$ -elementwise by (3.1)–(3.4). The definition of which is an extension of the definition in [NIC 05, Section

5] to the case of hanging nodes and mixed boundary conditions. For each free node  $a \in \mathcal{N}_T \cap (\mathcal{N} \setminus (\mathcal{N}_D \cup \mathcal{N}_N \cup \mathcal{N}_H))$ , we enforce

$$(3.1) \quad (\mathcal{I}u_h)|_T(a) = \sum_{T_a \in \bar{\omega}_a} \psi_{T_a}(a) u_h|_{T_a}.$$

For each boundary node, the value  $\mathcal{I}u_h(a)$  is prescribed by

$$(3.2) \quad (\mathcal{I}u_h)|_T(a) = \begin{cases} u_D(a) & \text{for } a \in \mathcal{N}_T \cap \mathcal{N}_D, \\ \bar{u}_a + \bar{g}_a & \text{for } a \in \mathcal{N}_T \cap \mathcal{N}_N. \end{cases}$$

For each hanging node  $a \in \mathcal{N}_T \cap \mathcal{N}_H$ , there holds

$$(3.3) \quad (\mathcal{I}u_h)|_T(a) = (\mathcal{I}u_h)|_{T_a}(a),$$

where  $T_a \in \mathcal{T}$  is the unique element with  $a \in \text{int}(E)$  for some (non-elementary) edge  $E \in \mathcal{E}_{T_a}$ . Finally, for each edge  $E \in \mathcal{E}_T$  holds

$$(3.4) \quad \int_E \frac{\partial(\mathcal{I}u_h)|_T}{\partial \mathbf{n}_{T,E}} ds = F_{T,E}(u_h).$$

We stress, that the Morley interpolant  $\mathcal{I}u_h$  is uniquely defined by (3.1)–(3.4). Moreover, the definition ensures the following orthogonality properties, which are essential for the analysis of the proposed error estimator: First, the residual  $R := f + \Delta(\mathcal{I}u_h)$  is  $L^2$ -orthogonal to  $\mathcal{P}^0(\mathcal{T})$ , i.e.

$$\int_T (f + \Delta(\mathcal{I}u_h)) dx = 0 \quad \text{for all } T \in \mathcal{T}.$$

In particular,  $R = f - f_T$ , where  $f_T$  denotes the  $\mathcal{T}$ -piecewise integral mean  $f_T|_T := |T|^{-1} \int_T f dx$ . Second, for boundary edges hold

$$\int_E \frac{\partial(u - \mathcal{I}u_h)}{\partial \mathbf{t}_{T,E}} ds = 0 \quad (E \in \mathcal{E}_D) \quad \text{resp.} \quad \int_E \frac{\partial(u - \mathcal{I}u_h)}{\partial \mathbf{n}_{T,E}} ds = 0 \quad (E \in \mathcal{E}_N).$$

Finally, for interior edges hold

$$\int_E \left[ \frac{\partial(\mathcal{I}u_h)}{\partial \mathbf{n}_{T,E}} \right] ds = 0 \quad (E \in \mathcal{E}_0) \quad \text{resp.} \quad \int_E \left[ \frac{\partial(\mathcal{I}u_h)}{\partial \mathbf{t}_{T,E}} \right] ds = 0 \quad (E \in \mathcal{E}_E).$$

Here,  $\mathcal{E}_0$  denotes the set of all interior edges which are not part of a non-elementary edge and we define the normal jump over an edge  $E \in \mathcal{E}_E$  by

$$[\partial(\mathcal{I}u_h)/\partial \mathbf{n}_{T,E}] := \partial(\mathcal{I}u_h)/\partial \mathbf{n}_{T,E} + \partial(\mathcal{I}u_h)/\partial \mathbf{n}_{T',E},$$

where  $T, T' \in \mathcal{T}$  and  $E = T \cap T'$ . Note that  $\mathbf{n}_{T,E} = -\mathbf{n}_{T',E}$  so that the sum in the definition is in fact a difference. For each non-elementary edge  $E = E_1 \cup E_2 \in \mathcal{E}$  with  $E_1, E_2 \in \mathcal{E}_E$ , we define the normal jump

$$[\partial(\mathcal{I}u_h)/\partial \mathbf{n}_{T,E}]_E(x) := [\partial(\mathcal{I}u_h)/\partial \mathbf{n}_{T,E}]_{E_i}(x) \quad \text{for all } x \in E_i, \quad i = 1, 2.$$

