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INTERPOLATION BETWEEN LOGARITHMIC SOBOLEV AND POINCARÉ INEQUALITIES

ANTON ARNOLD, JEAN-PHILIPPE BARTIER, AND JEAN DOLBEAULT

Abstract. This paper is concerned with intermediate inequalities which interpolate between the logarithmic Sobolev (LSI) and the Poincaré inequalities. Assuming that a given probability measure gives rise to a LSI, we derive *generalized Poincaré inequalities*, improving upon the known constants from the literature. We also analyze the special case when these inequalities are restricted to functions with zero components on the first eigenspaces of the corresponding evolution operator.

1. Introduction

In 1989 W. Beckner [B] derived a family of generalized Poincaré inequalities (GPI) for the Gaussian measure that yield a sharp interpolation between the classical Poincaré inequality and the logarithmic Sobolev inequality (LSI) of L. Gross [G]. For any $1 \le p < 2$ these GPIs read

$$(1.1)\quad \frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\mu_0 - \left(\int_{\mathbb{R}^d} |f|^p\,d\mu_0\right)^{2/p}\right] \le \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu_0 \quad \forall f \in H^1(d\mu_0)\ ,$$

where $\mu_0(x)$ denotes the normal centered Gaussian distribution on $\mathbb{R}^d$:

$$\mu_0(x) := (2\pi)^{-d/2}\,e^{-\frac{1}{2}|x|^2}\ .$$

For $p = 1$ the GPI (1.1) becomes the Poincaré inequality and in the limit $p \to 2$ it yields the LSI.

Theorem 2.4(b) improves upon (1.1) for functions f that are in the orthogonal of the first eigenspaces of the Ornstein-Uhlenbeck operator $\mathsf{N} := -\Delta + x\cdot\nabla$. Moreover, we extend this result to more general measures $d\nu$.

Generalizations of (1.1) to other probability measures and the quest for "sharpest" constants in such inequalities have attracted lots of interest in the last years ([AD, BCR, BR, LO, W]). In [AMTU] GPIs have been derived for strictly log-concave distribution functions $\nu(x)$:

$$(1.2)\quad \frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \le \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall f \in H^1(d\nu)\ ,$$

where κ is the uniform convexity bound of $-\log\nu(x)$, i.e. $\mathrm{Hess}(-\log\nu(x)) \ge \kappa\ \forall x \in \mathbb{R}^d$. This *Bakry-Emery condition* [BE] also implies a LSI with constant $\mathcal{C}_{LS} = \frac{1}{\kappa}$, i.e.,

$$(1.3)\qquad \frac{1}{2}\int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\nu}\right)d\nu \le \mathcal{C}_{LS}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall\ f \in H^1(d\nu)\ ,$$

which is a special case ($p \to 2$ limit) of (1.2).

Latała and Oleszkiewicz (see [LO]; Lemma 1, Corollary 1) derived such GPIs under the weaker assumption that $\nu(x)$ satisfies a LSI with constant $0 < \mathcal{C}_{LS} < \infty$. Under the assumption (1.3) they proved for $1 \le p < 2$:

$$(1.4)\ \frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \le \mathcal{C}_{LS}\min\left\{\frac{2}{p}, \frac{1}{2-p}\right\}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu\ .$$

In the limit $p \to 2$ one recovers again the LSI (1.3). Since this LSI implies a Poincaré inequality (with constant $\mathcal{C}_{LS}$), the second constant in the above *min* just follows

from Hölder's inequality $\left(\int_{\mathbb{R}^d} f\,d\nu\right)^2 \le \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p} = \|f\|^2_{L^p(d\nu)}$ (cf. §3 in [AD] and [LO]). For the special case of even distributions on $\mathbb{R}$ with $\nu(x) > 0$ and finite second moment, they also proved that the minimum on the r.h.s. of (1.4) can be replaced by 1 (cf. Lemma 2 in [LO]; its proof is related to our Theorem 2.4(a) below).

Theorem 2.4(a) improves upon the p-dependent constant on the r.h.s. of (1.4).

While (1.2), (1.4), and Theorem 2.4 deal with *linear* Beckner-type inequalities, §3 is concerned with *nonlinear* refinements. Such improvements are interesting for $1 < p < 2$, as there exist *no minimal functions* that would make (1.2) an equality. As our third result we shall derive in Theorem 3.1 "refined convex Sobolev inequalities" under the assumption that $\nu(x)$ satisfies a LSI. Such type of inequalities were introduced in [AD] under the (more restrictive) Bakry-Emery condition. Our new result in Theorem 3.1 is stronger than Inequality (1.2) in the sense of improving the functional dependance of the l.h.s. of (1.2) on the term $\|f\|_{L^2(d\nu)}/\|f\|_{L^p(d\nu)}$.

Apart of improving upon known inequalities, an additional motivation of this work stems from studying the large-time behavior of parabolic PDEs (like in [AMTU]). In this context one is interested in best possible decay estimates for the time-dependent solution to the steady state — seeking best rates, sharp constants, and exact decay functions. So, questions like tensorization or concentration properties of the measure are *not* our focus here.

2. Generalized Poincaré inequalities

Consider a probability measure on $\mathbb{R}^d$ with density

$$\nu(x) := e^{-V(x)}$$

with respect to Lebesgue's measure, that gives rise to a LSI (1.3) with a positive constant $\mathcal{C}_{LS}$. Here and in the sequel we assume $V \in W^{2,1}_{loc}(\mathbb{R}^d)$. The corresponding (positive) operator $\mathsf{N} := -\Delta + \nabla V \cdot \nabla$ then has the non-degenerate eigenvalue $\lambda_0 = 0$ and a spectral gap $\lambda_1 > 0$. This yields the sharp Poincaré constant $\mathcal{C}_P := 1/\lambda_1$, which satisfies

$$\mathcal{C}_P \le \mathcal{C}_{LS}\ .$$

This is easily recovered by taking $f = 1 + \varepsilon g$ in (1.3) and letting $\varepsilon \to 0$. As a special case, we can for instance consider the case of the Gaussian measure $\nu = \mu_0$ corresponding to $V(x) = |x|^2/2$ up to an additive constant. In this case the eigenvalues of N are $\lambda_k = k$.

For the subsequent analysis we make the following *assumption*:

(H) The eigenfunctions of the operator N form a basis of $L^2(d\nu)$.

We shall denote its (nonnegative) eigenvalues by λ_k, $k \in \mathbb{N}_0$. Eigenvalues with multiplicity larger than 1 are only counted once.

Remark 2.1. *We now give two simple conditions for (H) to hold:*

1. *Let N, defined on $C_0^\infty(\mathbb{R}^d)$, be essentially self-adjoint on $L^2(d\nu)$ and implying a LSI. Then it has a pure point spectrum without accumulation points, see Th. 2.1 of [W2]. Since $\lambda_k \nearrow \infty$, (H) holds, see Theorem XIII.64 in [RS].*
2. *With the following transformation, N can be rewritten as a Schrödinger operator: $f\,e^{-V/2} =: g$ so that $\int_{\mathbb{R}^d} |f|^2\,d\nu = \int_{\mathbb{R}^d} |g|^2\,dx$ and $\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu = \int_{\mathbb{R}^d} (|\nabla g|^2 + V_1\,|g|^2)\,dx$ with $V_1 := \frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V$.*
   Assume $V_1 \in L^1_{loc}(\mathbb{R}^d)$, bounded from below, and $\lim_{|x|\longrightarrow\infty} V_1(x) = \infty$. Then the eigenfunctions of the operator $-\Delta + V_1$ form a basis of $L^2(dx)$ and (H) holds (see Theorem XIII.67 of [RS]).