The tangential jump  $[\partial(\mathcal{I}u_h)/\partial\mathbf{t}_{T,E}]$  is defined analogously. For each element  $T \in \mathcal{T}$ , we now define the refinement indicator

$$(3.5) \quad \begin{aligned} \eta_T^2 &:= h_T^2 \|f - f_T\|_{L^2(T)}^2 + \sum_{E \in \mathcal{E}_T \setminus (\mathcal{E}_D \cup \mathcal{E}_N)} h_E \|[\nabla_{\mathcal{T}}(\mathcal{I}u_h)]\|_{L^2(E)}^2 \\ &+ \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_N} h_E \left\| \frac{\partial(u - \mathcal{I}u_h)}{\partial\mathbf{n}_{T,E}} \right\|_{L^2(E)}^2 + \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_D} h_E \left\| \frac{\partial(u - \mathcal{I}u_h)}{\partial\mathbf{t}_{T,E}} \right\|_{L^2(E)}^2. \end{aligned}$$

Note that  $\partial u / \partial\mathbf{n}_{T,E}$  is given by  $g$  and  $\partial u / \partial\mathbf{t}_{T,E}$  is approximated by a first order difference scheme. With techniques known from the finite element method [AIN 00], one proves reliability

$$(3.6) \quad C_{\text{rel}}^{-1} \|\nabla_{\mathcal{T}}(u - \mathcal{I}u_h)\|_{L^2(\Omega)} \leq \eta := \left( \sum_{T \in \mathcal{T}} \eta_T^2 \right)^{1/2}.$$

As for non-conforming FEM the proof employs a Helmholtz decomposition. The Galerkin orthogonality is replaced by the orthogonality properties of  $\mathcal{I}u_h$  stated before. The converse inequality holds even  $\mathcal{T}$ -elementwise, namely for all  $T \in \mathcal{T}$

$$(3.7) \quad C_{\text{loc}}^{-1} \eta_T \leq \left( \|\nabla_{\mathcal{T}}(u - \mathcal{I}u_h)\|_{L^2(\omega_T)}^2 + h_T^2 \|f - f_T\|_{L^2(\omega_T)}^2 \right)^{1/2},$$

where the involved patch reads  $\omega_T := \bigcup \{T' \in \mathcal{T} \mid T \cap T' \neq \emptyset\}$ . This estimate follows by usual techniques and the use of appropriate bubble functions. In particular, we obtain efficiency of  $\eta$  up to oscillation terms,

$$(3.8) \quad C_{\text{eff}}^{-1} \eta \leq \|\nabla_{\mathcal{T}}(u - \mathcal{I}u_h)\|_{L^2(\Omega)} + \|h(f - f_T)\|_{L^2(\Omega)},$$

where  $h \in L^\infty(\Omega)$ ,  $h|_T := h_T$  denotes the local mesh-width. Here, the constants  $C_{\text{rel}}, C_{\text{loc}}, C_{\text{eff}} > 0$  only depend on the shape of the elements in  $\mathcal{T}$ . We refer to [ERA 07] for the detailed proofs of the estimates (3.6)–(3.8).

#### 4. NUMERICAL EXPERIMENTS

We consider the Laplace problem (1.1) on the L-shaped domain  $\Omega = (-1, 1)^2 \setminus ([0, 1] \times [-1, 0])$ . The given exact solution is the harmonic function  $u(x, y) = \text{Im}((x + iy)^{2/3})$  (where  $\text{Im}$  denotes the imaginary part) which reads in polar coordinates

$$u(x, y) = r^{2/3} \sin(2\varphi/3) \quad \text{with} \quad (x, y) = r(\cos \varphi, \sin \varphi).$$

Note that  $u$  has a generic singularity at the reentrant corner  $(0, 0)$ . For the numerical computation, we prescribe the exact Neumann and Dirichlet data, where  $\Gamma_D = \Gamma \setminus \Gamma_N$  and  $\Gamma_N := \{0\} \times (-1, 0) \cup (0, 1) \times \{0\}$ . Note that  $\Gamma_N$  includes the reentrant corner, where the normal derivative  $\partial u / \partial\mathbf{n}$  is singular. We compute the energy error  $E_h := \left( \|u - u_h\|_{L^2(\Omega)}^2 + |u_{\mathcal{T}} - u_h|_{1,h}^2 \right)^{1/2}$ , with  $u_{\mathcal{T}} \in \mathcal{P}^0(\mathcal{T})$  the  $\mathcal{T}$ -piecewise integral mean of  $u$ , and where the discrete  $H^1$ -seminorm is defined by

$$|v_h|_{1,h} = \left( \sum_{E \in \mathcal{E}_E \cup \mathcal{E}_D} \left| (v_{T'} - v_T) / d_E \right|^2 h_E d_E \right)^{1/2}$$

for any  $v_h \in \mathcal{P}^0(\mathcal{T})$ . Provided that  $u \in H^2(\Omega)$ , the diamond path method leads to  $E_h = \mathcal{O}(N^{-1/2})$  with respect to the number  $N = |\mathcal{T}|$  of elements [COU 00]. Moreover, we compute the Morley error  $E_{\mathcal{I}} = \|\nabla_{\mathcal{T}}(u - \mathcal{I}u_h)\|_{L^2(\Omega)}$  and the corresponding error estimator  $\eta$ . All