Under the assumptions *(H)* and that N is a closed operator, we now make a spectral decomposition of any function $f \in H^1(d\nu)$ on the eigenspaces associated to λ_k. With $f = \sum_{k\in\mathbb{N}_0} f_k$, $a_k := \|f_k\|^2_{L^2(d\nu)}$, one obtains:

$$\|f\|^2_{L^2(d\nu)} = \sum_{k\in\mathbb{N}_0} a_k\,, \quad \|\nabla f\|^2_{L^2(d\nu)} = \sum_{k\in\mathbb{N}_0} \lambda_k\, a_k\,, \quad \|e^{-t\,\mathsf{N}}\, f\|^2_{L^2(d\nu)} = \sum_{k\in\mathbb{N}_0} e^{-2\,\lambda_k t} a_k\,.$$

With these notations, we can now prove the first preliminary result, which generalizes the one stated in [B] for the Gaussian measure in the case $k_0 = 1$.

Lemma 2.2. *Let $f \in H^1(d\nu)$. If $f_k = 0$ for $1 \le k < k_0$ for some $k_0 \ge 1$, then*

$$\int_{\mathbb{R}^d} |f|^2\, d\nu - \int_{\mathbb{R}^d} \left|e^{-t\,\mathsf{N}}\, f\right|^2\, d\nu \le \frac{1-e^{-2\lambda_{k_0}\,t}}{\lambda_{k_0}} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu\,.$$

Notice that the (constant) component f_0 of f does not contribute to the inequality.

Proof. We use the decomposition on the eigenspaces of N. For any f_k, $k \ge k_0$, we have

$$\int_{\mathbb{R}^d} |f_k|^2\, d\nu - \int_{\mathbb{R}^d} \left|e^{-t\,\mathsf{N}}\, f_k\right|^2\, d\nu = \left(1 - e^{-2\,\lambda_k\,t}\right) a_k\,.$$

For any fixed $t > 0$, the function $\lambda \mapsto (1-e^{-2\,\lambda\,t})/\lambda$ is monotone decreasing: if $k \ge k_0$, then

$$1 - e^{-2\,\lambda_k\,t} \le \frac{1-e^{-2\,\lambda_{k_0}\,t}}{\lambda_{k_0}}\,\lambda_k\,.$$

Thus we get

$$\int_{\mathbb{R}^d} |f_k|^2\, d\nu - \int_{\mathbb{R}^d} \left|e^{-t\,\mathsf{N}}\, f_k\right|^2\, d\nu \le \frac{1-e^{-2\,\lambda_{k_0}\,t}}{\lambda_{k_0}} \int_{\mathbb{R}^d} |\nabla f_k|^2\, d\nu\,,$$

which proves the result by summation. □

The second ingredient is Nelson's hypercontractive estimate, see [N]:

Lemma 2.3. *For any $f \in L^p(d\nu)$, $p \in (1,2)$, it holds*

$$\left\|e^{-t\mathsf{N}} f\right\|_{L^2(d\nu)} \le \|f\|_{L^p(d\nu)} \quad \forall\, t \ge -\frac{\mathcal{C}_{LS}}{2}\,\log(p-1)\,.$$

To make this note selfcontained we include a sketch of the proof given in [G]. We set $F(t) := \left(\int_{\mathbb{R}^d} |u(t)|^{q(t)}\, d\nu\right)^{1/q(t)}$ with $q(t)$ to be chosen later and $u(x,t) := (e^{-t\mathsf{N}} f)(x)$. A direct computation gives

$$\frac{F'(t)}{F(t)} = \frac{q'(t)}{q^2(t)} \int_{\mathbb{R}^d} \frac{|u|^q}{F^q} \log\left(\frac{|u|^q}{F^q}\right) d\nu - \frac{4}{F^q}\,\frac{q-1}{q^2} \int_{\mathbb{R}^d} \left|\nabla\left(|u|^{q/2}\right)\right|^2\, d\nu\,.$$

We set $v := |u|^{q/2}$, use the LSI (1.3) and choose q such that $4\,(q-1) = 2\,\mathcal{C}_{LS}\,q'$, $q(0) = p$ and $q(t) = 2$. This implies $F'(t) \le 0$ and the result holds with $2 = q(t) = 1 + (p-1)\,e^{2t/\mathcal{C}_{LS}}$.

A combination of Lemma 2.2 and 2.3 gives the following new result.

THEOREM 2.4. *Let ν satisfy the* LSI *(1.3) with the positive constant $\mathcal{C}_{LS}$ (hence having a Poincaré constant $0 < \mathcal{C}_P \le \mathcal{C}_{LS}$) and assume (H).*
(a) Then

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \le C(p)\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \qquad \forall f \in H^1(d\nu) \tag{2.1}$$

holds for $1 \le p < 2$, with

$$C(p) := \frac{1-(p-1)^\alpha}{(2-p)}\mathcal{C}_P, \quad \alpha := \frac{\mathcal{C}_{LS}}{\mathcal{C}_P} \ge 1\ . \tag{2.2}$$

(b) Moreover, if f satisfies $f_k = 0$ for $1 \le k < k_0$ for some $k_0 \ge 2$, then the constant in (2.1) improves to

$$C(p) := \frac{1-(p-1)^\alpha}{\lambda_{k_0}\,(2-p)}, \quad \alpha := \lambda_{k_0}\,\mathcal{C}_{LS} \ge \frac{\lambda_2}{\lambda_1} > 1\ . \tag{2.3}$$

REMARK 2.5 (on case (a)).

1. *In part (a) of this theorem, the constant $C(p)$ depends on both $\mathcal{C}_{LS}$ and $\mathcal{C}_P$, while the estimate (1.4) (obtained in [LO]) only depends on $\mathcal{C}_{LS}$. The resulting improvement of our Theorem 2.4(a) compared to (1.4) is illustrated in Fig. 2.1: It shows the p-dependent constant $C(p)/\mathcal{C}_{LS}$ for several values of α.*

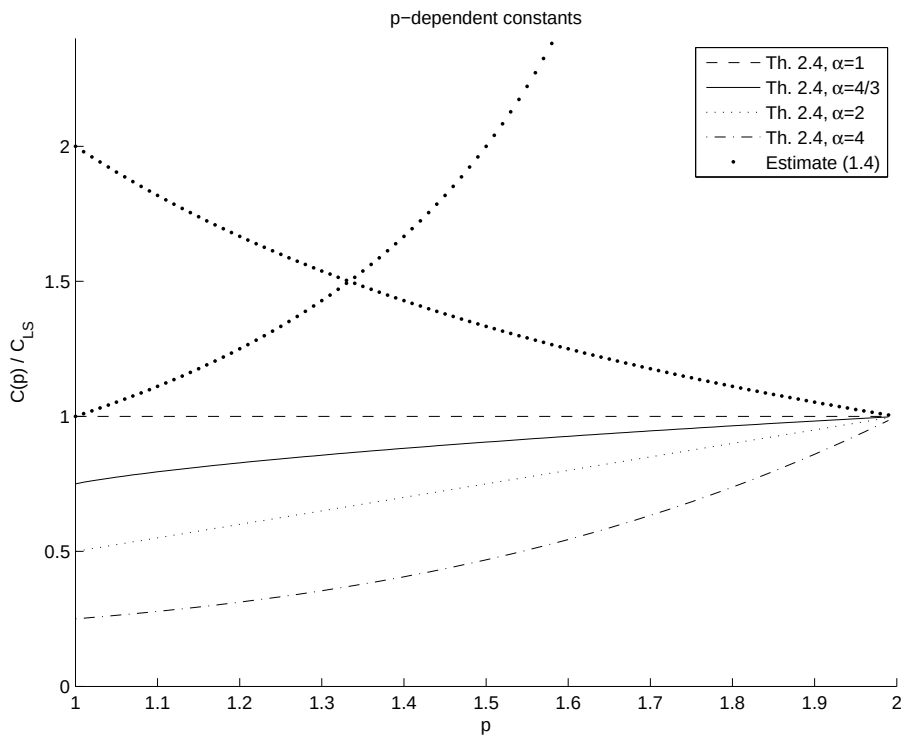

FIG. 2.1. *Comparison of the constants in the GPI for the known estimate (1.4) [$\cdots$] and the new estimates of Theorem 2.4 for various values of α.*