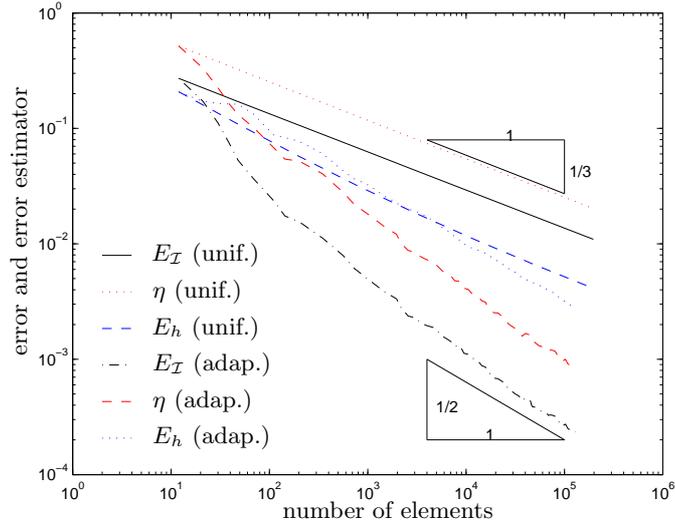


FIGURE 4.1. Errors  $E_{\mathcal{T}}$  and  $E_h$  as well as a posteriori error estimator  $\eta$  in Problem (1.1) for uniform and adaptive mesh-refinement.

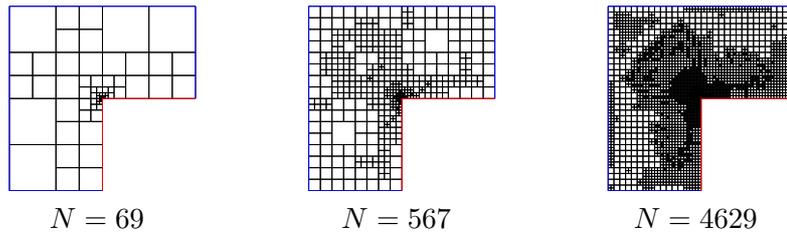


FIGURE 4.2. Adaptively generated meshes in problem (1.1).

computations are done in MATLAB in the spirit of [ALB 99, ERA 08-2]. We employ an adaptive mesh-refining algorithm, which is steered by the local refinement indicators  $\eta_T$  from (3.5): An element  $T_j \in \mathcal{T}$  is marked for refinement provided  $\eta_{T_j} \geq \theta \max\{\eta_{T_1}, \dots, \eta_{T_N}\}$ , where we use  $\theta = 0$  for uniform and  $\theta = 0.5$  for adaptive mesh-refinement, respectively. The numerical outcome is plotted in Figure 4.1 over the number  $N$  of elements. For uniform mesh-refinement, the energy error  $E_h$  converges with a suboptimal order which appears to be slightly better than  $\mathcal{O}(N^{-1/3})$ . The proposed adaptive strategy regains the optimal order of convergence  $\mathcal{O}(N^{-1/2})$ . As can be expected from the finite element method, the Morley error  $E_{\mathcal{T}}$  decreases like  $\mathcal{O}(N^{-1/3})$  for uniform mesh-refinement. The adaptive algorithm leads to an improved order of convergence  $\mathcal{O}(N^{-1/2})$ . As predicted by theory, the error estimator  $\eta$  is observed to be reliable and efficient.

## 5. OUTLOOK

The finite volume method is a well adapted method for the discretization of various convection dominated partial differential equations. It is very popular in the engineering community (fluid mechanics) because of its conservative properties. This method is contrary to the boundary element method (BEM), which can be applied to the most important linear

and homogeneous partial differential equations with constant coefficients also in unbounded domains. The coupling of FVM and BEM combines the advantages of both methods, e.g. in problems where stationary diffusive heat and convection in different media are coupled. While convection would be modelled by the finite volume method, diffusive heat (in a possibly unbounded domain) is solved using the boundary element method.

First numerical examples [ERA 08-1] show the efficiency of the symmetric coupling of FVM and BEM, where we used the results given here for the FVM and results from [CAR 99-1, CAR 99-2] to develop a stable discretization. The resulting system of linear equations is a  $4 \times 4$  block-system instead of a  $3 \times 3$  system for the FEM<sup>NC</sup>-BEM coupling. The additional blocks in the coupling of FVM and BEM result from a further continuous ansatz function on the interface to link the discontinuous displacement field to necessarily continuous boundary ansatz functions on the boundary. Quasi-optimal a priori error estimates and sharp a posteriori error estimates are almost established which justify adaptive mesh-refining algorithms. Additionally to the quantities shown in this paper, the a posteriori estimator contains a term from the BEM measuring the error of the Cauchy data through the Calderón system and the part of the normal resp. tangential jump on the coupling boundary is replaced by the coupling condition resp. tangential derivative of the Cauchy data. Numerical experiments show the adaptive coupling as an efficient tool for the numerical treatment of transmission problems.

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