2. *If the logarithmic Sobolev constant takes its minimal value $\mathcal{C}_{LS} = \mathcal{C}_P$ (i.e. $\alpha = 1$), we have $C(p) = \mathcal{C}_{LS}$, for all $p \in [1,2]$ which is the optimal constant (consider $f = 1 + \varepsilon g$ with $\varepsilon \to 0$ in (2.1)). However, even for the Gaussian measure, Inequality (2.1) does* not *admit a non-trivial* minimal function *for any $1 < p < 2$ (proved in §3.5 of [AMTU]). This motivates the interest in nonlinear improvements of (2.1), cf. §3 below.*
3. *For fixed $\alpha \geq 1$, $C(p)$ takes the sharp limiting values for the Poincaré inequality ($p = 1$) and the LSI ($p = 2$): $C(1) = \mathcal{C}_P$ and $\lim_{p\to 2} C(p) = \mathcal{C}_{LS}$ (this also holds for (2.3)).*
4. *For $\alpha > 1$, $C(p)$ is monotone increasing in p since it is a difference quotient of the convex function $p \mapsto (p-1)^\alpha$. Hence, $C(p) < \mathcal{C}_{LS}$ for $p < 2$ and $\alpha > 1$, and Theorem 2.4 strictly improves upon the constants of estimate (1.4).*
5. *The best constant, $c(p)$, on the r.h.s. of (2.1) satisfies $\mathcal{C}_P \leq c(p) \leq C(p)$, for any $p \in (1,2)$ (consider again $f = 1 + \varepsilon g$, $\varepsilon \to 0$).*
6. *Finally, Inequality (2.1) with (2.2) also gives the correct estimate for a distribution ν that only satisfies a Poincaré inequality (with constant $\mathcal{C}_P$) but* no *LSI, which formally corresponds to $\alpha \to \infty$: For fixed p we have from (2.2)*

$$\lim_{\alpha\to\infty} C(p) = \frac{\mathcal{C}_P}{(2-p)},$$

which corresponds to the second constant in the min *of inequality (1.4) (cf. also Theorem 4 in [AD] and §2.2 in [C]).*

Remark 2.6 (on case (b)). *Even in the Gaussian case $\nu = \mu_0$, Theorem 2.4(b) improves on Beckner's result for any $k_0 \geq 2$.*

Example 2.7.

1. *$\mathcal{C}_{LS} = \mathcal{C}_P$ clearly holds for the Gaussian distribution μ_0.*
2. *An example for $\mathcal{C}_{LS} > \mathcal{C}_P$ is obtained by the distribution $\nu(x) := c_\varepsilon \exp(-|x| - \varepsilon\, x^2)$, $x \in \mathbb{R}$ with $\varepsilon \to 0$, which was kindly suggested to us by Michel Ledoux. While $\mathcal{C}_P$ is bounded for $\varepsilon \in [0,1]$, $\mathcal{C}_{LS}$ blows up like $\mathcal{O}(1/\varepsilon)$, which can be estimated with Th. 1.1 of [BG] (also see [BR] for a simplified approach and §3 of [L] for a review of applications in geometry).*

3. A refined interpolation inequality

While (1.2) is a *linear* inequality between the *p–entropy* (or *p–variance*) on the l.h.s. and the *energy* on the r.h.s., we shall now derive *nonlinear* improvements of it. An inequality stronger than (1.2) has been shown by the first and the third author in [AD]. Under the Bakry-Emery condition on the measure $d\nu$, they proved that for all $p \in [1,2)$:

$$\frac{1}{(2-p)^2}\left[\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^p\, d\nu\right)^{2\left(\frac{2}{p}-1\right)} \left(\int_{\mathbb{R}^d} f^2\, d\nu\right)^{p-1}\right] \leq \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|^2\, d\nu \tag{3.1}$$

for any $f \in H^1(d\nu)$, where κ is the uniform convexity bound of $-\log \nu(x)$. The estimate (1.2) is a consequence of this inequality (and hence weaker). This can be shown using Hölder's inequality, $\left(\int_{\mathbb{R}^d} |f|^p\, d\nu\right)^{2/p} \leq \int_{\mathbb{R}^d} f^2\, d\nu$ and the inequality $\left(1 - t^{2-p}\right)/(2-p) \geq 1 - t$ for any $t \in [0,1]$, $p \in (1,2)$. With the same notations as in Section 2, we can prove the following new result:

THEOREM 3.1. *Let ν satisfy the* LSI *(1.3) with the positive constant $\mathcal{C}_{LS}$ and assume (H). If $f \in H^1(d\nu)$ is such that $f_k = 0$ for $1 \le k < k_0$ for some $k_0 \ge 1$, then*

$$\lambda_{k_0} \max\left\{ \frac{\|f\|^2_{L^2(d\nu)} - \|f\|^2_{L^p(d\nu)}}{1-(p-1)^\alpha}, \ \frac{\|f\|^2_{L^2(d\nu)}}{|\log(p-1)^\alpha|} \log\left(\frac{\|f\|^2_{L^2(d\nu)}}{\|f\|^2_{L^p(d\nu)}} \right) \right\} \le \|\nabla f\|^2_{L^2(d\nu)} \tag{3.2}$$

holds for $1 \le p < 2$, with $\alpha := \lambda_{k_0} \mathcal{C}_{LS} \ge 1$.

Proof. We shall proceed in two steps and derive first for all $\gamma \in (0,2)$ the following inequality, which is inspired by (3.1):

$$\frac{1}{(2-p)^2} \left[\int_{\mathbb{R}^d} f^2 \, d\nu - \left(\int_{\mathbb{R}^d} |f|^p \, d\nu \right)^{\frac{\gamma}{p}} \left(\int_{\mathbb{R}^d} f^2 \, d\nu \right)^{\frac{2-\gamma}{2}} \right] \le K_p(\gamma) \int_{\mathbb{R}^d} |\nabla f|^2 \, d\nu \ , \tag{3.3}$$

with

$$K_p(\gamma) := \frac{1-(p-1)^{\alpha\gamma/2}}{\lambda_{k_0} \, (2-p)^2} \ .$$

Step 1: The computations are analogous to the ones of Theorem 2.4. With the same notations as above, the squared bracket of (3.3) can be bounded from above by

$$\begin{aligned} \mathcal{N} &:= \|f\|^2_{L^2(d\nu)} - \left\| e^{-t\mathsf{N}} f \right\|^\gamma_{L^2(d\nu)} \|f\|^{2-\gamma}_{L^2(d\nu)} \\ &= a_0 + \sum_{k \ge k_0} a_k - \left(a_0 + \sum_{k \ge k_0} a_k \, e^{-2\lambda_k t} \right)^{\frac{\gamma}{2}} \left(a_0 + \sum_{k \ge k_0} a_k \right)^{\frac{2-\gamma}{2}} \end{aligned}$$

for any $t \ge -\frac{\mathcal{C}_{LS}}{2} \log(p-1)$ as in Lemma 2.3. By Hölder's inequality, we get

$$\begin{aligned} a_0 + \sum_{k \ge k_0} a_k \, e^{-\gamma\lambda_k t} &= a_0 + \sum_{k \ge k_0} \left(a_k \, e^{-2\lambda_k t} \right)^{\frac{\gamma}{2}} \cdot a_k^{\frac{2-\gamma}{2}} \\ &\le \left(a_0 + \sum_{k \ge k_0} a_k \, e^{-2\lambda_k t} \right)^{\frac{\gamma}{2}} \left(a_0 + \sum_{k \ge k_0} a_k \right)^{\frac{2-\gamma}{2}} . \end{aligned}$$

Then

$$\mathcal{N} \le \sum_{k \ge k_0} a_k \left(1 - e^{-\gamma\lambda_k t} \right)$$

can be bounded as in the proof of Theorem 2.4:

$$\mathcal{N} \le \frac{1 - e^{-\gamma\lambda_{k_0} t}}{\lambda_{k_0}} \sum_{k \ge k_0} \lambda_k \, a_k = \frac{1 - e^{-\gamma\lambda_{k_0} t}}{\lambda_{k_0}} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\nu$$

using the decay of $\lambda \mapsto \left(1 - e^{-\gamma\lambda t}\right)/\lambda$. The result then holds with

$$e^{-\gamma\lambda_{k_0} t} = (p-1)^{\gamma\lambda_{k_0}\mathcal{C}_{LS}/2} \ .$$

Step 2: Next we shall optimize Inequality (3.3) w.r.t. $\gamma \in (0,2)$. After dividing the l.h.s. of (3.3) by $K_p(\gamma)$ we have to find the maximum of the function

$$\gamma \mapsto h(\gamma) := \frac{1-a^\gamma}{1-b^\gamma} \ , \qquad \text{with} \quad a = \frac{\|f\|_{L^p(d\nu)}}{\|f\|_{L^2(d\nu)}} \le 1 \ , \quad b = (p-1)^{\alpha/2} \le 1$$

on $\gamma \in [0,2]$. We write $h(\gamma) = g(b^\gamma)$ with

$$g(y) := \frac{1 - y^{\frac{\log a}{\log b}}}{1-y} \ .$$

For $a < b < 1$ the function $g(y)$ is monotone increasing (since it is a difference quotient of the convex function $y^{\log a/\log b}$). Hence, $h(\gamma)$ is monotone decreasing. Analogously, h is monotone increasing for $b < a < 1$. Hence, the maximum of the function $h(\gamma)$ on $[0,2]$ is either $h(2)$ (if $a > b$) or $\lim_{\gamma\to 0} h(\gamma)$ (in the case $a < b$). This yields the two terms in the *max* of (3.2). □

Remark 3.2.

1. *The limiting cases of (3.2) are the sharp Poincaré inequality ($p \to 1$, for $k_0 = 1$) and the LSI ($p \to 2$).*
2. *Note that the first term in the* max *of (3.2) exactly corresponds to the linear Inequality (2.1). Hence, the statement of Theorem 3.1 is always at least as strong as Theorem 2.4. The second term in this* max *is dominant iff*

$$\frac{\|f\|_{L^p(d\nu)}}{\|f\|_{L^2(d\nu)}} < (p-1)^{\alpha/2} \ ,$$

   as it is seen from the above proof. In this case Inequality (3.2) strictly improves upon (2.1).
3. *For the special case when $\alpha = 1$ and $k_0 = 1$, we now compare the new Theorem 3.1 to the known Inequality (3.1). Since the latter was obtained in [AD] under the more restrictive Bakry-Emery condition, we assume here $\mathcal{C}_P = \mathcal{C}_{LS} = 1/\kappa$ which make this comparison very conservative. For $\|f\|_{L^p(d\nu)}/\|f\|_{L^2(d\nu)}$ "large" (i.e. $a > b$), (3.2) coincides with the linear Inequality (1.2), and (3.1) is strictly stronger. On the other hand, for "small" $\|f\|_{L^p(d\nu)}/\|f\|_{L^2(d\nu)}$, the new estimate (3.2) is stronger than (3.1).*
   In Fig. 3.1 we illustrate this comparison (still in the case $\alpha = 1$ and $\mathcal{C}_{LS} = 1/\kappa$) between the Inequalities (1.2), (3.1), and (3.2). To this end we rewrite them in terms of the non-negative functionals

$$\mathsf{e}_p[f] := \frac{\|f\|^2_{L^2(d\nu)}}{\|f\|^2_{L^p(d\nu)}} - 1 \ , \qquad \mathsf{I}_p[f] := \frac{2(2-p)}{\|f\|^2_{L^p(d\nu)}} \|\nabla f\|^2_{L^2(d\nu)} \ .$$

   *Note that $\mathsf{e}_p[f]$ is a scaled version of the p–*entropy *on the l.h.s. of (1.2). The GPI (1.2) is then the following* linear *lower bound for $\mathsf{I}_p[f]$:*

$$\mathsf{e}_p[f] \le \frac{1}{2\kappa} \mathsf{I}_p[f] \ ,$$

   while (3.1) and (3.2) are the nonlinear refinements*:*

$$k_1(\mathsf{e}_p[f]) \le \frac{1}{2\kappa} \mathsf{I}_p[f], \quad k_1(e) := \frac{1}{2-p}[e + 1 - (e+1)^{p-1}] \ge e \ ,$$

   and, respectively,

$$k_2(\mathsf{e}_p[f]) \le \frac{1}{2\kappa} \mathsf{I}_p[f], \quad k_2(e) := max\left\{e, \frac{2-p}{|\log(p-1)|}(e+1)\log(e+1)\right\} \ge e \ .$$

We remark that for the logarithmic entropy similar nonlinear estimates are discussed in §§1.3, 4.3 of [L].

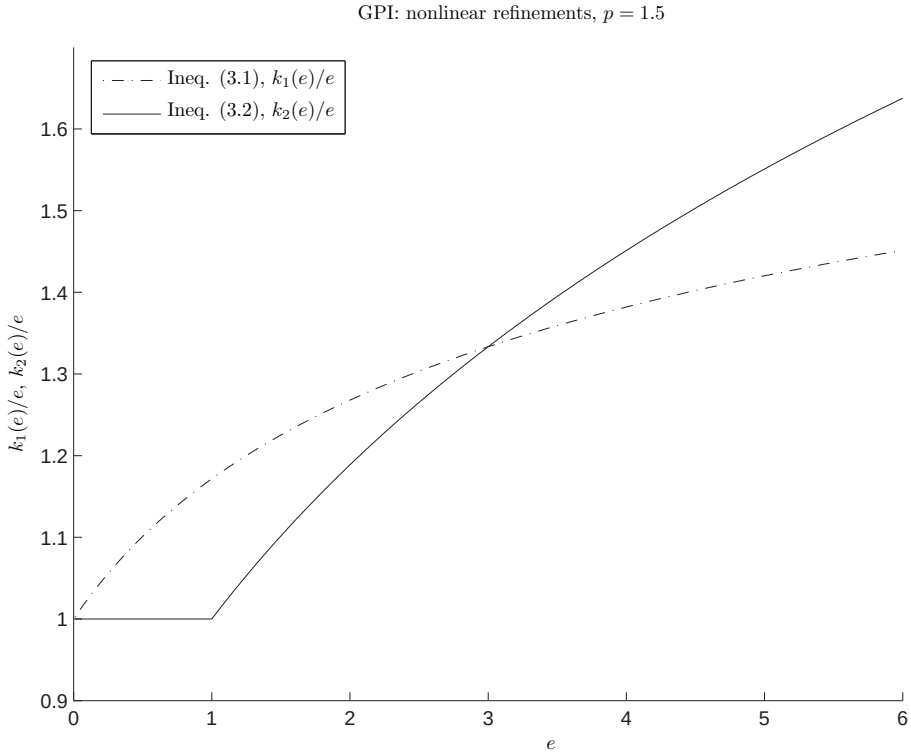

FIG. 3.1. *Comparison of the nonlinear refinements of the GPI (1.2) for $\alpha = 1$ and $p = 1.5$: The estimate (3.1) is known from [AD] and estimate (3.2) is new.*

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